SYDNEY GRAMMAR SCHOOL   ASSESSMENT EXAMINATION 2004

MATHEMATICS ASSESSMENT FOR CLASSES 6A–6H

Time allowed: Periods 6 and 7   **Exam date:** 2nd March 2004

Instructions:

All questions may be attempted.
All questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection:

Each question will be collected separately.
Start each question in a new answer booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your name, class, and master's initials on each answer booklet:

6A: WMP	6B: GJ	6C: JCM	6D: REP
6E: TCW	6F: MLS	6G: DS	6H: KWM

Checklist:

Folded A3 examination booklets required — six booklets per boy.
Canditature: 122 boys

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QUESTION ONE (10 marks) (Start a new answer booklet)

Marks

(a) Find tbe exact value of $\sin^{-1}\dfrac{\sqrt{3}}{2}$. 1

(b) Differentiate the following with respect to x:

(i) $y = \log_e(3x+1)$ 1

(ii) $y = \sin^{-1}\frac{x}{2}$ 1

(c) Find the equation of the tangent to the curve $y = e^x + 2$ at the point where $x = 0$. 3

(d) Write down primitive functions of:

(i) $\cos(2-5x)$ 1

(ii) $\sec^2(3x)$ 1

(e) Find $\displaystyle\lim_{x\to 0}\frac{\sin 2x}{3x}$, showing all working. 2

QUESTION TWO (10 marks) (Start a new answer booklet)

Marks

(a) Let α be the acute angle between the lines $y = x - 2$ and $y = \frac{1}{2}x + 2$.

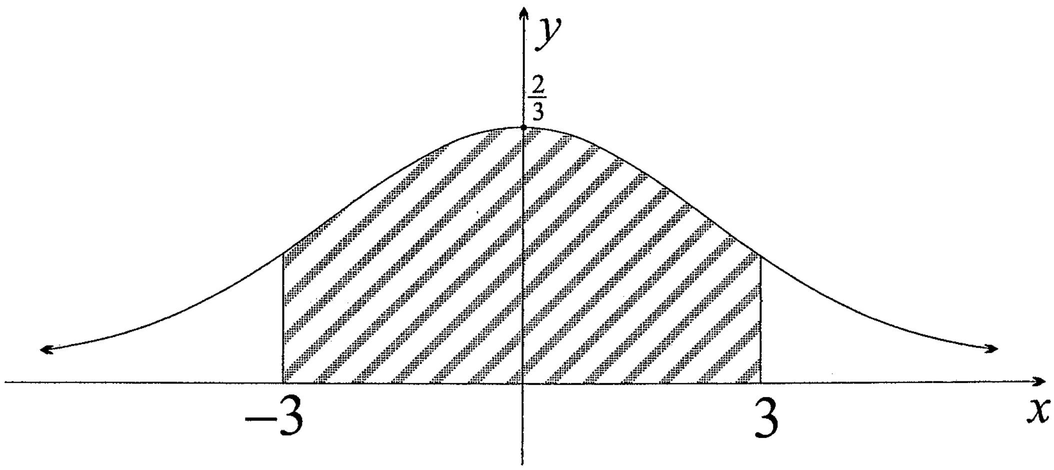
(i) Find the exact value of $\tan\alpha$. **1**

(ii) Find, correct to the nearest degree, the value of α. **1**

(b) (i) Write $\cos 2\theta$ in terms of $\sin\theta$. **1**

(ii) Hence find, showing all working, the exact value of $\cos\left(2\sin^{-1}\frac{1}{3}\right)$. **1**

(c)



Above is a graph of the function $y = \dfrac{6}{x^2+9}$. Find the shaded area. **3**

(d) Consider the function $f(x) = 2\sin^{-1}(3x+1)$.

(i) State the domain and range of $f(x)$. **2**

(ii) State the domain of the inverse function of $f(x)$. **1**

QUESTION THREE (10 marks) (Start a new answer booklet)

Marks

(a) Prove the identity $\dfrac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$. **2**

(b) (i) Differentiate $x\sin^{-1}x + \sqrt{1-x^2}$. **2**

(ii) Hence evaluate $\displaystyle\int_0^1 \sin^{-1}x\,dx$. **2**

(c) Consider the function $y = e^{kx}$, where k is a constant.

