George Lolas

Sydney Grammar School
Mathematics Department
Half-Yearly Examinations 2006

FORM VI

MATHEMATICS EXTENSION 1

Examination date

Wednesday 24th May 2006

Time allowed

2 hours

Instructions

All seven questions may be attempted.
All seven questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection

Write your candidate number clearly on each booklet.
Hand in the seven questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.

Checklist

SGS booklets: 7 per boy. A total of 1000 booklets should be sufficient.
Candidature: 121 boys.

Examiner

JNC

QUESTION ONE (14 marks) Use a separate writing booklet. **Marks**

(a) Find the exact value of $\tan^{-1}\frac{-1}{\sqrt{3}}$. $\boxed{1}$

(b) Find $\tan^{-1}\frac{1}{2}$ correct to three decimal places. $\boxed{1}$

(c) Solve $\sin 2x = -\dfrac{1}{\sqrt{2}}$, for $0 \le x \le \pi$. $\boxed{2}$

(d) Solve $e^x = 3$. Give your answer correct to three decimal places. $\boxed{2}$

(e) Find $\displaystyle\int \frac{2}{\sqrt{4-x^2}}\,dx$. $\boxed{1}$

(f) The polynomial $P(x) = x^3 + x + a$ has $(x-2)$ as a factor. Find the value of a. $\boxed{1}$

(g) Find primitives of:

(i) $\dfrac{1}{5x}$ $\boxed{1}$

(ii) $\dfrac{1}{3+x^2}$ $\boxed{2}$

(h) Differentiate with respect to x:

(i) $y = e^{1-2x}$ $\boxed{1}$

(ii) $y = xe^x$ $\boxed{2}$

Exam continues next page ...

QUESTION TWO (14 marks) Use a separate writing booklet. **Marks**

(a) Evaluate $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{8}} \sec^2 2x\,dx$. [3]

(b) (i) Without using calculus, draw a sketch of the polynomial function $y = x(x-1)(x+2)^2$. [2]

(ii) Use your graph to write down the solution of the inequation $x(x-1)(x+2)^2 < 0$. [1]

(c) Show that $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} = 2$. [3]

(d) The function $f(x)$ is given by $f(x) = \sin^{-1} x + \cos^{-1} x$, for $0 \le x \le 1$.

(i) Evaluate $f(0)$. [1]

(ii) Find $f'(x)$. [2]

(iii) Evaluate $\displaystyle\int_0^1 f(x)\,dx$. [2]

QUESTION THREE (14 marks) Use a separate writing booklet. **Marks**

(a) Find $\displaystyle\int \frac{x+1}{x^2+4}\,dx$. [3]

(b) (i) Express $2\sin\theta + \sqrt{5}\cos\theta$ in the form $R\sin(\theta+\alpha)$, where $R > 0$ and α is acute. Give α correct to the nearest degree. [3]

(ii) Hence, or otherwise, find the minimum value of the expression $2\sin\theta + \sqrt{5}\cos\theta$. [1]

(c) The displacement, x metres, of a particle moving in a straight line is given by $x = 3 - e^{-t}$, where t is the time in seconds.

(i) Find the initial displacement and velocity of the particle. [3]

