SYDNEY GRAMMAR SCHOOL

2010 Half-Yearly Examination

FORM VI
MATHEMATICS 2 UNIT

Thursday 11th March 2010

General Instructions

- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Structure of the paper

- Total marks — 84
- All seven questions may be attempted.
- All seven questions are of equal value.

Collection

- Write your candidate number clearly on each booklet.
- Hand in the seven questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place the question paper inside your answer booklet for Question One.

Checklist

- SGS booklets — 7 per boy
- Candidature — 83 boys

Examiner
DS

The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\ x > 0$

QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) Simplify:

(i) $e^x \times e^{2x}$ 1

(ii) $\ln 5 + \ln 2$ 1

(b) Write $\dfrac{1}{e}$ correct to 3 decimal places. 1

(c) Consider the equation $e^x = 100$.

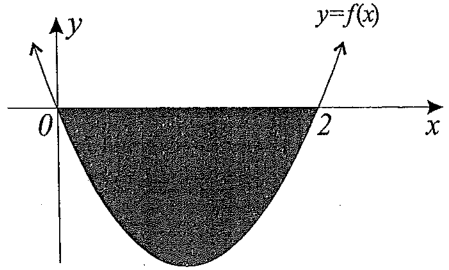
(i) Write the equation in logarithmic form. 1

(ii) Hence find x correct to 2 significant figures. 1

(d) Differentiate e^{x+4}. 1

(e) Evaluate $\displaystyle\int_0^3 2x\,dx$. 1

(f)



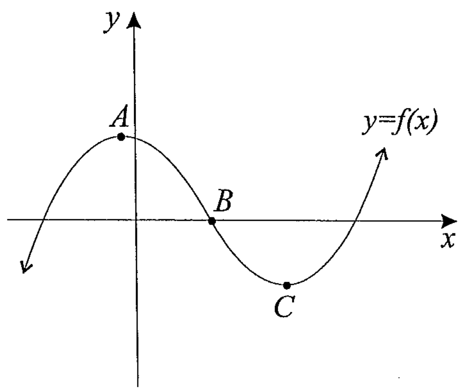
The curve $y = f(x)$ is graphed above and the region bounded by the curve and the x-axis is shaded. 1

If the area of the shaded region is 5 square units, state the value of $\displaystyle\int_0^2 f(x)\,dx$.

(g) Find the gradient of the tangent to the curve $y = 2\log_e x$ at the point $(e, 2)$. 2

QUESTION ONE (Continued)

(h)



The diagram above shows three points A, B and C on the curve $y = f(x)$.
At which of the points is:

(i) $f''(x) = 0$, 1

(ii) $f'(x) = 0$ and $f''(x) > 0$. 1

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) (i) Sketch the graph of $y = \log_e x$, showing any intercepts with the x or y axes. [1]

(ii) State the domain of $y = \log_e x$. [1]

(b)

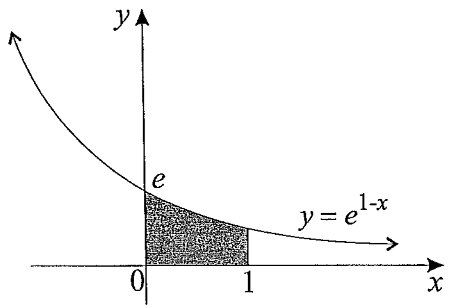
x	3	4	5
$f(x)$	8	10	9

[2]

Use the trapezoidal rule with the 3 function values in the table above to approximate $\int_3^5 f(x)\,dx$.

(c) Find the equation of the tangent to the curve $y = x^3 - 5x$ at the point where $x = -1$. [3]

(d)



Calculate the exact area of the shaded region in the diagram above. [3]

(e) (i) Differentiate $y = e^{x^3}$. [1]

(ii) Hence find $\int 6x^2 e^{x^3}\,dx$. [1]

Exam continues next page ...

