SYDNEY GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT
ASSESSMENT EXAMINATIONS 2005

FORM VI

MATHEMATICS EXTENSION 1

Examination date
Thursday 3rd March 2005

Time allowed
Periods 6 & 7

Instructions
All six questions may be attempted.
All six questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection
Write your candidate number clearly on each booklet.
Hand in the six questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.

6A: DS	6B: PKH	6C: DNW	6D: JNC
6E: KWM	6F: BDD	6G: REN	6H: MLS

Checklist
Folded A3 booklets: 6 per boy. A total of 1000 booklets should be sufficient.
Candidature: 122 boys.

Examiner
KWM

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

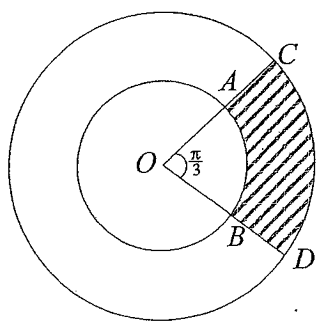
$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) Expand and simplify $\sin(\alpha + \frac{\pi}{6})$. 2

(b) Find $\lim_{x \to 0} \frac{\sin 4x}{x}$. 2

(c)



$OA = 2\,\text{cm}$
$OC = 4\,\text{cm}$

The diagram above shows two concentric circles with centre O.
The radius of the circles are $OA = 2\,\text{cm}$ and $OC = 4\,\text{cm}$.
The arc CD subtends an angle of $\frac{\pi}{3}$ at the centre O.

(i) Find the exact length of the arc CD. 1

(ii) Find the exact area of the shaded region. 2

(d) Given that $a = \log_e 2$ and $b = \log_e 3$, express $\log_e \frac{8}{9}$ in terms of a and b. 2

(e) (i) Write down the gradients of the lines $y = \frac{1}{2}x + 3$ and $2x + 8y + 5 = 0$. 1

(ii) Show that the acute angle θ between these lines is given by $\theta = \tan^{-1} \frac{6}{7}$. 2

Exam continues next page ...

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) Given that $\alpha = \cos^{-1} \frac{1}{2}$ and $\beta = \sin^{-1} \frac{1}{2}$, find $\alpha + \beta$. 2

(b) Differentiate with respect to x:

(i) $y = \sin^2 x$ 1

(ii) $y = \ln\left(\frac{x^2}{x+1}\right)$ 2

(c) Write down the domain and range of the function $f(x) = 2\cos^{-1}(x - 1)$. 2

(d) Given that $y = \sin^{-1} \frac{x}{2}$, find $\frac{dy}{dx}$ in its simplest form. 2

(e) (i) Prove that $\frac{1 - \cos 2x}{\sin 2x} = \tan x$. 2

(ii) Hence find the exact value of $\tan 15°$. 1

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a) Sketch the curve $y = 3\sin 2x$, for $-\pi \le x \le \pi$, showing all significant points. 2

(b) Evaluate the indefinite integral $\int \frac{x}{4 - x^2}\, dx$. 2

(c) Calculate the volume of the solid formed when the region between the curve $y = e^{-x}$ and the x-axis, from $x = 0$ to $x = \log_e 2$, is rotated about the x-axis. 3

(d) Find the exact value of the definite integral $\int_0^{\frac{\pi}{4}} \sin^2 x\, dx$. 3

(e) Solve the equation $\cos^2 x - \sin^2 x = 1$, for $0° \le x \le 360°$. 2

Exam continues overleaf ...

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a) Find the exact value of $\cos(2\sin^{-1}\frac{3}{5})$. [2]

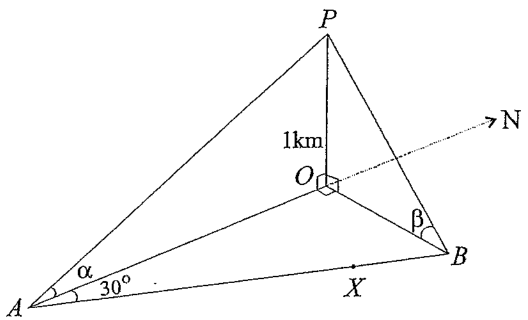
(b) (i) Express $\sqrt{3}\cos\theta - \sin\theta$ in the form $r\cos(\theta+\alpha)$, where $r > 0$ and $0 \le \alpha < 2\pi$. [2]

(ii) Hence solve the equation $\sqrt{3}\cos\theta - \sin\theta = 1$, for $0 \le \theta < 2\pi$. [2]

(c) Use the substitution $t = \tan\frac{x}{2}$ to express $\dfrac{1+\cos x}{1-\cos x}$ in terms of t. [2]

Hence prove that $\dfrac{1+\cos x}{1-\cos x} = \cot^2\frac{x}{2}$.

