Sydney Grammar School
Mathematics Department
Assessment Examinations 2006

FORM VI

MATHEMATICS EXTENSION 1

Examination date

Monday 6th March 2006

Time allowed

Periods 6 & 7

Instructions

All six questions may be attempted.
All six questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection

Write your candidate number clearly on each booklet.
Hand in the six questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.

Checklist

Folded A3 booklets: 6 per boy. A total of 1000 booklets should be sufficient.
Candidature: 122 boys.

Examiner

JNC

QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) Write down the exact value of $\cos\frac{3\pi}{4}$. [1]

(b) Write down a primitive of $\dfrac{1}{3x}$. [1]

(c) Sketch the graph of $y = \tan^{-1} x$. [1]

(d) Write down the derivative of $\cos 3x$. [1]

(e) Express $570°$ in radians in terms of π. [2]

(f) Sketch the graph of $y = \sin 2x$, for $0 \le x \le 2\pi$. [2]

(g) Find the exact value of $\displaystyle\int_0^1 e^{2x}\,dx$. [2]

(h) Differentiate $\sin^{-1}\frac{x}{2}$. [2]

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) Differentiate $x \ln x$. [2]

(b) Solve $2\cos\theta + \sqrt{3} = 0$, for $0 \le \theta \le 2\pi$. [2]

(c) Find the exact area of the sector which subtends an angle of $40°$ at the centre of a circle of radius 5 centimetres. [2]

(d) Find the equation of the tangent to the curve $y = \tan 2x$ at the point where $x = \frac{\pi}{8}$. [3]

(e) Find the exact value of $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{2x+1}\,dx$. [3]

Exam continues next page ...

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a) Express $\sin 2\theta$ in terms of t, where $t = \tan\theta$. [1]

(b) Find $\displaystyle\int \frac{x^2+2}{x}\,dx$. [2]

(c) (i) Express $x°$ in radians in terms of π. [1]

(ii) Find the derivative of $\sin x°$. [2]

(d) Find the exact value of:

(i) $\sin^{-1}\left(-\frac{1}{2}\right)$ [1]

(ii) $\cos\left(\tan^{-1}\left(-\frac{2}{3}\right)\right)$ [2]

(e) (i) Differentiate $y = e^{-x^2}$. [1]

(ii) Hence find the exact value of $\displaystyle\int_0^1 xe^{-x^2}\,dx$. [2]

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a) Find a general solution to the equation $\cos 2x = \sin x$. [3]

(b) (i) Express $4\cos x + 3\sin x + 5$ in simplest form in terms of t, where $t = \tan\frac{x}{2}$. [2]

(ii) Hence, or otherwise, solve $4\cos x + 3\sin x + 5 = 0$, for $0 \le x \le 360°$. Give your answer correct to the nearest minute. [2]

(c) Show that $\sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}} = \tan\theta$, for $0 \le \theta < \dfrac{\pi}{2}$. [2]

(d) Consider the function $f(x) = 1 + \dfrac{3}{x-2}$, for $x > 2$.
Find the inverse function. [3]

Exam continues overleaf ...

QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**

(a) The value, $\$V$, of a car decreases at a rate which is proportional to its value. That is, the value of the car decreases according to the equation $V = Ae^{-kt}$, where t is the time in years and A and k are constants. The purchase price of a car was \$70 000 and after 2 years its value dropped to \$50 000.

(i) Show that $\dfrac{dV}{dt} = -kV$. $\boxed{1}$

(ii) How long will it take for the value of the car to drop below \$30 000. Give your answer correct to the nearest month. $\boxed{3}$

(b) (i) Determine the domain and range of $y = 2\sin^{-1}(x-1)$. $\boxed{2}$

(ii) Sketch the graph of $y = 2\sin^{-1}(x-1)$. $\boxed{2}$

(iii) Make x the subject of the equation $y = 2\sin^{-1}(x-1)$. $\boxed{1}$

(iv) Find the exact area bounded by the curve $y = 2\sin^{-1}(x-1)$, the line $x = 2$ and the x-axis. $\boxed{3}$

QUESTION SIX (12 marks) Use a separate writing booklet. **Marks**

(a) The function $f(x) = x^2 \ln\left(\dfrac{1}{x^3}\right)$, for $x > 0$, has first derivative $-3x(1 + 2\ln x)$ and second derivative $-3(3 + 2\ln x)$.