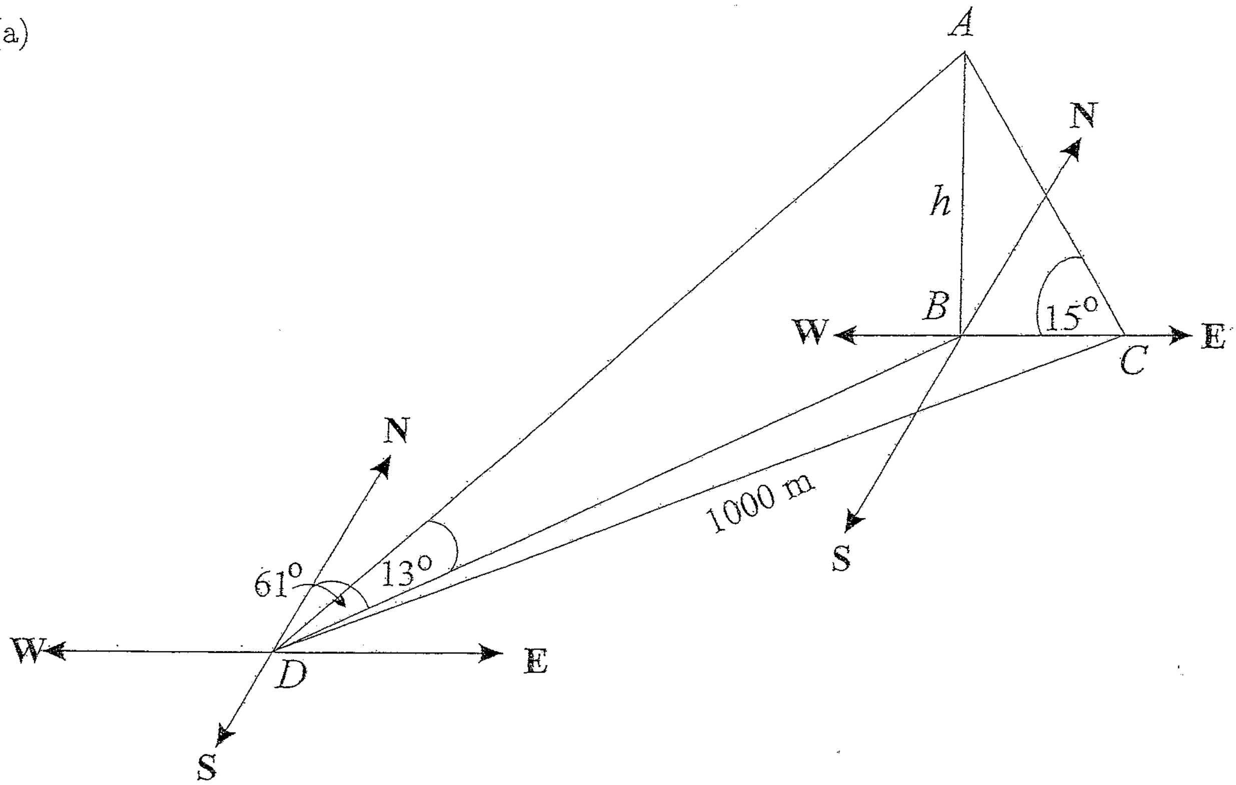
(ii) Briefly describe the motion of the particle as $t \to \infty$. [2]

(iii) On separate diagrams, sketch the displacement–time graph and the velocity–time graph. Clearly indicate the asymptotes on each graph. [2]

Exam continues overleaf ...

QUESTION FOUR (14 marks) Use a separate writing booklet. **Marks**

(a)



In the diagram above, AB represents a tower of height h metres. The angle of elevation of the tower from a point C due east of it is $15°$. From another point D, the bearing of the tower is $061°T$ and the angle of elevation is $13°$. The points C and D are 1000 metres apart and on the same level as the base of the tower.

(i) Show that $\angle DBC = 151°$. **2**

(ii) Use $\triangle ABC$ to show that $BC = h \tan 75°$. **1**

(iii) Find a similar expression for BD. **1**

(iv) Calculate the height of the tower, correct to the nearest metre. **3**

(b) The gradient of a curve is given by $\dfrac{dy}{dx} = \dfrac{2}{x+e}$. Find the equation of the curve if $y = 1$ when $x = 0$. **3**

(c) A solid of revolution is formed by rotating about the x-axis the region between the curve $y = 2(1 + e^{2x})$ and the x-axis, from $x = 0$ to $x = 1$. Show that the volume of the solid formed is $\pi(e^4 + 4e^2 - 1)$ cubic units. **4**

QUESTION FIVE (14 marks) Use a separate writing booklet. **Marks**

(a) Use long division to express $4x^3 - 7x - 8$ in the form $q(x)(x-1) + c$, where c is a constant. 3

(b) (i) Use the remainder theorem to find one factor of $f(x) = x(x+1) - a(a+1)$, where a is a constant. 2

(ii) By long division, or otherwise, find the other factor of $f(x)$. 2

(c) Let the roots of the equation $2x^2 + x + 3 = 0$ be α and β.

