SYDNEY GRAMMAR SCHOOL HALF-YEARLY EXAMINATION 2004

FORM VI MATHEMATICS

Time allowed: 2 hours

Exam date: 10th May 2004

Instructions

All questions may be attempted.
All questions are of equal value.
Part marks are shown in boxes in the right margin.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection

The writing booklets will be collected in one bundle.
Start each question in a new writing booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your candidate number on each booklet.

Checklist

SGS writing booklets required — 7 per boy.
Candidature: 118 boys.

Examiner

PKH

QUESTION ONE (14 marks) Use a separate writing booklet. **Marks**

(a) Use your calculator to evaluate each of the following, correct to three decimal places:

(i) e^{-1} [1]

(ii) $\ln 5{\cdot}4$ [1]

(iii) $\sin 0{\cdot}8$ [1]

(b) Change $\frac{3\pi}{4}$ radians to degrees. [2]

(c) Write down the exact value of $\tan \frac{\pi}{4}$. [1]

(d) Differentiate the following:

(i) $y = 5x^2 - 3$ [1]

(ii) $y = e^{-2x}$ [1]

(iii) $y = 5 \log_e x$ [1]

(iv) $y = \tan x$ [1]

(e) Find:

(i) $\int e^{2x+1}\, dx$ [2]

(ii) $\int \cos 2x\, dx$ [2]

Exam continues next page ...

QUESTION TWO (14 marks) Use a separate writing booklet. **Marks**

(a) Clearly sketch the following functions on separate axes. Show all intercepts with the x-axis and y-axis.

(i) $y = 2x - 4$ **2**

(ii) $y = 2\sin x$, for $0 \leq x \leq 2\pi$ **2**

(iii) $y = e^{-x}$ **2**

(b) Solve $\cos\theta = \dfrac{\sqrt{3}}{2}$, for $0 \leq \theta \leq 2\pi$. **2**

(c) Differentiate:

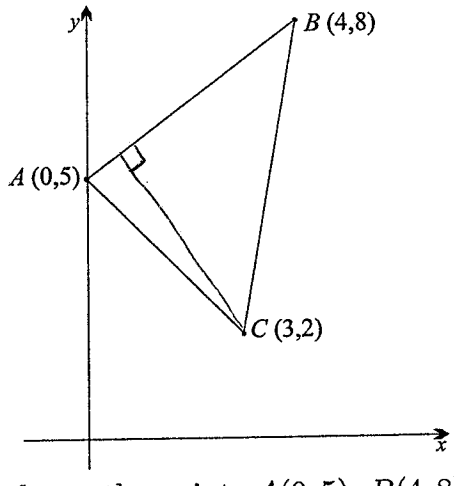
(i) $y = e^{x^2+x}$ **2**

(ii) $y = x\ln x$ **2**

(iii) $y = \dfrac{x}{\cos x}$ **2**

QUESTION THREE (14 marks) Use a separate writing booklet. **Marks**

(a)



In the diagram above, the points $A(0,5)$, $B(4,8)$ and $C(3,2)$ are shown.

(i) Find the length of the interval AB. **2**

(ii) Find the gradient of the line AB. **1**

(iii) Show that the equation of the line AB is $3x - 4y + 20 = 0$. **2**

(iv) Find the perpendicular distance from C to the line AB. **2**

(v) Find the area of the triangle ABC. **2**

Exam continues overleaf ...

(b) A parabola has equation $x^2 = -8(y-3)$.

(i) What are the coordinates of the vertex? **1**

(ii) What is the focal length? **1**

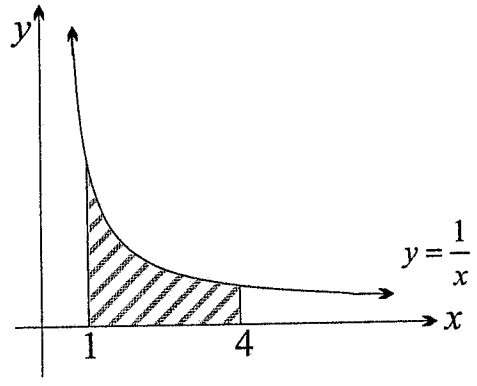
(iii) Sketch the parabola, clearly indicating the focus, vertex and directrix. **3**

QUESTION FOUR (14 marks) Use a separate writing booklet. **Marks**

(a) (i) Write down the amplitude and the period of $y = 4\cos 2x$. **2**

(ii) Sketch $y = 4\cos 2x$, for $0 \leq x \leq 2\pi$. **2**

(b) **3**



Find the shaded area in the diagram above.