(i) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. **2**

(ii) Determine the values of k for which $y = e^{kx}$ satisfies the equation **2**

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0.$$

QUESTION FOUR (10 marks) (Start a new answer booklet)

Marks

(a) (i) Prove that $\sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}} = \tan\theta$, for $0 \le \theta < \frac{\pi}{2}$. **2**

(ii) Hence show that the exact value of $\tan\frac{\pi}{8}$ is $\sqrt{2}-1$. **2**

(b) Find the general solution of the equation $\sin 2x = \cos x$. **3**

(c) Write down $\sin\theta$ and $\cos\theta$ in terms of $t = \tan\frac{\theta}{2}$. Hence or otherwise prove that **3**

$$\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\frac{\theta}{2}.$$

QUESTION FIVE (10 marks) (Start a new answer booklet)

Marks

(a) Given that $\log_8 2 = \log_x 5$, find x. **2**

(b) (i) Find the values of the constants R and α if **2**

$$R\cos(\theta-\alpha) = \sqrt{3}\cos\theta + \sin\theta, \text{ for all } \theta,$$

where $R > 0$ and $0 \le \alpha < 2\pi$.

(ii) Hence solve the equation $\sqrt{3}\cos\theta + \sin\theta = 1$, for $0 \le \theta \le 2\pi$. **2**

(c) A company assumes that the proportion P of viewers who will buy a new product after it is advertised n times on television satisfies the function $P = 1 - e^{kn}$, where k is a constant.

(i) If 50% of viewers buy the product after 10 advertisments appear, show that $k = -\frac{1}{10}\log_e 2$. **1**

(ii) How many times should the company advertise the product if it wants at least 90% of its viewers to buy it? **3**

QUESTION SIX (10 marks) (Start a new answer booklet)

(a) Consider the function $f(x) = \dfrac{\log_e x}{x}$.

Marks

(i) Find the coordinates of the stationary point on the curve $y = f(x)$ and determine its nature. **3**

(ii) Hence show that $\pi^e < e^\pi$. **2**

(b) Consider the equation

$$\tan 2\theta = \frac{\tan\theta}{b + a\tan\theta}, \quad \text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

where a and b are positive constants.

(i) Find the conditions on a and b for the equation to have two distinct non-zero solutions for θ. **3**

(ii) Suppose now that the equation has two distinct non-zero solutions, $\theta = \alpha$ and $\theta = \beta$.

(α) Prove that if $b \neq 1$, then $\tan(\alpha + \beta) = \dfrac{a}{b-1}$. **1**

(β) What is the situation when $b = 1$? **1**

MLS

Solutions – 2004 3U Form 6 Ass. March.

(a) $\frac{\pi}{3}$ ✓

(b) (i) $\frac{dy}{dx} = \frac{3}{3x+1}$ ✓

(ii) $\frac{dy}{dx} = \frac{1}{\sqrt{4-x^2}}$ or $\frac{1}{2\sqrt{1-(\frac{x}{2})^2}}$ ✓

(c) when $x=0$, $y=3$ ✓

$\frac{dy}{dx} = e^x$ and $\frac{dy}{dx} = 1$ when $x=0$ ✓

The tangent is $y-3=x$ or $y=x+3$ ✓

(d) (i) $\int \cos(2-5x)\,dx = -\frac{1}{5}\sin(2-5x) + c$ ✓

(ii) $\int \sec^2 3x\,dx = \frac{1}{3}\tan 3x + c$ ✓

(e) $\lim_{x\to 0} \frac{\sin 2x}{3x} = \lim_{x\to 0} \frac{2}{3}\frac{\sin 2x}{2x}$ ✓