QUESTION THREE (12 marks) Use a separate writing booklet.

(a) Differentiate:

(i) $y = \dfrac{2}{x}$

(ii) $y = (3x-1)^3$ [1]

(iii) $y = \log_e(1 + x^4)$ [1]

(b) Find:

(i) $\int e^{-4x}\,dx$ [1]

(ii) $\int \dfrac{5}{5x+3}\,dx$ [1]

(c) (i) Show that the function $y = x\log_e x$ has derivative $\dfrac{dy}{dx} = 1 + \log_e x$. [2]

(ii) Hence find the x-coordinate of the point on the curve $y = x\log_e x$ where the tangent is horizontal. [2]

(d) Evaluate $\int_0^1 \left(1 - \tfrac{1}{2}x\right)^{-2}\,dx$. [3]

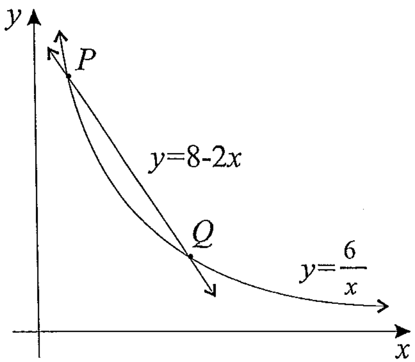
Exam continues overleaf ...

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a) A curve has gradient function $\frac{dy}{dx} = 8x^3 - 6x + 3$ and a y-intercept of -5. Find the equation of the curve. [2]

(b) Use Simpson's rule with 5 function values to find the value of $\int_2^6 \ln x \, dx$ correct to 2 decimal places. [3]

(c)



The diagram above shows the hyperbola $y = \frac{6}{x}$ and the line $y = 8 - 2x$ intersecting at P and Q.

(i) By solving the equations simultaneously, show that P is $(1, 6)$ and Q is $(3, 2)$. [3]

(ii) Find, correct to 2 significant figures, the area of the region enclosed by the hyperbola and the line. [4]

Exam continues next page ...

QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**

(a) Find the value of k given that $\int_{\frac{1}{2}}^{k} \frac{1}{x^2} \, dx = \frac{4}{5}$. [3]

(b) (i) Show that $\frac{4}{1+2x} - 1 = \frac{3-2x}{1+2x}$. [1]

(ii) Hence find the exact value of $\int_0^1 \frac{3-2x}{1+2x} \, dx$. [3]

(c) The region $\mathcal{R}$ is bounded by the curve $y = 2 + e^{-2x}$, the x-axis and the vertical lines $x = 0$ and $x = 2$.

(i) Expand $\left(2 + e^{-2x}\right)^2$. [1]

(ii) Show that the solid of revolution formed when $\mathcal{R}$ is rotated about the x-axis has volume $\frac{\pi}{4}\left(41 - 8e^{-4} - e^{-8}\right)$ cubic units. [4]

QUESTION SIX (12 marks) Use a separate writing booklet. **Marks**

(a) The curve $y = ax^3 + bx^2 - 9$ passes through the point $(2, 31)$ and is stationary at $x = -2$. Find the values of a and b. [4]

(b) Consider the curve with equation $y = e^{1 - \frac{1}{2}x^2}$.

(i) Find and classify the stationary point on the curve. [3]

(ii) Find the second derivative using the product rule, and hence find the two points of inflexion. [3]

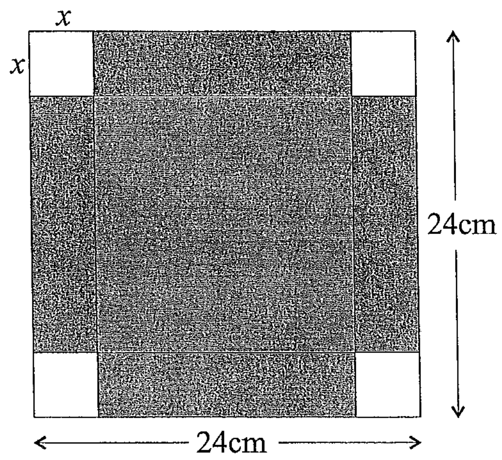
(iii) As x becomes large, what limiting value does y approach? [1]

(iv) Sketch the curve using the information found in the parts above. [1]

Exam continues overleaf ...