(c) Consider the function $y = 4x^3 - x^4$.

(i) Show that the function has stationary points at $x = 0$ and $x = 3$. **2**

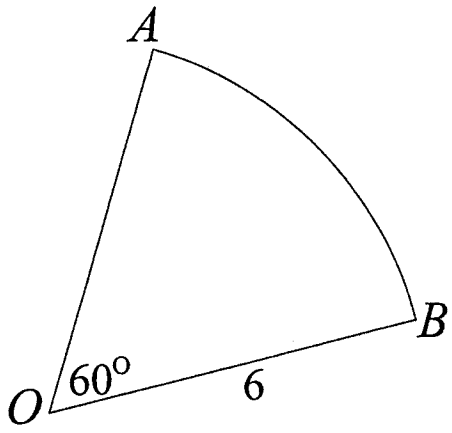
(ii) Determine the nature of the stationary points. **2**

(iii) Sketch the function showing the stationary points and x-intercepts. **3**

Exam continues next page ...

QUESTION FIVE (14 marks) Use a separate writing booklet. **Marks**

(a)



The sector drawn above has radius 6 units and $\angle AOB$ is $60°$.

(i) Find the size of $\angle AOB$ in radians. **1**

(ii) Find the exact area of the sector AOB. **2**

(b) Solve $\log_3(2x-1) = 2$. **2**

(c) Consider the function $y = \ln(x^2+1)$.

(i) Copy and complete the table below, giving the values of y correct to three decimal places: **2**

x	1	2	3
y			

(ii) Use one application of Simpson's rule to estimate $\displaystyle\int_1^3 \ln(x^2+1)\,dx$. **3**

(d) Find the volume of the solid formed when the curve $y = \dfrac{2}{\sqrt{x+2}}$, between $x = -1$ and $x = 2$, is rotated about the x-axis. **4**

Exam continues overleaf ...

QUESTION SIX (14 marks) Use a separate writing booklet. **Marks**

(a) The derivative of a function is given by $\frac{dy}{dx} = e^{-\frac{1}{2}x} - \frac{1}{x+1}$. **3**

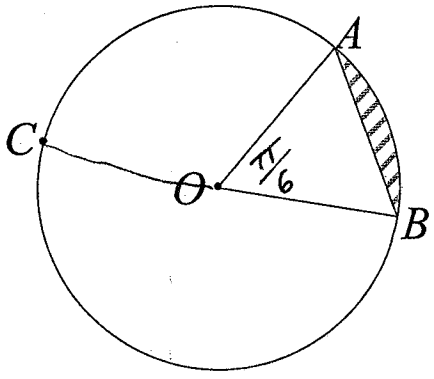
The curve $y = f(x)$ passes through $(0, 0)$. Find y as a function of x.

(b) A function $y = f(x)$ satisfies all the following conditions: **3**

$$f(-1) > 0$$
$$f(4) < 0$$
$$f'(x) < 0$$
$$f''(x) > 0$$

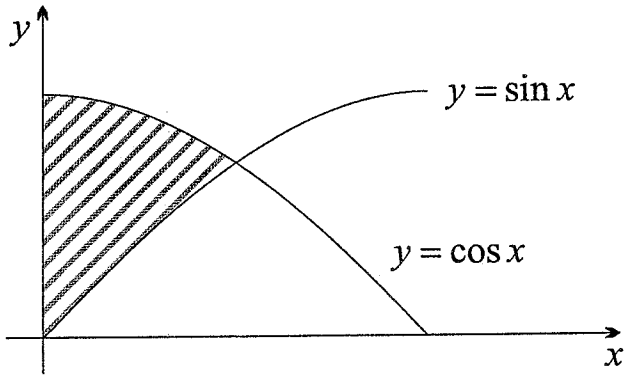
Draw a possible sketch of the function, for $-1 \leq x \leq 4$.

(c) **4**



In the diagram above, the area of the shaded minor segment is 8 square units, and $\angle AOB = \frac{\pi}{6}$. Find the length of the major arc ACB, correct to two decimal places.

(d)



In the diagram above, the curves $y = \cos x$ and $y = \sin x$ are sketched.

