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Sydney Grammar School
Mathematics Department
Half-Yearly Examinations 2006

FORM VI

MATHEMATICS

Examination date

Monday 15th May 2006

Time allowed

2 hours

Instructions

All seven questions may be attempted.
All seven questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection

Write your candidate number clearly on each booklet.
Hand in the seven questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.

Checklist

SGS booklets: 7 per boy. A total of 1000 booklets should be sufficient.
Candidature: 105 boys.

Examiner

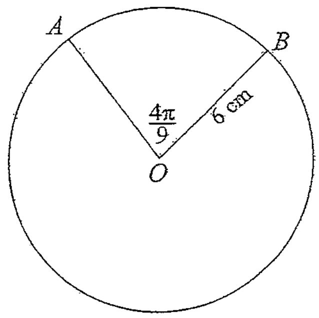
REN

QUESTION ONE (16 marks) Use a separate writing booklet. **Marks**

(a) (i) Express 45° in radians. 1

(ii) Find the exact value of $\cos \frac{5\pi}{6}$. 2

(b) 2



In the diagram above, find the area of sector AOB.

(c) Solve $\sin x = \frac{1}{\sqrt{2}}$ for $0 \le x \le 2\pi$. 2

(d) Simplify the following:

(i) $\ln e$ 1

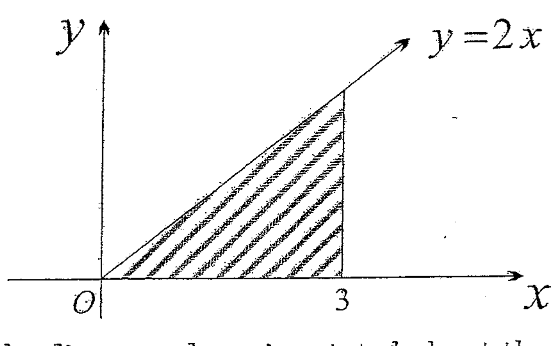
(ii) $\ln \sqrt{e}$ 1

(e) Simplify $\log_{12} 18 + \log_{12} 8$. 2

(f) Use the change of base law to calculate $\log_2 94$ correct to one decimal place. 1

(g) Find $\int (x^4 + 1)\, dx$. 1

(h) 3



The shaded region in the diagram above is rotated about the x-axis to form a solid. Find the volume of the solid.

QUESTION TWO (16 marks) Use a separate writing booklet. **Marks**

(a) Differentiate the following:

(i) $\log_e(4x-1)$ **1**

(ii) $\log_e(x+2)^6$ **1**

(iii) $\dfrac{x^2}{\log_e x}$ **2**

(b) Find the equation of the tangent to the curve $y = \log_e(x-1)$ at the point $(2, 0)$. **3**

(c) Find the following:

(i) $\displaystyle\int \frac{4}{x}\,dx$ **1**

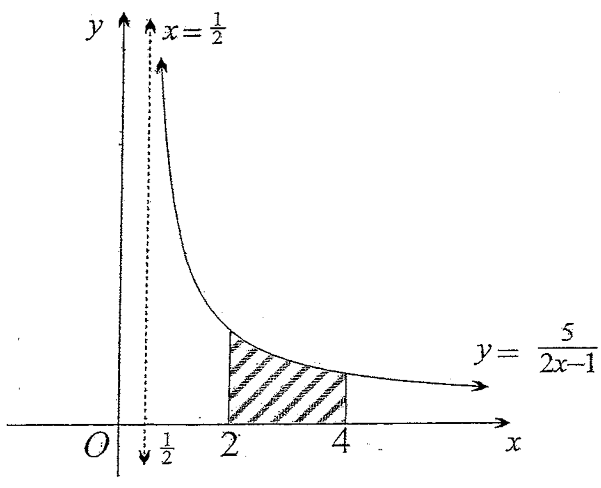
(ii) $\displaystyle\int \frac{1}{1-3x}\,dx$ **1**

(d) Evaluate the following:

(i) $\displaystyle\int_1^2 \frac{1}{2x}\,dx$ **2**

(ii) $\displaystyle\int_1^3 \frac{x^2-2x}{x^2}\,dx$ **2**

(e) **3**



Find the area of the shaded region in the diagram above.