(d)



The diagram above shows a mountain peak P that rises 1 km above a level plain. A bushwalker parks his car at a point A on the plain due south of the peak. The angle of elevation of the peak from A is α.

He then walks on a bearing of N30°E on level ground, until he reaches a point B due east of the mountain. The angle of elevation of the mountain peak from the point B is β.

(i) Show that $\tan\beta = \sqrt{3}\tan\alpha$. [2]

(ii) During his walk from A to B, the greatest angle of elevation from his position to the mountain peak occurs at a point X. Find an expression for the distance AX in terms of α. [2]

Exam continues next page ...

QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**

(a) A cylindrical tank that initially holds 1000 litres of water is penetrated by a rifle shot during an insurgent attack, and begins to leak. The volume V litres of water in the tank at any time t hours afterwards is given by

$$V = 1000e^{-kt}.$$

(i) Show that [1]

$$\frac{dV}{dt} = -kV.$$

(ii) During the first hour, 20% of the initial volume of water leaks from the tank. Show that $k = \log_e \frac{5}{4}$. [1]

(iii) How long will it take for the initial volume to decrease by 50%? Give your solution correct to the nearest minute. [2]

(b) Find $\displaystyle\int_{\frac{3}{4}}^{\frac{3}{2}} \frac{3}{\sqrt{9-4x^2}}\,dx$. [3]

(c) Given that $\cos 3x = 4\cos^3 x - 3\cos x$, find the general solution of the equation [2]

$$4\cos^3 x = 3\cos x.$$

(d) Find the exact value of $\displaystyle\int_{e}^{e^2} \frac{1}{x\ln x}\,dx$. [3]

QUESTION SIX (12 marks) Use a separate writing booklet. **Marks**

(a) (i) Find $\dfrac{d}{dx}(x^2\ln x)$. [2]

(ii) Hence, or otherwise, evaluate $\displaystyle\int_1^e x\ln x\,dx$. [1]

(b) (i) Sketch the curve $y = \tan^{-1}(x-1)$. [2]

(ii) Calculate the volume of the solid formed when the region between the curve $y = \tan^{-1}(x-1)$ and the y-axis, from $y = -\frac{\pi}{4}$ to $y = \frac{\pi}{4}$, is rotated about the y-axis. [4]

(c) The function $g(x)$ is defined by $g(x) = \dfrac{e^x - e^{-x}}{2}$. Show that for all x, [3]

$$g^{-1}(x) = \log_e\left(x + \sqrt{x^2+1}\right).$$

END OF EXAMINATION

QUESTION 1

$P(x) \equiv Kx^3 + x^2 - (2K-1)x + 2$

$P(-1) = 4$ (Using the remainder theorem.)

$-K + 1 + (2K-1) + 2 = 4$

$K + 2 = 4$ ✓

$K = 2$

i) $y = e^x \ln x$

$\frac{dy}{dx} = e^x \ln x + e^x \times \frac{1}{x}$ ✓

$= e^x(\ln x + \frac{1}{x})$

(ii) $y = \sin^{-1} 2x$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-4x^2}} \times 2$ ✓

$= \frac{2}{\sqrt{1-4x^2}}$ ✓

$\tan\alpha = \frac{1}{4}$ and $\tan\beta = \frac{3}{5}$

$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \frac{\frac{1}{4} + \frac{3}{5}}{1 - \frac{3}{20}}$ ✓

$= \frac{5 + 12}{20 - 3}$

$= 1$ ✓

I)

$x(2x-1)(x+1) \equiv 2x^3 + bx^2 + cx + 3$

$(2x^2 + x - 1) \equiv 2x^3 + bx^2 + cx + 3$

$2x^3 + x^2 - x + 3 \equiv 2x^3 + bx^2 + cx + 3$ ✓

equating co-efficients:

$b = 1$ and $c = -1$. ✓

(e) (i) $y = e^{-x^2}$

$\frac{dy}{dx} = -2x e^{-x^2}$ ✓

(ii) $\frac{d^2y}{dx^2} = -2e^{-x^2} + -2xe^{-x^2} \times -2x$

$= -2e^{-x^2} + 4x^2 e^{-x^2}$ ✓

$= 2e^{-x^2}(2x^2 - 1)$

(iii) find the x-coordinates of the points of inflexion.