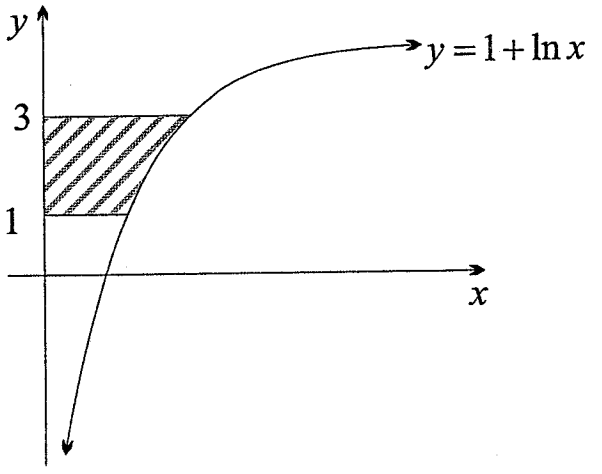
(i) Show that the curves intersect where $x = \frac{\pi}{4}$. **1**

(ii) Find the exact area shaded in the diagram above. **3**

Exam continues next page ...

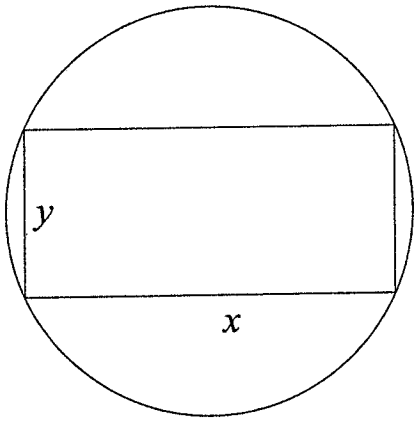
QUESTION SEVEN (14 marks) Use a separate writing booklet. **Marks**

(a)



Find the volume of the solid formed when the shaded region above is rotated about the y-axis. (Note that the rotation is about the y-axis.) **4**

(b)



The circular cross-section of a large log is shown above. The log has radius 1 metre. A beam with rectangular cross-section is cut from the log. Let x be the width of the cross-section and y the height of the cross-section. Let A be the area of the rectangular cross-section.

(i) Show that $A = x(4 - x^2)^{\frac{1}{2}}$. **2**

(ii) Find the maximum cross-sectional area of the beam. Justify your answer. **4**

(c) Use a graphical approach to determine how many solutions each of the following equations have.

(i) $\sin x = 2\tan x$, for $0 \leq x \leq 2\pi$ **2**

(ii) $\sin 5x = \log_{10} x$ **2**

END OF EXAMINATION

Solution to S.C.S. Mathematics H.Y. 2004

Question 1

(a) (i) 0·368 ✓
(ii) 1·687 ✓
(iii) 0·717 ✓

(b) $\frac{3\pi}{4} = 135^\circ$ ✓ ✓

(c) $\tan\frac{\pi}{4} = 1$ ✓

(d) (i) $y = 5x^2 - 3$
$y' = 10x$ ✓

(ii) $y = e^{-2x}$
$y' = -2e^{-2x}$ ✓

(iii) $y = 5\ln x$
$y' = \frac{5}{x}$ ✓

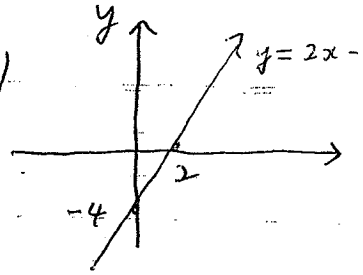
(iv) $y = \tan x$
$y' = \sec^2 x$ ✓

(e) (i) $\int e^{2x+1}\,dx$
$= \frac{1}{2}e^{2x+1} + C$ ✓✓

(ii) $\int \cos 2x\,dx$
$= \frac{1}{2}\sin 2x + C$ ✓✓

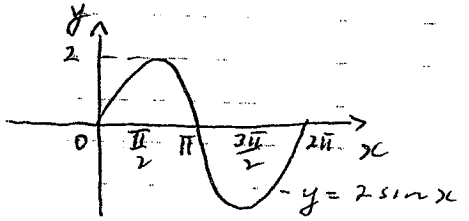
Question 2

(a) (i)



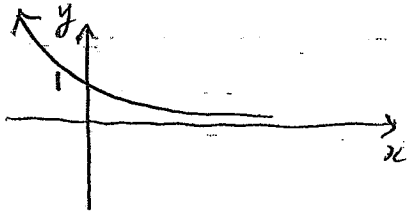
✓✓

(ii)