QUESTION THREE (16 marks) Use a separate writing booklet. **Marks**

(a) Consider the function $f(x) = x^4 + 2$.

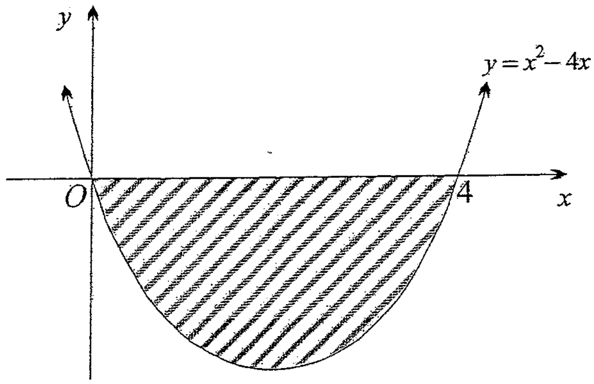
(i) Show that $f''(x) = 0$ at the point $(0, 2)$. **2**

(ii) Show that the point $(0, 2)$ is not a point of inflexion. **2**

(b) Find the value(s) of k for which the function $y = x^2 - 2kx + 1$ is increasing at $x = 3$. **2**

(c) Evaluate the following: $\displaystyle\int_1^2 \frac{1}{x^2}\,dx$. **2**

(d) **3**



Find the area of the shaded region in the diagram above.

(e) Differentiate the following:

(i) e^{x^2} **1**

(ii) $e^x \ln x$ **1**

(f) Find $\displaystyle\int (e^{2x} - e^{-3x})\,dx$. **1**

(g) Evaluate $\displaystyle\int_0^1 e^{2x+1}\,dx$. **2**

QUESTION FOUR (16 marks) Use a separate writing booklet. **Marks**

(a) Differentiate the following:

(i) $\cos 5x$ [1]

(ii) $\tan(x^2 + 1)$ [1]

(iii) $x^2 \sin x$ [2]

(b) Find the equation of the normal to the curve $y = \sin 2x + \cos 2x$ at the point $(\frac{\pi}{4}, 1)$. [4]

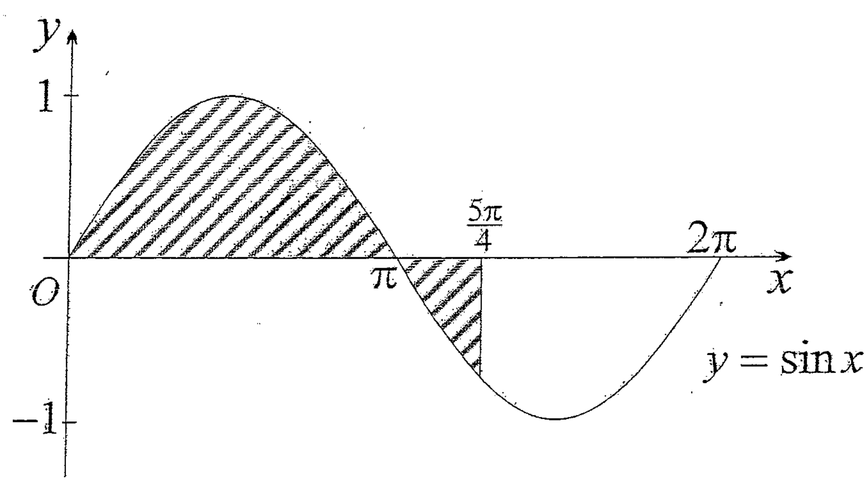
(c) Find the following:

(i) $\int \sec^2(2 - 3x)\, dx$ [1]

(ii) $\int (3 \sin x - 4 \sin 4x)\, dx$ [1]

(d) Evaluate $\int_0^{\frac{\pi}{3}} 3 \cos \frac{1}{2}x\, dx$. [2]

(e) [4]

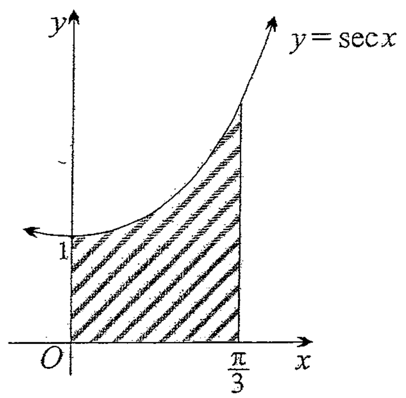


Find the exact area of the shaded region in the diagram above.