$= \frac{2}{3}\lim_{x\to 0}\frac{\sin 2x}{2x}$

$= \frac{2}{3}$ ✓

(c) Area $= 2\int_0^3 \frac{6}{x^2+9}\,dx$ ✓

$= \frac{12}{3}\left[\tan^{-1}\frac{x}{3}\right]_0^3$ ✓

$= 4\left[\tan^{-1}1 - \tan^{-1}0\right]$

$= 4 \times \frac{\pi}{4}$

$= \pi\ u^2$ ✓

(d) (i) $-1 \le 3x+1 \le 1$

$-2 \le 3x \le 0$

$-\frac{2}{3} \le x \le 0$

Domain is $-\frac{2}{3} \le x \le 0$ ✓

$-\frac{\pi}{2} \le \frac{y}{2} \le \frac{\pi}{2}$

$-\pi \le y \le \pi$

Range is $-\pi \le y \le \pi$ ✓

(ii) $-\pi \le x \le \pi$ ✓

(a) (i) $\tan\alpha = \frac{1-\frac{1}{2}}{1+\frac{1}{2}}$

$= \frac{\frac{1}{2}}{\frac{3}{2}}$

$= \frac{1}{3}$ ✓

(ii) So, $\alpha \doteqdot 18°$ ✓

(b) (i) $\cos 2\theta = 1 - 2\sin^2\theta$ ✓

(ii) Let $\theta = \sin^{-1}\frac{1}{3}$

Then $\sin\theta = \frac{1}{3}$

And $\cos 2\theta = 1 - 2(\frac{1}{3})^2 = \frac{7}{9}$ ✓

Q3. (a) $LHS = \frac{2\tan\theta}{1+\tan^2\theta}$

$= \frac{2\tan\theta}{\sec^2\theta}$

$= \frac{2\sin\theta}{\cos\theta}\cdot\frac{\cos^2\theta}{1}$ ✓

$= 2\sin\theta\cos\theta$

$= \sin 2\theta$ ✓

$= RHS$.

or use 't' results.

b (i) $y = x\sin^{-1}x + (1-x^2)^{\frac{1}{2}}$

$\frac{dy}{dx} = x\frac{1}{\sqrt{1-x^2}} + \sin^{-1}x + \frac{1}{2}\frac{-2x}{\sqrt{1-x^2}}$

$= \sin^{-1}x$

(ii) $\int_0^1 \sin^{-1}x\,dx = \left[x\sin^{-1}x + \sqrt{1-x^2}\right]_0^1$

$= (1\sin^{-1}1 + 0) - (0 + \sqrt{1})$

$= \frac{\pi}{2} - 1$

(c) (i) $y = e^{kx}$

$\frac{dy}{dx} = ke^{kx}$

$\frac{d^2y}{dx^2} = k^2e^{kx}$

(ii) $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0$

$k^2e^{kx} + 7ke^{kx} + 12e^{kx} = 0$

$e^{kx}(k^2 + 7k + 12) = 0, \quad e^{kx} > 0$

$k^2 + 7k + 12 = 0$

$(k+4)(k+3) = 0$

$k = -4$ or -3

Q4. (a) (i) $\cos 2\theta = 1 - 2\sin^2\theta$

$= 2\cos^2\theta - 1$

$1 - \cos 2\theta = 2\sin^2\theta$

$1 + \cos 2\theta = 2\cos^2\theta$

$LHS = \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}$

$= \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}$

$= \sqrt{\tan^2\theta}$

$= \tan\theta$, since $0 \le \theta \le \frac{\pi}{2}$.

$= RHS$

OR let $t = \tan\theta$

$LHS = \sqrt{\frac{1 - \frac{1-t^2}{1+t^2}}{1 + \frac{1-t^2}{1+t^2}}}$

$= \sqrt{\frac{1+t^2-1+t^2}{1+t^2+1-t^2}}$

$= \sqrt{\frac{2t^2}{2}}$

$= \sqrt{t^2}$

$= t$, since $0 \le \theta \le \frac{\pi}{2}$

$= \tan\theta$

$= RHS$

(ii) $\tan\frac{\pi}{8} = \sqrt{\frac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}}$

$= \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}}$

$= \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}}$

$= \sqrt{(\sqrt{2}-1)^2}$

$= \sqrt{2} - 1$

(b) $\sin 2x = \cos x$

$2\sin x\cos x = \cos x$

$2\sin x\cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$ or $2\sin x - 1 = 0$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{2} + n\pi$ or $x = \frac{\pi}{6} + 2n\pi$ or $\frac{5\pi}{6} + 2n\pi$

where n is an integer.