(i) Find $\alpha + \beta$ and $\alpha\beta$. 2

(ii) Find $(\alpha - 1)(\beta - 1)$. 2

(iii) Find a quadratic equation whose roots are α^2 and β^2. 3

QUESTION SIX (14 marks) Use a separate writing booklet. **Marks**

(a) A particle is moving in a straight line. At the time t seconds, its velocity is v metres per second and its displacement from the origin is x metres. Prove that $\ddot{x} = \dfrac{d}{dx}(\tfrac{1}{2}v^2)$. 2

(b) The acceleration of a particle is given by $\ddot{x} = \frac{3}{2}(1 - x^2)$. Initially the particle is at the origin and moving with a velocity of 6 metres per second.

(i) Show that the velocity of the particle is given by $v^2 = 36 + 3x - x^3$. 2

(ii) Explain why the particle will never reach the position $x = 4$. 2

(c) A particle is oscillating in simple harmonic motion such that its displacement x metres from the origin satisfies the equation $\ddot{x} = -4x$, where the time t is in seconds.

(i) Show that $x = a\cos(2t + \beta)$ is a possible equation of the motion, where a and β are constants. 2

(ii) Initially the particle has a velocity of $2\,\text{m/s}$ and a displacement from the origin of 4 metres.

(α) Find the period of oscillation. 1

(β) Show that the amplitude of oscillation is $\sqrt{17}$ metres. 3

(γ) Determine the maximum speed of the particle. 1

(δ) Where does the particle first come to rest? 1

Exam continues overleaf ...

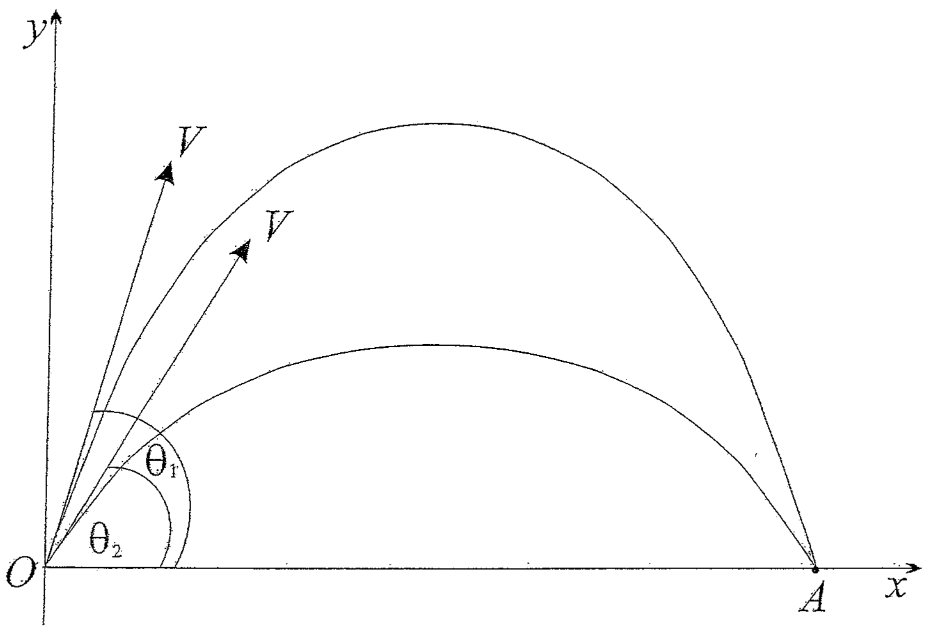
QUESTION SEVEN (14 marks) Use a separate writing booklet. **Marks**

(a) (i) A particle is projected from the origin with velocity V at an angle of projection θ. The equations of displacement are $x = Vt\cos\theta$ and $y = -\dfrac{gt^2}{2} + Vt\sin\theta$.