✓✓

(iii)



✓✓

(b) $\cos\theta = \frac{\sqrt{3}}{2}$, $0 \le \theta \le 2\pi$

$\theta = 30^\circ$ or 330° ✓✓

$\theta = \frac{\pi}{6}$ or $\frac{11\pi}{6}$

(c) (i) $y = e^{x^2+x}$
$y' = (2x+1)e^{x^2+x}$ ✓✓

(ii) $y = x\ln x$
$y = x \times \frac{1}{x} + 1 \times \ln x$
$= 1 + \ln x$ ✓✓

(iii) $y = \frac{x}{\cos x}$ ✓
$y' = \frac{\cos x - x(-\sin x)}{\cos^2 x}$ ✓
$= \frac{\cos x + x\sin x}{\cos^2 x}$

Question 3

(i) $d(AB) = \sqrt{(4-0)^2+(8-5)^2}$

$= \sqrt{4^2+3^2}$

$= 5$ ✓✓

(ii) $m(AB) = \frac{8-5}{4-0}$

$= \frac{3}{4}$ ✓

(iii) Equ of line AB is $y - y_1 = m(x-x_1)$

$y - 5 = \frac{3}{4}(x-0)$ ✓

$4y - 20 = 3x$

$3x - 4y + 20 = 0$ ✓

(iv) Let perpendicular distance from C to line AB be d: $d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$

$C = (3,2)$ $d = \frac{|3\times 3 + (-4)\times 2 + 20|}{\sqrt{3^2+4^2}}$ ✓

$= \frac{21}{5}$ units ✓

(v) Area $= \frac{1}{2} \times b \times h$

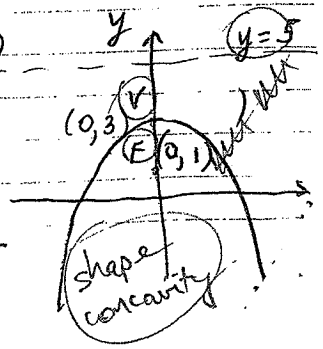
$= \frac{1}{2} \times 5 \times \frac{21}{5}$ ✓ $= 10\frac{1}{2}$ units^2 ✓

(b) (i) $V = (3, 0)$ ✓

(ii) $(x-3)^2 = -4(2)y$

focal length is 2 units ✓

(iii)



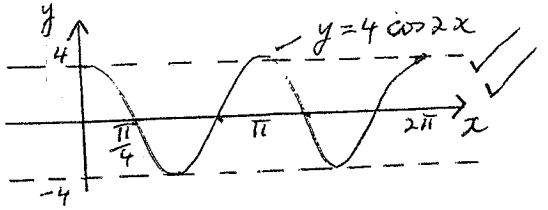
-1/error

Question 4

(a) amplitude $= 4$ ✓

period $= \frac{2\pi}{2}$

$= \pi$ ✓



(b) $A = \int_1^4 \frac{1}{x}\,dx$ ✓

$A = [\ln x]_1^4$ ✓

$= \ln 4 - \ln 1$ ✓

$A = \ln 4$ units^2

(c) (i) $y = 4x^3 - x^4$

$y' = 12x^2 - 4x^3$ ✓

Stat pts where $y' = 0$

$12x^2 - 4x^3 = 0$

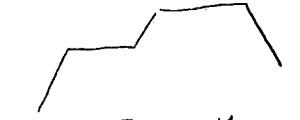
$4x^2(3-x) = 0$

$x = 0$ or $x = 3$

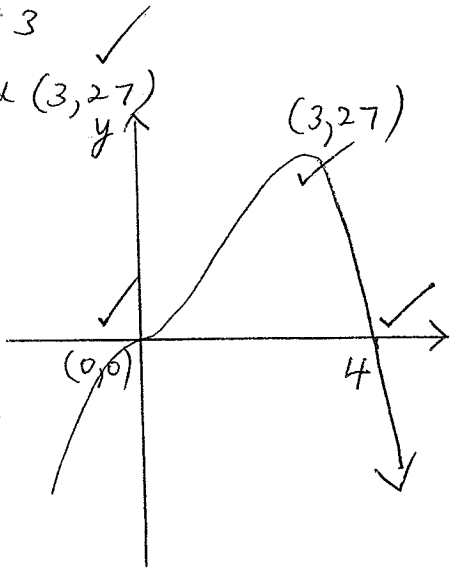
Stat pts at $(0,0)$ and $(3,27)$ ✓

(ii)