$2e^{-x^2}(2x^2-1) = 0$ ✓

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm\frac{1}{\sqrt{2}}$

The curve is concave down $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ ✓

(from the graph.)

OR.

Solve $2e^{-x^2}(2x^2-1) < 0$

$2x^2 - 1 < 0$

$x^2 < \frac{1}{2}$

$-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

(12)

QUESTION 2

(a) $x = 5 + 4t - t^2$

(i) when $t = 0$, $x = 5$ ✓

(ii) $x = 5 + 4t - t^2$

$\dot{x} = 4 - 2t$

$4 - 2t = 0$

$2t = 4$

$t = 2s.$ ✓

The particle changes direction at $x = 5 + 8 - 4$

$x = 9.$ ✓

(iii) when $t = 0$, $x = 5$

when $t = 2$, $x = 9$

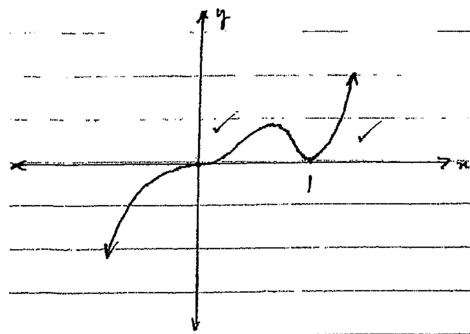
when $t = 6$, $x = -7$

distance travelled $= 5 + 9 + 7$

$= 20\text{m}.$ ✓

(b) (i) $y = x^3(x-1)^2$

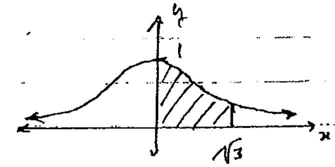
x intercepts at $x = 0$ and $x = 1$.



i) $x^3(x-1)^2 > 0$

$x > 0,\ x \neq 1.$ ✓

(c)



$A = \int_0^{\sqrt{3}} \frac{dx}{1+x^2}$ ✓

$= \left[\tan^{-1}x\right]_0^{\sqrt{3}}$

$= \tan^{-1}\sqrt{3} - \tan^{-1}0$

$= \frac{\pi}{3}$ square units. ✓

(d)

(i) $\int \sec^2 3x\, dx = \frac{1}{3}\tan 3x + c$ ✓

(ii) $\int \sin^3 x \cos x\, dx = \frac{1}{4}\sin^4 x + c$ ✓✓

(12)

QUESTION 3

$\cos^2 x - \sin^2 x = \frac{\sqrt{3}}{2}$

for $0 \le x \le 2\pi$.

$\cos 2x = \frac{\sqrt{3}}{2}$ ✓

✓ $2x = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}$

✓ $x = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$

) (i) $v = 2 - 4e^{-t}$

when $t = 0$, $v = 2 - 4$

$v = -2$ m/s. ✓

(ii) when $t = \ln 2$

$v = 2 - 4e^{-\ln 2}$

$v = 2 - 4e^{\ln\frac{1}{2}}$ ✓

$v = 2 - 4 \times \frac{1}{2}$

$v = 0$.

(iii) $\int_0^{\ln 2} 2 - 4e^{-t}\,dt$

$= \left[2t + 4e^{-t}\right]_0^{\ln 2}$ ✓

$= (2\ln 2 + 4e^{-\ln 2}) - (0 + 4)$

$= 2\ln 2 + 4 \times \frac{1}{2} - 4$ ✓

$= |\ln 4 - 2|$ m

) as $t \to \infty$, $e^{-t} \to 0$

$v = 2 - 4e^{-t}$

$v \to 2$ m/s. ✓

(e)

(i) $f(x) = \ln\frac{(1 + \sin x)}{\cos x}$

$f(x) = \ln(1 + \sin x) - \ln\cos x$

$f'(x) = \frac{\cos x}{1 + \sin x} + \frac{\sin x}{\cos x}$ ✓

$= \frac{\cos^2 x + \sin x(1 + \sin x)}{\cos x(1 + \sin x)}$

$= \frac{\cos^2 x + \sin^2 x + \sin x}{\cos x(1 + \sin x)}$

$= \frac{1 + \sin x}{\cos x(1 + \sin x)}$ ✓

$= \frac{1}{\cos x}$

$f'(x) = \sec x$.