QUESTION SEVEN (12 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above shows a square piece of cardboard with side length 24 cm. An open box is to be made by cutting a square of side x cm from each corner, and then folding up the flaps to form the sides.

(i) Write down, in terms of x, the side length of the square base of the box. **1**

(ii) Show that the volume V cm^3 of the box is given by **1**

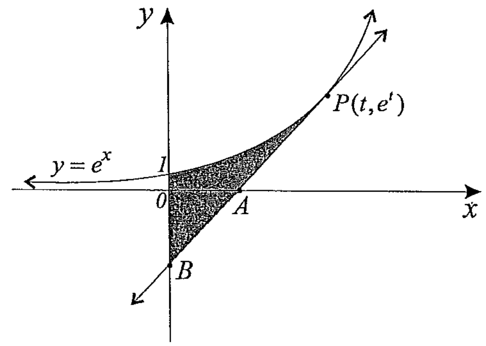
$$V = 4x^3 - 96x^2 + 576x.$$

(iii) Find the maximum possible volume of the box. **4**

Exam continues next page ...

QUESTION SEVEN (Continued)

(b)



The diagram above shows the curve $y = e^x$ and the tangent to the curve at the point $P(t, e^t)$, where $t > 1$. The tangent meets the x and y axes at the points A and B respectively.

(i) Find the coordinates of A and B in terms of t. **3**

(ii) Show that the shaded region bounded by the curve, the tangent at P and the y-axis has area $\left(\frac{1}{2}e^t(t^2 - 2t + 2) - 1\right)$ square units. **3**

END OF EXAMINATION

FORM 6 2 UNIT : HALF YEARLY 2010 (1x12)

(1)(a)(i) $e^{x} \cdot e^{2x} = e^{3x}$

(ii) $\ln 5 + \ln 2 = \ln 10$

(b) $\frac{1}{e} \doteqdot 0.368$ (must be correct to 3 dec. pl.)

(c)(i) $e^{x} = 100$

$\therefore x = \log_e 100$

(ii) $x \doteqdot 4.6$ (must be correct to 2 sig. figs)

(d) $\frac{d}{dx}\left(e^{x+4}\right) = e^{x+4}$

(e) $\int_0^3 2x\,dx = \left[x^2\right]_0^3$

$= 3^2 - 0^2$

$= 9$

(f) $\int_0^2 f(x)\,dx = -5$

(g) $y = 2\log_e x$

$\therefore y' = \frac{2}{x}$

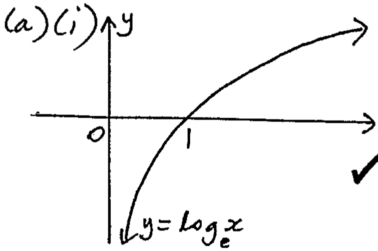
At $x = e$, $y' = \frac{2}{e}$.

So the gradient is $\frac{2}{e}$.

(h)(i) B

(ii) C

(2)(a)(i)



(ii) Domain is $x > 0$.