(i) Find the exact value of x at which the function has its only stationary point. $\boxed{1}$

(ii) Determine the nature of the stationary point. $\boxed{2}$

(iii) Find the exact value of x at which the function has a point of inflection. $\boxed{2}$

(iv) Given that $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} f'(x) = 0^+$ sketch the graph of $y = f(x)$ for the domain $0 < x \le 1$. $\boxed{2}$

(b) (i) Show that $8\cos^4 x = 3 + 4\cos 2x + \cos 4x$. $\boxed{2}$

(ii) Find the volume of the solid generated by rotating the area enclosed between $y = \cos x$ and $y = \cos^2 x$, for $0 \le x \le \dfrac{\pi}{2}$, about the x-axis. $\boxed{3}$

END OF EXAMINATION

SOLUTIONS: FORM VI 3UNIT ASSESSMENT 2006

QUESTION ONE

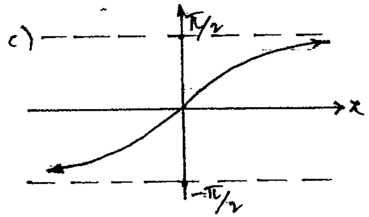
TOTAL: 12×6

$= 72.$

a) $\cos \dfrac{3\pi}{4} = -\dfrac{1}{\sqrt{2}}$ ✓

b) $\dfrac{1}{3} \ln x + c$ ✓

c)



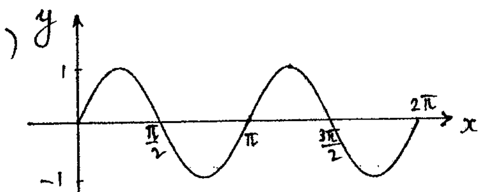
✓

d) $\dfrac{d}{dx}(\cos 3x) = -3\sin 3x$ ✓

e) $570^\circ = 570 \times \dfrac{\pi}{180}$ ✓

$= \dfrac{19\pi}{6}$ radians ✓

f)



✓✓

g) $\displaystyle\int_0^1 e^{2x}\,dx = \frac{1}{2}\left[e^{2x}\right]_0^1$ ✓

$= \dfrac{1}{2}(e^2 - 1)$ ✓

h) $\dfrac{d}{dx} \sin^{-1}\dfrac{x}{2} = \dfrac{1}{\sqrt{1 - \frac{x^2}{4}}} \cdot \dfrac{1}{2}$ ✓

$= \dfrac{1}{\sqrt{\frac{4 - x^2}{4}}} \cdot \dfrac{1}{2}$

$= \dfrac{1}{\sqrt{4 - x^2}}$ ✓

QUESTION TWO

a) $\frac{d}{dx}(x \ln x) = x \cdot \frac{1}{x} + 1 \cdot \ln x$ ✓

$= 1 + \ln x$ ✓

b) $2\cos\theta + \sqrt{3} = 0$

$\cos\theta = -\frac{\sqrt{3}}{2}$

Related angle $= \frac{\pi}{6}$ ✓

$\therefore \quad \theta = \frac{5\pi}{6}$ or $\frac{7\pi}{6}$ ✓

c) Area $= \frac{1}{2} r^2 \theta$

$= \frac{1}{2} \times 5^2 \times 40 \times \frac{\pi}{180}$ ✓

$= \frac{25\pi}{9}$ cm^2 ✓

d) $y = \tan 2x$

$\frac{dy}{dx} = 2\sec^2 2x$ ✓

At $x = \frac{\pi}{8}$, $y' = 2\left(\sec\frac{\pi}{4}\right)^2$

$= 4$ ✓

and $y = 1$.