(ii) $\int_0^{\frac{\pi}{4}} \sec x\,dx = \left[\ln\left(\frac{1 + \sin x}{\cos x}\right)\right]_0^{\frac{\pi}{4}}$ ✓

$= \ln\left(\frac{1 + \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right) - \ln 1$ ✓

$= \ln(\sqrt{2} + 1)$

(12)

QUESTION 4

(a) $x^3 - 4x^2 + 3x - 1 = 0$

(i) $\alpha + \beta + \gamma = -\frac{b}{a}$

$= 4$ ✓

(ii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

• $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

$= 3$

• $\alpha\beta\gamma = -\frac{d}{a}$

$= 1$

$\therefore \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{3}{1}$

$= 3$. ✓

(iii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$ ✓

$= 4^2 - 2 \times 3$

$= 10$ ✓

(b) (i) $x = 6\cos 2t$

$\dot{x} = -12\sin 2t$

Maximum velocity $|\dot{x}| = 12$ m/s ✓

(ii) $\ddot{x} = -24\cos 2t$

So $\ddot{x} = 0$ when

$-24\cos 2t = 0$ ✓

at $2t = \frac{\pi}{2}$

$t = \frac{\pi}{4}$ s. ✓

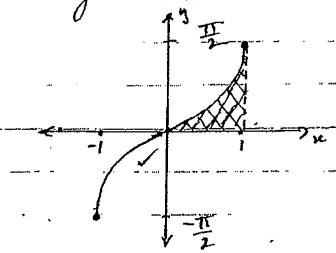
(iii) $\ddot{x} = -24\cos 2t$

$= -4(6\cos 2t)$

$= -4x$ ✓

(c) $y = \sin^{-1} x$

(i)



✓✓

(ii) $y = \sin^{-1} x$

$x = \sin y$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

Area shaded = Rectangle − area bounded by the curve and the y-axis.

$A = 1 \times \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin y\,dy$ ✓

$= \frac{\pi}{2} + \left[\cos y\right]_0^{\frac{\pi}{2}}$

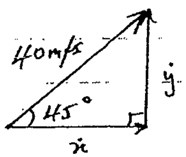
$= \frac{\pi}{2} + (0 - 1)$ ✓

$= \frac{\pi}{2} - 1$ square units.

(12)

QUESTION 5

initial components of velocity:



$\dot{x} = 20 \cos 45°$ $\qquad$ $\dot{y} = 40 \sin 45°$

$\dot{x} = 20\sqrt{2}$ m/s $\qquad$ $\dot{y} = 20\sqrt{2}$ m/s. ✓

$\ddot{x} = 0$ $\qquad$ $\ddot{y} = -10$

$\dot{x} = c_1$ $\qquad$ $\dot{y} = -10t + c_2$

when $t = 0$, $\dot{x} = 20\sqrt{2}$ and $\dot{y} = 20\sqrt{2}$.

$c_1 = c_2 = 20\sqrt{2}$

$\dot{x} = 20\sqrt{2}$ $\qquad$ $\dot{y} = -10t + 20\sqrt{2}$

$x = \int 20\sqrt{2}\, dt$ $\qquad$ $y = \int -10t + 20\sqrt{2}\, dt$

$x = 20t\sqrt{2} + c_3$ ✓ $\qquad$ $y = -5t^2 + 20t\sqrt{2} + c_4$

when $t = 0$, $x = 0$ and $y = 22\tfrac{1}{2}$.

$c_3 = 0$ $\qquad$ $c_4 = 22\tfrac{1}{2}$

$x = 20t\sqrt{2}$ ✓ $\qquad$ $y = -5t^2 + 20t\sqrt{2} + 22\tfrac{1}{2}$

are the equations of motion for the shell.

Put $x = 180$ m.

$20t\sqrt{2} = 180$

$t = \frac{9\sqrt{2}}{2}$ s ✓

then $x = 180$.