x	-1	0	1	3	4
y'	16	0	8	0	



Horizontal stat pt ✓ Max stat pt ✓



Question 5

(a) (i) Angle $\angle AOB = \frac{\pi}{3}$ ✓

(ii) Area $= \frac{1}{2} r^2 \theta$

$= \frac{1}{2} \times 36 \times \frac{\pi}{3}$ ✓ ✓

$= 6\pi$ units^2 ✓

(b) $\log_3(2x-1) = 2$ ✓

$2x - 1 = 9$ ✓

$x = 5$ ✓

(c)

x	1	2	3
y	0.693	1.609	2.303

✓✓ −1 per error

$A = \frac{3-1}{6} [0.693 + 4 \times 1.609 + 2.303]$ ✓

$= 3.144$ ✓

(d) Volume $= \pi \int_a^b y^2 \, dx$

$= \pi \int_{-1}^{2} \frac{4}{x+2} \, dx$ ✓✓

$= \pi \left[4 \ln(x+2) \right]_{-1}^{2}$ ✓

$= 4\pi (\ln 4 - \ln 1)$

$= 4\pi \ln 4$ units^3 ✓

Question 6

(a) $y' = e^{-\frac{x}{2}} - \frac{1}{x+1}$

$y = -2e^{-\frac{x}{2}} - \ln(x+1) + C$

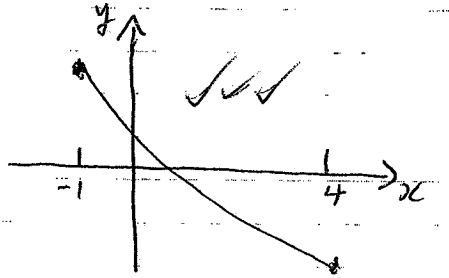
Passes through $(0, 0)$

$0 = -2 - \ln 1 + C$

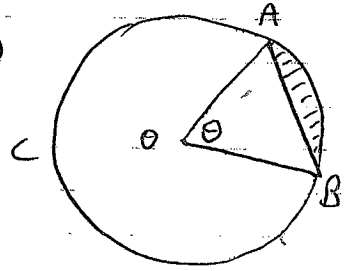
$C = 2$

$y = -2e^{-\frac{x}{2}} - \ln(x+1) + 2$ ✓

(b)



(c)



$\frac{1}{2} r^2 \theta - \frac{1}{2} r^2 \sin\theta = 8$ ✓

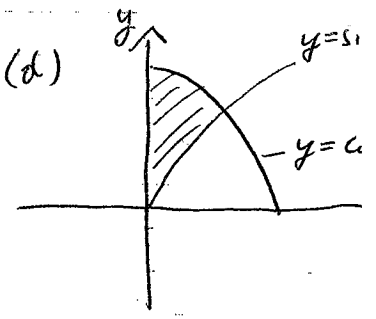
$r^2 \left(\frac{\pi}{6} - \frac{1}{2} \right) = 8$

$r = 26.03845766$ ✓

Circumference of the circle $= 2\pi r$ ✓ ✓

Arc ACB $= \frac{11}{12} \times 2\pi r = 149.97$ u

(d)



(i) When $x = \frac{\pi}{4}$

$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

∴ Curves intersect ✓ where $x = \frac{\pi}{4}$.

$A = \int_0^{\frac{\pi}{4}} \cos x - \sin x \, dx$ ✓

$= \sin x + \cos x \Big]_0^{\frac{\pi}{4}}$ ✓

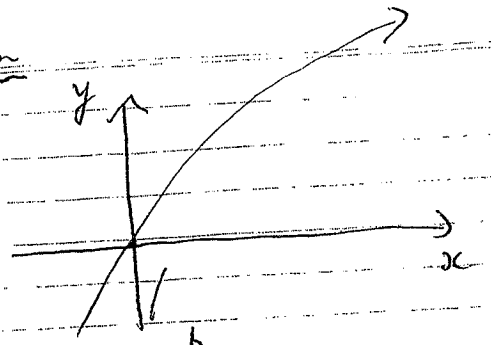
$= \sin \frac{\pi}{4} + \cos \frac{\pi}{4} - (0 + 1)$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1$

$= \sqrt{2} - 1$ units^2

Seven

(a)