Equation is $y - 1 = 4\left(x - \frac{\pi}{8}\right)$

ie, $y = 4x - \frac{\pi}{2} + 1$ ✓

e) $\int_0^{\frac{1}{2}} \frac{1}{2x+1}\,dx = \frac{1}{2}\left[\ln(2x+1)\right]_0^{\frac{1}{2}}$ ✓✓ (−1 per error)

$= \frac{1}{2}(\ln 2 - \ln 1)$

$= \frac{1}{2}\ln 2$ ✓

QUESTION THREE

a) Let $t = \tan\theta$, $\sin 2\theta = \frac{2t}{1+t^2}$ ✓

b) $\int \frac{x^2+2}{x}\,dx = \int x + \frac{2}{x}\,dx$ ✓

$= \frac{x^2}{2} + 2\ln x + c$ ✓

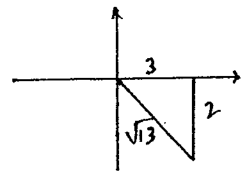
c)(i) $x^\circ = \frac{\pi x}{180}$ radians ✓

(ii) $\frac{d}{dx}(\sin x^\circ) = \frac{d}{dx}\left(\sin\frac{\pi x}{180}\right)$

$= \frac{\pi}{180}\cos\frac{\pi x}{180}$ ✓✓ (−1 per error)

d)(i) $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ ✓

(ii) Let $\alpha = \tan^{-1}\frac{2}{3}$



$\cos\alpha = \frac{3}{\sqrt{13}}$ ✓✓ (1 for $\sqrt{13}$, 1 for ratio)

e)(i) $\frac{d}{dx}\left(e^{-x^2}\right) = -2x\,e^{-x^2}$ ✓

(ii) $\int_0^1 x e^{-x^2}\,dx = -\frac{1}{2}\left[e^{-x^2}\right]_0^1$ ✓

$= -\frac{1}{2}\left(e^{-1} - 1\right)$ ✓

QUESTION FOUR

a) $\cos 2x = \sin x$

$\cos 2x = \cos\left(\frac{\pi}{2} - x\right)$

So $2x = 2n\pi + \left(\frac{\pi}{2} - x\right)$ or $2x = 2n\pi - \left(\frac{\pi}{2} - x\right)$

$3x = 2n\pi + \frac{\pi}{2}$ or $x = 2n\pi - \frac{\pi}{2}$

$\therefore$ $x = \frac{2n\pi}{3} + \frac{\pi}{6}$ or $x = 2n\pi - \frac{\pi}{2}$ for n an integer

b) (i) Let $t = \tan\frac{x}{2}$,

$4\cos x + 3\sin x + 5$

$= 4 \cdot \frac{1-t^2}{1+t^2} + \frac{3 \cdot 2t}{1+t^2} + 5$

$= \frac{4 - 4t^2 + 6t + 5 + 5t^2}{1+t^2}$

$= \frac{t^2 + 6t + 9}{1+t^2}$

$= \frac{(t+3)^2}{1+t^2}$

(ii) $\frac{(t+3)^2}{1+t^2} = 0$

$\therefore t = -3$

$\tan\frac{x}{2} = -3$

Related angle $= 71^\circ 34'$

$\frac{x}{2} = 108^\circ 26'$

$\therefore x = 216^\circ 52'$

(Note: $x = 180^\circ$ is not a solution)

c) $\sqrt{\frac{1 - \cos 2\theta}{1 + \cos 2\theta}} = \sqrt{\frac{\sin^2\theta}{\cos^2\theta}}$

$= \sqrt{\tan^2\theta}$

$= \tan\theta$, since $\tan\theta \geq 0$, for $0 < \theta < \frac{\pi}{2}$

d) Let $y = 1 + \frac{3}{x-2}$,

then f^{-1}: $x = 1 + \frac{3}{y-2}$

$x - 1 = \frac{3}{y-2}$

$\therefore y - 2 = \frac{3}{x-1}$

$y = 2 + \frac{3}{x-1}$

The range of $f(x)$ is $y > 1$, so the domain of $f^{-1}(x)$ is $x > 1$.

(Domain must be indicated for full marks)

QUESTION FIVE

a) (i) $V = Ae^{-kt}$

$\frac{dV}{dt} = -kAe^{-kt}$

$= -kV$

(ii) When $t = 0$, $A = 70000$

When $t = 2$, $V = 50000$

$50000 = 70000\,e^{-2k}$

$\therefore k = -\frac{1}{2}\ln\frac{5}{7} \doteqdot 0.168236\ldots$

When $V = 30000$; $30000 = 70000e^{-kt}$

$\therefore t = -\frac{1}{k}\ln\frac{3}{7}$

$= 5.03636\ldots$

The cars' value will drop below \$30000 in the 61st month after purchase (or 5 years)

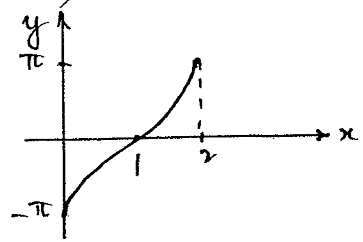
b)

(i) $y = 2\sin^{-1}(x-1)$.