OR
$$\sin 2x = \cos x$$
$$\sin 2x = \sin\left(\tfrac{\pi}{2} - x\right)$$
$$2x = \left(\tfrac{\pi}{2} - x\right) + 2n\pi \quad \text{or} \quad 2x = \pi - \left(\tfrac{\pi}{2} - x\right) + 2n\pi$$
$$3x = \tfrac{\pi}{2} + 2n\pi \qquad 2x = \tfrac{\pi}{2} + x + 2n\pi$$
$$x = \tfrac{\pi}{6} + \tfrac{2}{3}n\pi \qquad x = \tfrac{\pi}{2} + 2n\pi$$

where n is an integer.

(c) $\sin\theta = \frac{2t}{1+t^2}$ and $\cos\theta = \frac{1-t^2}{1+t^2}$

$$LHS = \frac{1 + \frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}}$$
$$= \frac{1+t^2 + 2t - 1 + t^2}{1+t^2 + 2t + 1 - t^2}$$
$$= \frac{2t^2 + 2t}{2 + 2t}$$
$$= \frac{2t(t+1)}{2(t+1)}$$
$$= t$$
$$= \tan\frac{\theta}{2}$$

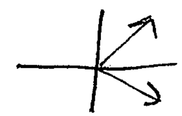
Q5. (a) Let $y = \log_8 2$

Then $8^y = 2$
$$2^{3y} = 2^1$$
$$y = \tfrac{1}{3}$$
Then $\frac{1}{3} = \log_x 5$
$$x^{\frac{1}{3}} = 5$$
$$x = 5^3$$
$$= 125$$

(b) (i) $R\cos(\theta - \alpha) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$
$$= \sqrt{3}\cos\theta + \sin\theta$$
So, $R\cos\alpha = \sqrt{3}$ and $R\sin\alpha = 1$

Giving $R = 2$ and $\tan\alpha = \frac{1}{\sqrt{3}}$
$$\alpha = \frac{\pi}{6}$$

(ii) Solving $2\cos\left(\theta - \frac{\pi}{6}\right) = 1$
$$\cos\left(\theta - \tfrac{\pi}{6}\right) = \tfrac{1}{2}$$



related angle is $\frac{\pi}{3}$
$$\theta - \tfrac{\pi}{6} = \tfrac{\pi}{3} \quad \text{or} \quad \tfrac{5\pi}{3}$$
$$\theta = \tfrac{\pi}{2} \quad \text{or} \quad \tfrac{11\pi}{6}$$

(c) (i) When $n = 10$, $P = 50\%$

So
$$\tfrac{1}{2} = 1 - e^{10k}$$
$$e^{10k} = \tfrac{1}{2}$$
$$k = \tfrac{1}{10}\log_e \tfrac{1}{2}$$
$$= -\tfrac{1}{10}\log_e 2$$

(ii) We want $P \geq 90\%$

So solve $1 - e^{kn} \geq \frac{9}{10}$
$$e^{kn} \leq \tfrac{1}{10}$$
$$kn \leq \ln\tfrac{1}{10}$$

$n \geq \frac{1}{k} \ln \frac{1}{10}$

$n \geq 33.2$

The company should advertise 34 times

Q6.

a) (i) $f(x) = \frac{\log_e x}{x}$, $x > 0$

$f'(x) = \frac{x \cdot \frac{1}{x} - \log_e x}{x^2}$

$= \frac{1 - \log x}{x^2}$

at a stationary point $1 - \log x = 0$

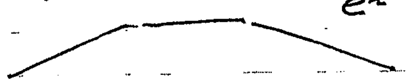
$x = e$

and $y = \frac{1}{e}$

So $(e, \frac{1}{e})$ is a stationary point

Now we look at the gradient of $f(x)$ on either side of $(e, \frac{1}{e})$ noting there are no discontinuities for $x > 0$.