(α) Show that the time of flight T is given by $T = \dfrac{2V\sin\theta}{g}$. **2**

(β) Hence, or otherwise, show that the range R on the horizontal plane through the point of projection is $R = \dfrac{V^2 \sin 2\theta}{g}$. **2**

(ii)



In the diagram above, two particles are projected from the origin with the same velocity V at different angles of projection θ_1 and θ_2. They strike the same point A, on the horizontal plane through the point of projection, at times T_1 and T_2 respectively.

(α) Show that $\theta_1 + \theta_2 = \frac{\pi}{2}$. **2**

(β) Show that the distance from O to A is $\frac{1}{2}gT_1T_2$. **3**

(b) A particle is moving in simple harmonic motion about the origin. It starts at $x = 1$ and at $t = 1$ it is at $x = 5$ for the first time. The particle next returns to the position $x = 5$ at $t = 2$.

(i) Let $x = a\cos(nt + \alpha)$, and show that $\cos n = \frac{3}{5}$. **3**

(ii) Find the amplitude of the motion. **2**

END OF EXAMINATION

FORM VI : EXTENSION 1 – HALF YEARLY '06

QUESTION ONE

a) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$ ✓

b) $\tan^{-1}\left(\frac{1}{2}\right) = 0.464$ ✓

c) $\sin 2x = -\frac{1}{\sqrt{2}}$

$2x = \frac{5\pi}{4}, \frac{7\pi}{4}$ ✓

$\therefore x = \frac{5\pi}{8}, \frac{7\pi}{8}$ ✓

d) $e^x = 3$

$x = \ln 3$ ✓

$\doteqdot 1.099$ ✓ (Penalise incorrect or no rounding)

e) $\int \frac{2\,dx}{\sqrt{4-x^2}} = 2\sin^{-1}\frac{x}{2} + c$ ✓

f) $P(2) = 10 + a$

$\therefore a + 10 = 0$

$a = -10$ ✓

g) (i) $\int \frac{dx}{5x} = \frac{1}{5}\ln x + c$ ✓

(ii) $\int \frac{dx}{3+x^2} = \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + c$ ✓✓ (−1/error)

h) (i) $\frac{d}{dx}\, e^{1-2x} = -2e^{1-2x}$ ✓

(ii) $\frac{d}{dx}\left(xe^x\right) = 1\times e^x + x\,e^x$

$= e^x(1+x)$

✓✓ either (−1/error)

QUESTION TWO

a) $\int_{\frac{\pi}{12}}^{\frac{\pi}{8}} \sec^2 2x\,dx = \frac{1}{2}\Big[\tan 2x\Big]_{\frac{\pi}{12}}^{\frac{\pi}{8}}$ ✓

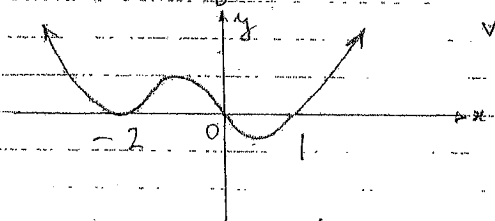
$= \frac{1}{2}\left(\tan\frac{\pi}{4} - \tan\frac{\pi}{6}\right)$ ✓

$= \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right)$ ✓

or

$= \frac{1}{6}\left(3 - \sqrt{3}\right)$

b) (i)



✓✓ (−1/error)

(ii) $0 < x < 1$ ✓

c) $\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = \frac{\sin 3\theta\cos\theta - \cos 3\theta\sin\theta}{\sin\theta\cos\theta}$ ✓

$= \frac{\sin 2\theta}{\frac{1}{2}(2\sin\theta\cos\theta)}$ ✓

$= 2$ ✓

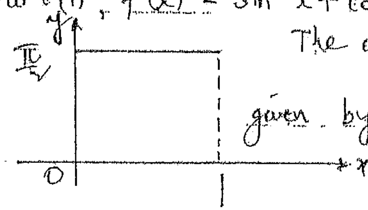
d) (i) $f(0) = 0 + \frac{\pi}{2}$

$= \frac{\pi}{2}$ ✓

(ii) $f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}}$ ✓

$= 0$ ✓

(iii) By part (i), $f(x) = \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$