(b) $\int_3^5 f(x)\,dx$

$\doteqdot \frac{h}{2}\left(y_0 + y_2 + 2y_1\right)$

$= \frac{1}{2}\left(8 + 9 + 2(10)\right)$

$= \frac{37}{2}$ or 18.5

(c) When $x = -1$,

$y = (-1)^3 - 5(-1)$

$= 4.$

$y' = 3x^2 - 5$

When $x = -1$,

$y' = -2.$

So the tangent has equation

$y - 4 = -2(x+1)$

$y = -2x + 2.$

(d) Area $= \int_0^1 e^{1-x}\,dx$

$= -\left[e^{1-x}\right]_0^1$

$= -\left(e^0 - e\right)$

$= (e - 1)\ u^2$

(e)(i) $\frac{d}{dx}\left(e^{x^3}\right) = 3x^2 e^{x^3}$

(ii) So $2\int 3x^2 e^{x^3}\,dx = 2e^{x^3} + c.$

(No penalty for omitting c.)

(3)(a)(i) $y = 2x^{-1}$

$y' = -2x^{-2}$

$= \frac{-2}{x^2}$

(ii) $y = (3x-1)^3$

$y' = 3(3x-1)^2 . 3$

$= 9(3x-1)^2$

(iii) $y = \log_e(1 + x^4)$

$y' = \frac{4x^3}{1 + x^4}$

(b)(i) $\int e^{-4x}\,dx = -\frac{1}{4}e^{-4x} + c$

(ii) $\int \frac{5}{5x+3}\,dx = \log_e(5x+3) + c$

(no penalty for omitting c)

(c)(i) $y = x\log_e x$

By the product rule,

$y' = x \cdot \frac{1}{x} + \log_e x \cdot 1$

$= 1 + \log_e x$

(ii) Let $y' = 0$.

$\therefore 1 + \log_e x = 0$

$\log_e x = -1$

$x = e^{-1}$ or $\frac{1}{e}$

(d) $\int_0^1 \left(1 - \frac{1}{2}x\right)^{-2} dx$

$= \left[\frac{\left(1 - \frac{1}{2}x\right)^{-1}}{-1\left(-\frac{1}{2}\right)}\right]_0^1$

$= 2\left(\left(\frac{1}{2}\right)^{-1} - 1^{-1}\right)$

$= 2(2 - 1)$

$= 2$

(4)(a) $\frac{dy}{dx} = 8x^3 - 6x + 3$

$\therefore y = \int (8x^3 - 6x + 3)\,dx$

$= 2x^4 - 3x^2 + 3x + c$

When $x = 0$, $y = -5$.

$\therefore c = -5$

So the curve has equation

$y = 2x^4 - 3x^2 + 3x - 5.$

(b)

x	2	3	4	5	6
y	$\ln 2$	$\ln 3$	$\ln 4$	$\ln 5$	$\ln 6$
	y_0	y_1	y_2	y_3	y_4

$\int_2^6 \ln x\,dx \doteqdot \frac{h}{3}\left(y_0 + y_4 + 4(y_1 + y_3) + 2(y_2)\right)$

$= \frac{1}{3}\left(\ln 2 + \ln 6 + 4(\ln 3 + \ln 5) + 2\ln 4\right)$

$= 5.3632\ldots$

$\doteqdot 5.36$

(Allow any reasonable approximation.)

(c)(i) $y = \frac{6}{x}$ ①

$y = 8 - 2x$ ②

Substitute ① into ②:

$\frac{6}{x} = 8 - 2x$

$6 = 8x - 2x^2$

$x^2 - 4x + 3 = 0$

$(x-1)(x-3) = 0$

$x = 1$ or $x = 3$

When $x = 1$, $y = 6$.

When $x = 3$, $y = 2$.

So $P = (1,6)$ and $Q = (3,2)$.