x	1	e	e^2
$\frac{dy}{dx}$	$\frac{1 - \ln 1}{1} = 1$	0	$\frac{1 - 2\ln e}{e^2} = -\frac{1}{e^2} < 0$



So we have a maximum point at $(e, \frac{1}{e})$

ii) Since $f(x)$ is a maximum at $(e, \frac{1}{e})$ then $f(e) > f(\pi)$

That is $\frac{\log e}{e} > \frac{\log \pi}{\pi}$

$\frac{1}{e} > \frac{\log \pi}{\pi}$

$\frac{\pi}{e} > \log \pi$

$e^{\frac{\pi}{e}} > \pi$

$e^{\pi} > \pi^{e}$

(b) (i) $\tan 2\theta = \frac{\tan\theta}{b + a\tan\theta}$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

and $a > 0$, $b > 0$

So, $\frac{2\tan\theta}{1 - \tan^2\theta} = \frac{2\tan\theta}{2b + 2a\tan\theta}$

* Note, $\tan\theta \neq 1$ or -1
and $\tan\theta \neq 0$ since we are looking for non zero solutions.

So, $1 - \tan^2\theta = 2b + 2a\tan\theta$

$\tan^2\theta + 2a\tan\theta + 2b - 1 = 0$

for two distinct solutions, $\Delta > 0$

$4a^2 - 4(2b - 1) > 0$

$a^2 > 2b - 1$

each solution gives exactly one value for θ since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

* But there are problems!

Firstly, if $\tan\theta = 1$, which we know is impossible then $1 = \frac{-2a - \sqrt{4(a^2-2b+1)}}{2}$

$= -a - \sqrt{a^2-2b+1}$, which is impossible since $a > 0$

or $1 = \frac{-2a + \sqrt{4(a^2-2b+1)}}{2}$

$= -a + \sqrt{a^2-2b+1}$

$(a+1)^2 = a^2 - 2b + 1$

$a^2 + 2a + 1 = a^2 - 2b + 1$

$a = -b$, which is impossible since $a > 0$ and $b > 0$

Secondly, if $\tan\theta = -1$, which we know is impossible then $-1 = -a - \sqrt{a^2-2b+1}$

$(a-1)^2 = a^2 - 2b + 1$

$a^2 - 2a + 1 = a^2 - 2b + 1$

$a = b$

or $-1 = -a + \sqrt{a^2-2b+1}$

$(a-1)^2 = a^2 - 2b + 1$

and $a = b$ again

So, we have the further condition $a \neq b$

Thirdly, if $\tan\theta = 0$, which is impossible

$0 = -a \pm \sqrt{a^2-2b+1}$

$a^2 = a^2 - 2b + 1$

$b = \frac{1}{2}$

So, we have a further condition $b \neq \frac{1}{2}$

(ii) (α) $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \frac{\text{sum of roots}}{1 - \text{product of roots}}$

$= \frac{-2a}{1-(2b-1)}$

$= \frac{-2a}{2b+2}$

$= \frac{a}{b-1}$

(ii) (β) If $b = 1$, then $\tan(\alpha+\beta)$ is undefined, so $\alpha + \beta = n\frac{\pi}{2}$, where n is an odd integer.

Now, $\tan\theta = -a \pm \sqrt{a^2-2b+1}$

$= -a \pm \sqrt{a^2-1}$ when $b = 1$

so, $\tan\alpha = -a - \sqrt{a^2-1}$ or $\tan\beta = -a + \sqrt{a^2-1}$

are these roots positive or negative?

product of roots $= a^2 - (a^2-1) = 1$, positive

sum of roots $= -2a$, negative since $a > 0$

So $(\alpha+\beta)$ is $-\frac{\pi}{2}$