The area of the rectangle is given by $\int_0^1 f(x)\,dx = \frac{\pi}{2}\times 1$ ✓

$= \frac{\pi}{2}$ ✓

(explanation is not required)

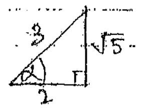
QUESTION THREE

a) $\int \frac{x+1}{x^2+4}\,dx = \int \frac{x}{x^2+4} + \frac{1}{x^2+4}\,dx$ ✓

$= \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\frac{x}{2} + c$ ✓✓

b) (i) $2\sin\theta + \sqrt{5}\cos\theta = R(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$

$\left.\begin{aligned} R\cos\alpha &= 2 \\ R\sin\alpha &= \sqrt{5} \end{aligned}\right\}$ ✓

$\tan\alpha = \frac{\sqrt{5}}{2} \Rightarrow$ 

✓ $\alpha \doteqdot 48^\circ$

$R\cos\alpha = 2$

$R\cdot\frac{2}{3} = 2$

$\therefore R = 3$ ✓

So $2\sin\theta + \sqrt{5}\cos\theta = 3\sin(\theta + 48^\circ)$ (Not necessary)

(ii) Minimum value is -3 since the minimum value of $\sin(\theta+48^\circ)$ is -1. ✓

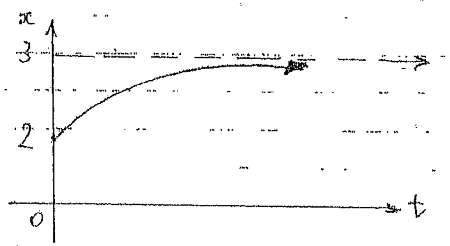
c) (i) $x = 3 - e^{-t}$

$\dot{x} = e^{-t}$ ✓

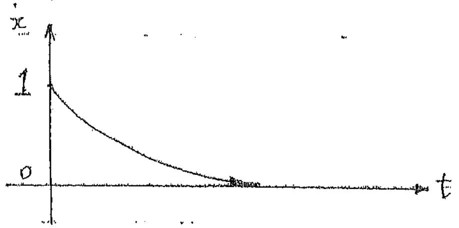
When $t=0$; $x = 3-1 = 2\text{m}$ ✓ and $\dot{x} = 1$ m/s ✓

(ii) As $t \to \infty$, $x \to 3^-$ and $\dot{x} \to 0^+$ ✓✓

(iii)



✓ (Asymptote necessary)



✓

QUESTION FOUR

a) (i) $\angle DBC = 61 + 90$ ✓ (alternate $\angle$s on $\|$ lines and C is due east ✓)

$= 151$

(ii) $\left.\begin{aligned} \frac{BC}{h} &= \tan 75 \\ \therefore BC &= h\tan 75 \end{aligned}\right\}$ ✓ (both necessary)

(iii) $BD = h\tan 77$ ✓

(iv) In ΔBDC:

$BC^2 + BD^2 - 2.BC.BD\cos 151 = 1000^2$

$h^2(\tan^2 75 + \tan^2 77 - 2\tan 75\tan 77\cos 151) = 1000^2$ ✓

$\therefore h = \frac{1000}{\sqrt{\tan^2 75 + \tan^2 77 - 2\tan 75\tan 77\cos 151}}$ ✓

$h \doteqdot 128$ m ✓ (penalise incorrect rounding)

b) $y' = \frac{2}{x+e}$

$y = 2\ln(x+e) + c$ ✓

Substituting $x = 0$, $y = 1$:

$1 = 2\ln e + c$

$\therefore c = -1$ ✓

So $y = 2\ln(x+e) - 1$ ✓

c) $V = \pi\int_0^1 (2(1+e^{2x}))^2\,dx$ ✓

$= 4\pi\int_0^1 1 + 2e^{2x} + e^{4x}\,dx$ ✓

$= 4\pi\left[x + e^{2x} + \frac{e^{4x}}{4}\right]_0^1$ ✓

$\left.\begin{aligned} &= 4\pi\left[\left(1+e^2+\frac{e^4}{4}\right) - \left(1+\frac{1}{4}\right)\right] \\ &= \pi(e^4 + 4e^2 - 1) \end{aligned}\right\}$ ✓ (Must show substitution).