(ii) Area $= \int_1^3 \left(8 - 2x - \frac{6}{x}\right) dx$

$= \left[8x - x^2 - 6\ln x\right]_1^3$

$= 24 - 9 - 6\ln 3 - (8 - 1)$

$= 8 - 6\ln 3$

$\doteqdot 1.4083\ldots$

$\doteqdot 1.4\ u^2$

(Allow any reasonable approximation)

5)(a) $\int_{\frac{1}{2}}^{k} x^{-2}\,dx = \frac{4}{5}$

$\left[-\frac{1}{x}\right]_{\frac{1}{2}}^{k} = \frac{4}{5}$ ✓

$-\frac{1}{k} + 2 = \frac{4}{5}$ ✓

$-\frac{1}{k} = -\frac{6}{5}$

$\therefore k = \frac{5}{6}$ ✓

(b)(i)

$LHS = \frac{4}{1+2x} - 1$

$= \frac{4 - 1(1+2x)}{1+2x}$ ✓

$= \frac{3-2x}{1+2x}$

$= RHS$

(ii) $\int_0^1 \frac{3-2x}{1+2x}\,dx$

$= \int_0^1 \left(\frac{4}{1+2x} - 1\right) dx$ ✓

$= \left[2\ln(1+2x) - x\right]_0^1$ ✓

$= 2\ln 3 - 1 - (2\ln 1 - 0)$

$= 2\ln 3 - 1$ ✓

(c)(i)

$(2+e^{-2x})^2 = 4 + 4e^{-2x} + e^{-4x}$ ✓

(ii)

$V = \pi\int_0^2 (4 + 4e^{-2x} + e^{-4x})\,dx$ ✓

$= \pi\left[4x - 2e^{-2x} - \frac{1}{4}e^{-4x}\right]_0^2$ ✓

$= \pi\left(8 - 2e^{-4} - \frac{1}{4}e^{-8} - (0 - 2 - \frac{1}{4})\right)$

$= \pi\left(10\frac{1}{4} - 2e^{-4} - \frac{1}{4}e^{-8}\right)$ ✓

$= \frac{\pi}{4}\left(41 - 8e^{-4} - e^{-8}\right)\ u^3$ ✓

6)(a) $y = ax^3 + bx^2 - 9$

Substitute $x = 2$, $y = 31$: ✓

$31 = 8a + 4b - 9$

$8a + 4b = 40$

$2a + b = 10$ ① ✓

$y' = 3ax^2 + 2bx$

When $x = -2$, $y' = 0$.

$\therefore 0 = 12a - 4b$

$3a - b = 0$ ② ✓

①+②: $5a = 10$

$\therefore a = 2$ and $b = 6$ ✓

(b)(i) $y = e^{1 - \frac{1}{2}x^2}$

$y' = -xe^{1-\frac{1}{2}x^2}$ ✓

Let $y' = 0$ for stationary points.

$\therefore -xe^{1-\frac{1}{2}x^2} = 0$

$\therefore x = 0$ ✓ (note that $e^{1-\frac{1}{2}x^2} > 0$ for all x)

When $x = 0$, $y = e$.

So $(0, e)$ is a stationary point.

x	-1	0	1
y'	$e^{\frac{1}{2}} > 0$	0	$-e^{\frac{1}{2}} < 0$

$(0, e)$ is a maximum turning point. ✓

(ii) $y' = -xe^{1-\frac{1}{2}x^2}$ ✓

$y'' = -x \cdot -xe^{1-\frac{1}{2}x^2} + e^{1-\frac{1}{2}x^2}\cdot(-1)$

$= x^2e^{1-\frac{1}{2}x^2} - e^{1-\frac{1}{2}x^2}$

$= e^{1-\frac{1}{2}x^2}(x^2 - 1)$ ✓

Let $y'' = 0$ for possible points of inflexion.

$\therefore x = \pm 1$ ✓ ($e^{1-\frac{1}{2}x^2} > 0$ for all x)

When $x = 1$, $y = e^{\frac{1}{2}}$.

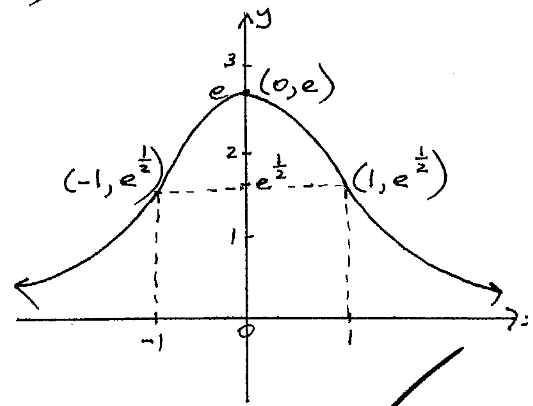
When $x = -1$, $y = e^{\frac{1}{2}}$.