Domain is $-1 \leq x - 1 \leq 1$ and range is $-\frac{\pi}{2} \leq \frac{y}{2} \leq \frac{\pi}{2}$

ie, $0 \leq x \leq 2$ $\quad -\pi \leq y \leq \pi$

(ii)



−1 per error; all intercepts need to be shown

(iii) $\frac{y}{2} = \sin^{-1}(x-1)$

$x = 1 + \sin\frac{y}{2}$

(iv) $A = 2\pi - \int_0^{\pi} 1 + \sin\frac{y}{2}\,dy$

$= 2\pi - \left[y - 2\cos\frac{y}{2}\right]_0^{\pi}$

$= 2\pi - [(\pi - 0) - (0 - 2)]$

$= \pi - 2$ units square.

QUESTION SIX

a) (i)

$$-3x(1+2\ln x)=0$$

$1+2\ln x=0$ only since $x>0$.

$x=e^{-\frac{1}{2}}$ is the

(ii) At $x=e^{-\frac{1}{2}}$, $f''(e^{-\frac{1}{2}})=-3.(3+2.(-\frac{1}{2}))$

$=-6$,

So $x=e^{-\frac{1}{2}}$ is a maximum stationary point.

(iii) Point of inflection at :

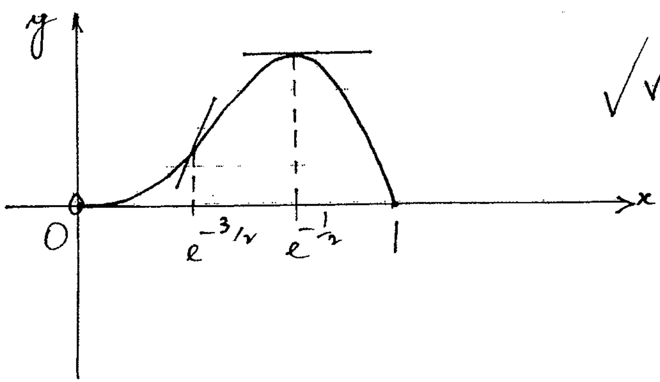
$$-3(3+2\ln x)=0$$

$x=e^{-\frac{3}{2}}$

x	e^{-2}	$e^{-3/2}$	e^{-1}
$f''(x)$	$+3$	0	-3
	$\cup$	.	$\cap$

So concavity changed and there is a point of inflection at $x=e^{-3/2}$

(iv) At $x=1$, $y=0$.



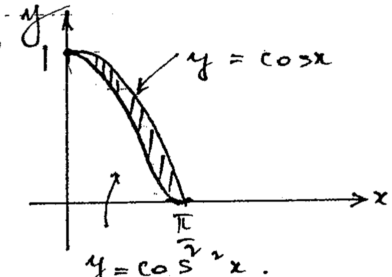
b) (i) $8\cos^4 x=8\left(\frac{1}{2}(1+\cos 2x)\right)^2$

$=2(1+2\cos 2x+\cos^2 2x)$

$=2(1+2\cos 2x+\frac{1}{2}(1+\cos 4x))$

$=3+4\cos 2x+\cos 4x$

(ii)



$$V=\pi\int_0^{\frac{\pi}{2}}(\cos x)^2dx-\pi\int_0^{\pi}(\cos^2 x)^2dx$$

$$=\pi\int_0^{\frac{\pi}{2}}\frac{1}{2}(1+\cos 2x)-\cos^4 x\,dx$$

$$=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}1+\cos 2x-2\left(\frac{1}{8}\left(3+4\cos 2x+\cos 4x\right)\right)dx$$

$$=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\frac{1}{4}-\frac{1}{4}\cos 4x\,dx$$

$$=\frac{\pi}{8}\int_0^{\frac{\pi}{2}}1-\cos 4x\,dx$$

$$=\frac{\pi}{8}\left[x-\frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{2}}$$

$$=\frac{\pi}{8}\left[\left(\frac{\pi}{2}-0\right)-(0-0)\right]$$

$=\frac{\pi^2}{16}$ cubic units