$y = 1 + \ln x$

$\ln x = y - 1$

$x = e^{y-1}$ ✓

$V = \pi \int_a^b x^2 \, dy$ ✓

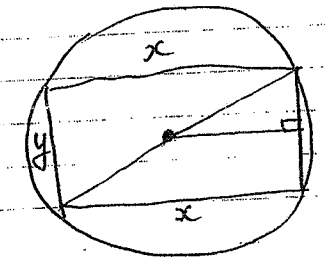
$= \pi \int_1^3 e^{2y-2} \, dy$

$= \pi \left[\frac{e^{2y-2}}{2} \right]_1^3$ ✓

$= \frac{\pi}{2} \left[e^4 - 1 \right]$ ✓

(b)

(i)



By Pythagoras $x^2 + y^2 = 4$

$y^2 = 4 - x^2$

$y = \sqrt{4 - x^2}$ ✓

Area $= xy$

$A = x(4 - x^2)^{\frac{1}{2}}$ ✓

(ii) $\frac{dA}{dx} = \sqrt{4-x^2} + x \times \frac{1}{2}(4-x^2)^{-\frac{1}{2}} \times -2x$

$= \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}}$ ✓

Stationary pts where $\frac{dA}{dx} = 0$

$\sqrt{4-x^2} = \frac{x^2}{\sqrt{4-x^2}}$

$x^2 = 4 - x^2$

$2x^2 = 4$

$x = \pm\sqrt{2}$ ✓

When $x = 1$, $\frac{dA}{dx} = \sqrt{3} - \frac{1}{\sqrt{3}}$

$x = 1.5$ $\frac{dA}{dx} = \sqrt{1.75} - \frac{1.5^2}{\sqrt{1.75}} < 0$ ✓

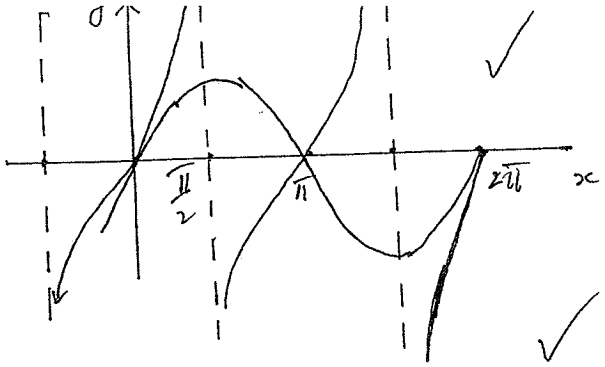
So Max occurs where $x = \sqrt{2}$

Max Area $= \sqrt{2}\left(4 - (\sqrt{2})^2\right)^{\frac{1}{2}}$

$= \sqrt{2} \times \sqrt{2}$

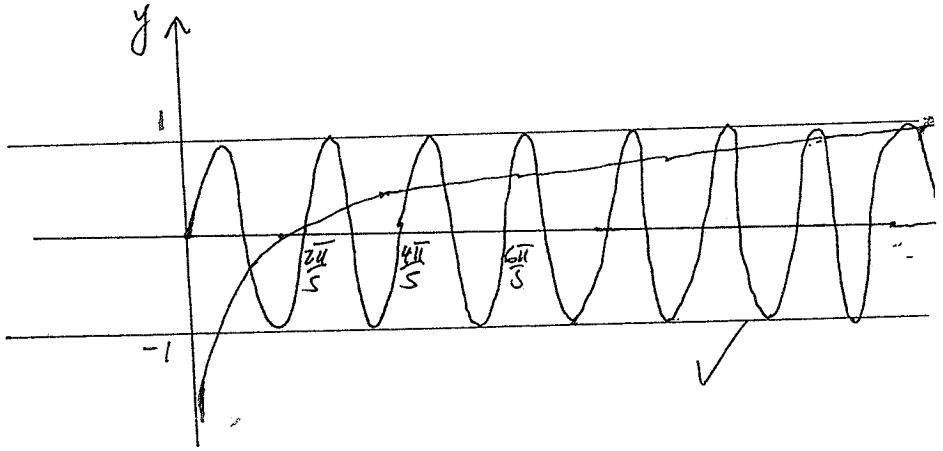
$= 2 \text{ m}^2$ ✓

(c) (i)



✓ ✓

From graph there are 3 solutions



✓

With a careful graph there are 15 solutions. ✓

Note $\frac{16\pi}{5} \div 10.05$

So $\log_{10} \frac{16\pi}{5}$ is just above the line $y =$