QUESTION FIVE (16 marks) Use a separate writing booklet. **Marks**

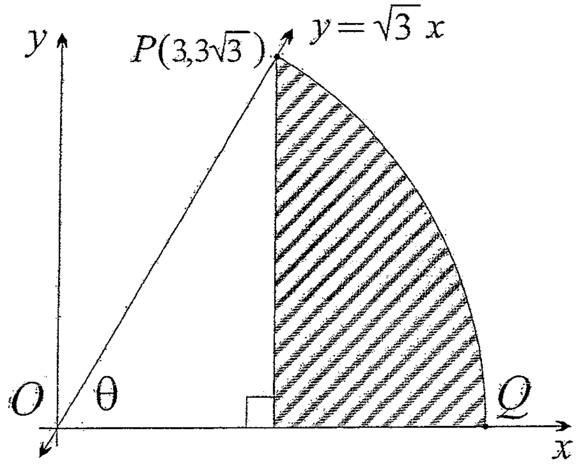
(a) Sketch the graph of $y = 3\sin 2x$ for $0 \le x \le 2\pi$. **2**

(b) **3**



The shaded region shown in the diagram above is rotated about the x-axis to form a solid. Find the volume of the solid.

(c)



In the diagram above, the point $P(3, 3\sqrt{3})$ lies on the line $y = \sqrt{3}x$. With centre O and radius OP the arc PQ is drawn, cutting the x-axis at Q. Let $\angle POQ = \theta$.

(i) Show that $\theta = \frac{\pi}{3}$. **1**

(ii) Find the coordinates of Q. **2**

(iii) Find the shaded area. **3**

(d) The gradient of a curve is given by $\frac{dy}{dx} = \frac{1}{4x-3}$.

(i) If the curve passes through the point $(1, -2)$, find the equation of the curve. **2**

(ii) Find the value of y when $x = 2$. **1**

(iii) Show that the curve is concave down for all values of x in its domain. **2**

Exam continues next page ...

QUESTION SIX (16 marks) Use a separate writing booklet. **Marks**

(a) Consider the function $f(x) = xe^{2x}$.

(i) Find the x-intercept. **1**

(ii) Show that $f'(x) = e^{2x}(2x+1)$. **1**

(iii) Show that there is a stationary point at $(-\frac{1}{2}, -\frac{1}{2e})$ and determine its nature. You may use the fact that $f''(x) = 4e^{2x}(x+1)$. **3**

(iv) Find the point of inflexion. **2**

(v) Describe the behaviour of y as $x \to -\infty$. **1**

(vi) Sketch the graph of $y = xe^{2x}$, clearly labelling the stationary point and point of inflexion. **3**

(b) The portion of the curve $x = e^{\frac{1}{y}}$ from $x = 2$ to $x = 5$ is rotated about the x-axis to form a solid.

(i) Show that the volume V of the solid is given by **2**

$$V = \pi \int_2^5 \frac{1}{(\ln x)^2}\, dx.$$

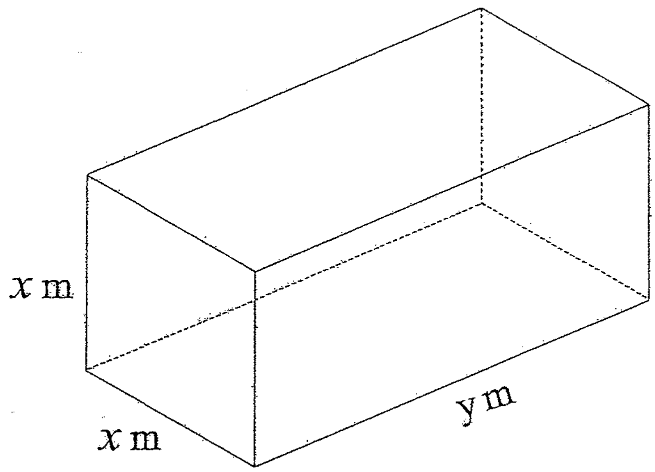
(ii) Use the trapezoidal rule with four function values to find an approximation for V, correct to one decimal place. **3**

Exam continues overleaf ...