QUESTION FIVE

a)
$$\begin{array}{r} 4x^2+4x-3 \\ (x-1)\overline{)\,4x^3-7x-8} \\ \underline{4x^3-4x^2} \\ 4x^2-7x \\ \underline{4x^2-4x} \\ -3x-8 \\ \underline{-3x+3} \\ -11 \end{array}$$

So $4x^3-7x-8 = (4x^2+4x+3)(x-1)-11$

b) (i) $f(a) = a(a+1) - a(a+1)$
$= 0$
$\therefore x-a$ is a factor

(ii)
$$\begin{array}{r} x+(1+a) \\ x-a\overline{)\,x^2+x-a(a+1)} \\ \underline{x^2-ax} \\ x(1+a)-a(a+1) \\ \underline{x(1+a)-a(1+a)} \\ 0 \end{array}$$

OR $f(-(1+a))$
$= -(1+a)(-1-a-1)-a(a+1)$
$= -(1+a)(-a)-a(1+a)$
$= 0$

$\therefore (x+(a+1))$ is the other factor of $f(x)$.

c) (i) $\alpha+\beta = -\frac{1}{2}$
$\alpha\beta = \frac{3}{2}$

(ii) $(\alpha-1)(\beta-1) = \alpha\beta - (\alpha+\beta)+1$
$= \frac{3}{2}+\frac{1}{2}+1$
$= 3$

(iii) $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$
$= \left(-\frac{1}{2}\right)^2 - 2\cdot\frac{3}{2}$
$= -\frac{11}{4}$

Equation is $x^2 - (\alpha^2+\beta^2)x + (\alpha\beta)^2 = 0$

$x^2 + \frac{11}{4}x + \frac{9}{4} = 0$

$4x^2+11x+9 = 0$

QUESTION SIX

a)(i) $\ddot{x} = \frac{dv}{dt}$
$= \frac{dv}{dx}\cdot\frac{dx}{dt}$
$= v\frac{dv}{dx}$
$= \frac{d}{dv}\left(\frac{1}{2}v^2\right)\cdot\frac{dv}{dx}$
$= \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

(b) (i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{3}{2}(1-x^2)$

$\frac{1}{2}v^2 = \frac{3}{2}\left(x-\frac{x^3}{3}\right)+c$

When $x=0$, $v=6$ so $\frac{1}{2}\times 36 = c$
$\therefore c = 18$

$\frac{1}{2}v^2 = \frac{3}{2}\left(x-\frac{x^3}{3}\right)+18$

ie $v^2 = 36+3x-x^3$

(ii) At $x=4$, $v^2 = 36+12-64$
$= -16$, which is impossible.

So the particle does not reach $x=4$.

(c) (i) $x = a\cos(2t+\beta)$

$\dot{x} = -2a\sin(2t+\beta)$

$\ddot{x} = -4a\cos(2t+\beta)$

$= -4x$

(ii) (α) $T = \frac{2\pi}{2}$

$= \pi$

(β) When $t=0$; $x=4$ and $\dot{x}=2$:

$$\left.\begin{aligned} 4 &= a\cos\beta \\ 2 &= -2a\sin\beta \end{aligned}\right\}$$

Squaring and adding gives

$16+1 = a^2(\cos^2\beta+\sin^2\beta)$

$a^2 = 17$

$a = \pm\sqrt{17}$

So the amplitude is $\sqrt{17}$ m since particle oscillates about the origin.

(γ) Maximum ~~speed~~ velocity will occur when $\sin(2t+\beta) = \pm 1$.

$\dot{x} = \pm 2\sqrt{17}$

So maximum speed is $2\sqrt{17}$ m/s.

(δ) Since $a = \sqrt{17}$ and the particle oscillates about the origin and moves in a positive direction from $x=4$, it first comes to rest at $x = \sqrt{17}$ m.

QUESTION SEVEN

a) (i) (α) $x = Vt\cos\theta$ $\quad y = -\frac{gt^2}{2} + Vt\sin\theta$

Time of flight occurs when $y=0$.