$y = -5t^2 + 20t\sqrt{2} + 22\tfrac{1}{2}$

$= -5 \times \frac{81 \times 2}{4} + 20 \times \frac{9\sqrt{2}}{2} \times \sqrt{2} + 22\tfrac{1}{2}$

$= -\frac{401}{2} + 180 + 22\tfrac{1}{2}$

$= 202\tfrac{1}{2} + 202\tfrac{1}{2}$

$= 0$ ✓

when $x = 180$, $y = 0$. The shell hits the ground 180m from the base of the cliff.

(iii) The maximum height is reached when $\dot{y} = 0$.

$-10t + 20\sqrt{2} = 0$

$t = 2\sqrt{2}$ s. ✓

$y = -5t^2 + 20t\sqrt{2} + 22\tfrac{1}{2}$

$y = -5 \times 8 + 80 + 22\tfrac{1}{2}$

$y = 62\tfrac{1}{2}$ m ✓

The maximum height above the ground reached by the shell is $62\tfrac{1}{2}$ m.

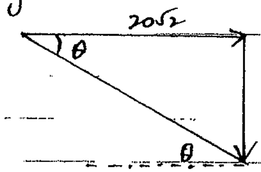
) The shell strikes the ground at $t = \frac{9\sqrt{2}}{2}$ s. (part (i))

Components of velocity:

$\dot{x} = 20\sqrt{2}$

$\dot{y} = -10 \times \frac{9\sqrt{2}}{2} + 20\sqrt{2}$

$\dot{y} = -25\sqrt{2}$ ✓



$\tan\theta = \frac{25\sqrt{2}}{20\sqrt{2}}$

$\tan\theta = \frac{5}{4}$ ✓

The equations of motion are:

$x = 40t\cos\alpha$ and $y = 40t\sin\alpha - 5t^2 + 22\tfrac{1}{2}$

at $x = 160$ and $y = 0$:

$0t\cos\alpha = 160$

$t = \frac{4}{\cos\alpha}$ substitute. ✓

$0t\sin\alpha - 5t^2 + 22\tfrac{1}{2} = 0$

$60\tan\alpha - \frac{80}{\cos^2\alpha} + 22.5 = 0$

$60\tan\alpha - 80\sec^2\alpha + 22.5 = 0$

$60\tan\alpha - (1 + \tan^2\alpha)80 + 22.5 = 0$

$80\tan^2\alpha + 160\tan\alpha - 80 + 22.5 = 0$

$80\tan^2\alpha - 160\tan\alpha + 57.5 = 0$ ✓

using the quadratic formula

$\tan\alpha = \frac{160 \pm \sqrt{160^2 - 4 \times 80 \times 57.5}}{160}$

$\tan\alpha = \frac{160 \pm \sqrt{7200}}{160}$

$\tan\alpha = \frac{160 - 60\sqrt{2}}{160}$

$\alpha = 25°\,9'$ ✓

12

QUESTION 6

(a) $v^2 = 6 + 10x - 4x^2$

(i) $\frac{1}{2}v^2 = 3 + 5x - 2x^2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 5 - 4x$ ✓

$\therefore \ddot{x} = -4\left(x - \frac{5}{4}\right)$

$\ddot{x} \propto x$, the motion is SHM.

(ii) (α) centre of motion is $x = \frac{5}{4}$. ✓

(β) $n = 2$. $T = \frac{2\pi}{n}$

$T = \pi$ s ✓

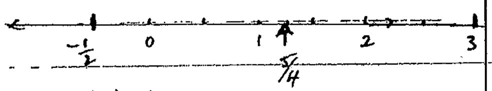
(γ) when $v = 0$.