QUESTION SEVEN (16 marks) Use a separate writing booklet. **Marks**

(a) Evaluate $\displaystyle\int_{\pi}^{\frac{4\pi}{3}} \frac{\sin x}{1-\cos x}\,dx$. **3**

(b)



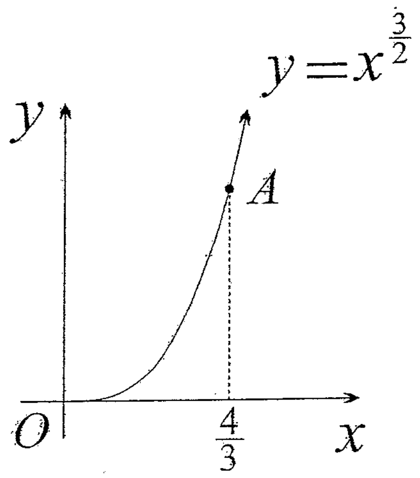
The diagram above shows a large rectangular container with no lid. The container has square ends of side x metres and a length of y metres. The volume of the container is 36 cubic metres.

(i) If S square metres is the surface area of the container, show that $S = 2x^2 + \dfrac{108}{x}$. **3**

(ii) Find the values of x and y for which S is a minimum. **4**

(iii) Find the minimum value of S. **1**

(c)



The diagram above shows the curve $y = x^{\frac{3}{2}}$. At the point A, $x = \frac{4}{3}$.

(i) Write down the derivative of $x^{\frac{3}{2}}$. **1**

(ii) The length L of the arc of the curve $y = f(x)$ from $x = a$ to $x = b$ is given by the formula **4**

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

Find the length of the arc OA.

END OF EXAMINATION

SOLUTIONS FOR MATHEMATICS HALF-YEARLY

Question 1

(a) (i) $45^\circ = \frac{\pi}{4}$ ✓

(ii) $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$ ✓✓

(b) $\text{Area} = \frac{1}{2}\times 6^2 \times \frac{4\pi}{9}$ ✓

$= 8\pi \text{ cm}^2$ ✓

(c) related angle $= \frac{\pi}{4}$ ✓

$x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ ✓

(d) (i) $\ln e = 1$ ✓

(ii) $\ln\sqrt{e} = \frac{1}{2}\ln e$ ✓

$= \frac{1}{2}$

(e) $\log_{12} 18 + \log_{12} 8 = \log_{12} 144$ ✓

$= 2$ ✓

(f) $\log_2 94 = \frac{\ln 94}{\ln 2}$

$\doteqdot 6.6$ ✓

(g) $\int (x^4+1)\,dx = \frac{x^5}{5} + x + c$ ✓

(h) $\text{Volume} = \pi\int_0^3 4x^2\,dx$ ✓

$= \frac{4\pi}{3}\left[x^3\right]_0^3$ ✓

$= \frac{4\pi}{3}\left[3^3 - 0\right]$

$= 36\pi \text{ units}^3$ ✓

Question 2

(a)