$\Rightarrow -\frac{gt^2}{2} + Vt\sin\theta = 0$

$t\left(V\sin\theta - \frac{gt}{2}\right) = 0$

$\therefore t=0$ or $V\sin\theta - \frac{gt}{2} = 0$

$\therefore t = \frac{2V\sin\theta}{g}$

Since $t=0$ is initial time, the time of flight is $T = \frac{2V\sin\theta}{g}$

(β) $R = Vt\cos\theta$

$= V\left(\frac{2V\sin\theta}{g}\right)\cos\theta$

$= \frac{V^2\sin 2\theta}{g}$

(ii) (α) $OA = \frac{V^2\sin 2\theta_1}{g} = \frac{V^2\sin 2\theta_2}{g}$

$\Rightarrow \sin 2\theta_1 = \sin 2\theta_2$

So $2\theta_1 = 2\theta_2 + 2n\pi$ or $2\theta_1 = \pi - 2\theta_2$

$\therefore \theta_1 = \theta_2$ $\qquad 2(\theta_1+\theta_2) = \pi$

$\therefore \theta_1+\theta_2 = \frac{\pi}{2}$

Since $\theta_1 \neq \theta_2$, $\theta_1+\theta_2 = \frac{\pi}{2}$ only.

(β) $T_1 = \dfrac{2V\sin\theta_1}{g}$ and $T_2 = \dfrac{2V\sin\theta_2}{g}$

$$OA = \frac{V^2 \sin 2\theta_1}{g}$$

$$= \frac{V^2\, 2\sin\theta_1 \cos\theta_1}{g}$$

$$= \frac{V^2\, 2\sin\theta_1 \sin\theta_2}{g}, \quad \text{since } \cos\theta_1 = \sin\theta_2 \text{ as } \theta_1 + \theta_2 = \frac{\pi}{2}.$$

$$= \frac{g}{2}\left(\frac{2V\sin\theta_1}{g}\right)\left(\frac{2V\sin\theta_2}{g}\right)$$

$$= \frac{1}{2} g T_1 T_2$$

b)(i) $t = 0,\ x = 1 \Rightarrow a\cos\alpha = 1$ —— (1)

$t = 1,\ x = 5 \qquad a\cos(n+\alpha) = 5$ —— (2)

$t = 2,\ x = 5 \qquad a\cos(2n+\alpha) = 5$ —— (3)

From (1) : $a = \dfrac{1}{\cos\alpha}$ —— (4)

Substituting (4) into (2) and expanding gives:

$$\frac{\cos n\cos\alpha - \sin n \sin\alpha}{\cos\alpha} = 5$$

ie $\cos n - \sin n \tan\alpha = 5$ —— (5)

Substituting (4) into (3) gives:

$\cos 2n - \sin 2n \tan\alpha = 5$ —— (6)

either

From (5) $\tan\alpha = \dfrac{\cos n - 5}{\sin n}$, substituting into (6),

$$\cos 2n - \sin 2n\left(\frac{\cos n - 5}{\sin n}\right) = 5$$

$$2\cos^2 n - 1 - \frac{2\cancel{\sin n}\cos n(\cos n - 5)}{\cancel{\sin n}} = 5$$

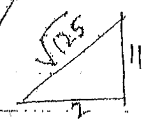
$$2\cancel{\cos^2 n} - 1 - 2\cancel{\cos^2 n} + 10\cos n = 5$$

$$\therefore \cos n = \frac{3}{5}.$$

(ii) Substituting $\cos n = \frac{3}{5}$ into (5) gives:

$$\frac{3}{5} - \frac{4}{5}\tan\alpha = 5$$

$$\tan\alpha = -\frac{11}{2}.$$



Since $\tan\alpha$ is negative, α is in quadrants 2 or 4 and $\cos\alpha = \pm\dfrac{2}{5\sqrt5}$. From equation (4) and sinc

amplitude is positive, $a = \left|\dfrac{1}{\cos\alpha}\right|$

$$\therefore a = \frac{5\sqrt5}{2}.$$