$6 + 10x - 4x^2 = 0$

$2x^2 - 5x - 6 = 0$

$(2x + 1)(x - 3) = 0$

$x = -\frac{1}{2}$ or $x = 3$ ✓



amplitude $= 3 - \frac{5}{4}$

$= \frac{7}{4}$ ✓

(b) $\int_0^K \frac{6}{\sqrt{25 - 9x^2}}\,dx = \frac{\pi}{3}$

$6\left[\frac{1}{3}\sin^{-1}\frac{3x}{5}\right]_0^K = \frac{\pi}{3}$ ✓✓

$\left[\sin^{-1}\frac{3x}{5}\right]_0^K = \frac{\pi}{6}$

$\sin^{-1}\frac{3K}{5} = \frac{\pi}{6}$

$\frac{3K}{5} = \frac{1}{2}$

$6K = 5$

$K = \frac{5}{6}$ ✓

(c) $t = \tan\frac{\theta}{2}$

$\cos\theta = \frac{1 - t^2}{1 + t^2}$

$\cos\theta + \tan\frac{\theta}{2} - 1 = 0$

$\frac{1 - t^2}{1 + t^2} + t - 1 = 0$ ✓

$1 - t^2 + t(1 + t^2) - (1 + t^2) = 0$

$1 - t^2 + t + t^3 - 1 - t^2 = 0$

$t^3 - 2t^2 + t = 0$

$t(t^2 - 2t + 1) = 0$

$t(t - 1)(t - 1) = 0$

$t = 0$ or $t = 1$ ✓

$\tan\frac{\theta}{2} = 0$ or $\tan\frac{\theta}{2} = 1$

$\frac{\theta}{2} = n\pi$ ✓ $\frac{\theta}{2} = \frac{\pi}{4} + \pi n$ ✓

$\theta = 2\pi n$ or $\theta = \frac{\pi}{2} + 2\pi n$

(where n is an integer)

(12)

QUESTION 7.

(a) Using the division algorithm:

$P(x) = (x^2 + x - 2)Q(x) + ax + b$

$P(x) = (x - 1)(x - 2)Q(x) + ax + b$

$P(1) = 3$ and $P(-2) = -2$ ✓

① $a + b = 3$

② $-2a + b = -2$

①−② $3a = 5$

$a = \frac{5}{3}$ and $b = \frac{4}{3}$, hence the remainder is $\frac{5x}{3} + \frac{4}{3}$ ✓

(b) (i) Let the tangent be $y = mx + b$. Since it passes through the point $(2, -8)$.

$2m + b = -8$

$b = -2m - 8$. ✓

(ii) $\begin{cases} y = mx - 8 - 2m \\ y = x^3 - 4x^2 - x + 2. \end{cases}$

Solving simultaneously:

$x^3 - 4x^2 - (1 + m)x + (10 + 2m) = 0$. ✓

The roots of this cubic α, α and 2 correspond to the x co-ordinates of the points of intersection.

Sum of roots: $\alpha + \alpha + 2 = 4$

$2\alpha = 2$

$\alpha = 1$. ✓

Therefore $Q(1, -2)$.

(c) $\sqrt{\frac{1+x}{1-x}} \times \sqrt{\frac{1+x}{1+x}} = \frac{1+x}{\sqrt{1-x^2}}$ ✓

$\int\sqrt{\frac{1+x}{1-x}}\,dx = \int\frac{1+x}{\sqrt{1-x^2}}\,dx$

$= \int\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\,dx$ ✓

$= \sin^{-1}x - \sqrt{1-x^2} + c$ ✓

(d) (i) $f(x) = e^x$

at $x = 0$, $f'(x) = e^x$, $f'(0) = 1$

Using the definition. ✓

$f'(x) = \lim_{h\to 0}\frac{e^{x+h} - e^x}{h}$

$f'(0) = \lim_{h\to 0}\frac{e^h - 1}{h}$. ✓

Therefore $\lim_{h\to 0}\frac{e^h - 1}{h} = 1$ ....(1)

(ii) $\lim_{n\to\infty}\frac{e^{1/n} + e^{2/n} + e^{3/n} + \dots + e^{n/n}}{n}$

The numerator is a GP.

$S_n = \frac{a(r^n - 1)}{r - 1}$

$= e^{1/n}\left\{\frac{(e^{1/n})^n - 1}{e^{1/n} - 1}\right\}$

$= (e - 1)\frac{e^{1/n}}{e^{1/n} - 1}$

$\lim_{n\to\infty}\frac{e^{1/n} + e^{2/n} + e^{3/n} + \dots + e^{n/n}}{n}$

$= \lim_{n\to\infty}\frac{(e-1)e^{1/n}}{n(e^{1/n} - 1)}$ ✓

(Now let $\frac{1}{n} = h$ as $n \to \infty$, $h \to 0$)

$$\lim_{h\to 0} \frac{(e-1)\,e^{h}\cdot h}{e^{h}-1}$$

$$\lim_{h\to 0} (e-1)\,e^{h}\cdot \frac{h}{e^{h}-1}$$

$$(e-1) \lim_{h\to 0} \frac{h}{e^{h}-1} \cdot e^{h}$$

from (1)

✓

$= e-1.$

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