We are told that there are two points of inflexion, so $(1, e^{\frac{1}{2}})$ and $(-1, e^{\frac{1}{2}})$ are the points of inflexion.

(iii) As $x \to \infty$, $y \to 0$. ✓

(So $y = 0$ (the x-axis) is a horizontal asymptote.)

(iv) $e^{\frac{1}{2}} \doteq 1\cdot65$



✓

7) (a)(i) $(24-2x)$ cm ✓

(ii) $V = (24-2x)^2 . x$ ✓

$= x(576 - 96x + 4x^2)$

$= 4x^3 - 96x^2 + 576x$

(iii) $\frac{dV}{dx} = 12x^2 - 192x + 576$

$= 12(x^2 - 16x + 48)$

$= 12(x-4)(x-12)$ ✓

Let $\frac{dV}{dx} = 0$ for stationary points.

$\therefore x = 4$ or $x = 12$

But when $x = 12$, $V = 0$.

So $x \neq 12$. ✓

$\frac{d^2V}{dx^2} = 24x - 192$

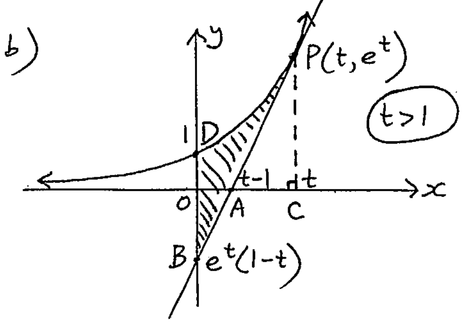
$= -96$ when $x = 4$.

So $x = 4$ maximises V. ✓

So the maximum volume is

$4 \times 16^2 = 1024 \text{ cm}^3$. ✓

(b)



(i) $y' = e^x$

At $x = t$, $y' = e^t$.

So the tangent has equation

$y - e^t = e^t(x - t)$, ✓

[i.e. $y = e^t x + e^t(1-t)$].

When $y = 0$,

$x - t = -1$

$x = t - 1$. ✓

So $A = (t-1, 0)$.

When $x = 0$,

$y = e^t - te^t$

$= e^t(1-t)$. ✓

So $B = (0, e^t(1-t))$.

(ii) Area of region OCPD is

$\int_0^t e^x dx = [e^x]_0^t$

$= e^t - 1$. ✓

Area of ΔAOB is

$\frac{1}{2}(t-1).e^t(t-1)$ (note $t>1$)

$= \frac{1}{2}e^t(t-1)^2$. ✓

Area of ΔACP is

$\frac{1}{2}.1.e^t$.

So the shaded region has area

$(e^t - 1) + \frac{1}{2}e^t(t-1)^2 - \frac{1}{2}e^t$

$= \frac{1}{2}e^t((t-1)^2 + 1) - 1$

$= \frac{1}{2}e^t(t^2 - 2t + 2) - 1$ as required. ✓

Alternatively,

$\text{area} = \int_0^t (e^x - (e^t x + e^t(1-t)))dx$

$= \int_0^t (e^x - e^t x - e^t(1-t))dx$

$= \left[e^x - \frac{e^t}{2}x^2 - e^t(1-t)x\right]_0^t$

$= e^t - \frac{t^2 e^t}{2} - t(1-t)e^t - 1$

$= \frac{1}{2}e^t(2 - t^2 - 2t + 2t^2) - 1$

$= \frac{1}{2}e^t(t^2 - 2t + 2) - 1$.