(i) $\frac{d}{dx}(\log_e(4x-1)) = \frac{4}{4x-1}$ ✓

(ii) $\frac{d}{dx}(6\log(x+2)) = \frac{6}{x+2}$ ✓

(iii) $u = x^2 \qquad v = \log_e x$

$\therefore u' = 2x \qquad v' = \frac{1}{x}$ ✓

$$y' = \frac{2x \times \log_e x - x^2 \times \frac{1}{x}}{(\log_e x)^2}$$ ✓

$$= \frac{2x\log_e x - x}{(\log_e x)^2}$$ ✓

(b) $y' = \frac{1}{x-1}$ ✓

At $(2,0)$, $y' = \frac{1}{2-1} = 1$ ✓

Eqn of tangent: $y - 0 = 1(x-2)$ ✓

$y = x - 2$

(c) (i) $\int \frac{4}{x}\,dx = 4\ln x + c$ ✓

(ii) $\int \frac{1}{1-3x}\,dx = -\frac{1}{3}\ln(1-3x)+c$ ✓

(d) (i) $\int_1^2 \frac{1}{2x}\,dx = \frac{1}{2}\left[\ln x\right]_1^2$ ✓

$= \frac{1}{2}\ln 2$ ✓

(ii) $\int_1^3 \frac{x^2-2x}{x^2}\,dx$

$= \int_1^3 \left(1 - \frac{2}{x}\right)dx$

$= \left[x - 2\ln x\right]_1^3$ ✓

$= 3 - 2\ln 3 - (1 - 2\ln 1)$

$= 3 - 2\ln 3 - 1$ ✓

$= 2 - 2\ln 3$

(e)

$\text{Area} = \int_2^4 \frac{5}{2x-1}\,dx$ ✓

$= \frac{5}{2}\left[\ln(2x-1)\right]_2^4$ ✓

$= \frac{5}{2}\left[\ln 7 - \ln 3\right]$

$= \frac{5}{2}\ln\left(\frac{7}{3}\right)$ ✓

– [Last step required for full marks]

Question 3

(a) (i) $f'(x) = 4x^3$

$f''(x) = 12x^2$ ✓

At $(0,2)$, $f''(0) = 12 \times 0^2 = 0$ ✓

(ii)

x	-1	0	1
$f''(x)$	12	0	12

✓ Since sign of $f''(x)$ doesn't change, $(0,2)$ not a point of inflexion. ✓

(b) $y' = 2x - 2k$ ✓

when $x = 3$, $6 - 2k > 0$

$6 > 2k$

$k < 3$ ✓

(c) $\int_1^2 \frac{1}{x^2}\,dx = \int_1^2 x^{-2}\,dx$

$= -\left[\frac{1}{x}\right]_1^2$ ✓

$= -\left[\frac{1}{2} - 1\right]$

$= \frac{1}{2}$ ✓

(d) Area $= \left|\int_0^4 (x^2 - 4x)\,dx\right|$ ✓

$= \left|\left[\frac{x^3}{3} - 2x^2\right]_0^4\right|$ ✓

$= \left|\frac{4^3}{3} - 2\times 4^2\right|$

$= 10\frac{2}{3}$ units^2 ✓

(e) (i) $\frac{d}{dx}(e^{x^2}) = 2xe^{x^2}$ ✓

(ii) $\frac{d}{dx}(e^x \ln x) = e^x \ln x + e^x \times \frac{1}{x}$ ✓

$= e^x\left(\ln x + \frac{1}{x}\right)$

(f) $\int e^{2x} - e^{-3x}\,dx = \frac{1}{2}e^{2x} + \frac{1}{3}e^{-3x} + c$ ✓

(g) $\int_0^1 e^{2x+1}\,dx = \frac{1}{2}\left[e^{2x+1}\right]_0^1$ ✓

$= \frac{1}{2}\left[e^3 - e^1\right]$

$= \frac{1}{2}(e^3 - e)$ ✓

Question 4

(a) (i) $\frac{d}{dx}(\cos 5x) = -5\sin 5x$ ✓

(ii) $\frac{d}{dx}(\tan(x^2+1)) = 2x\sec^2(x^2+1)$ ✓

(iii) $\frac{d}{dx}(x^2 \sin x) = 2x\sin x + x^2\cos x$ ✓

$= x(2\sin x + x\cos x)$ ✓

(b) $y' = 2\cos 2x - 2\sin 2x$ ✓

At $(\frac{\pi}{4}, 1)$, $y' = 2\cos\frac{\pi}{2} - 2\sin\frac{\pi}{2} = -2$ ✓

grad of normal $= \frac{1}{2}$ ✓

Eqn of normal: $y - 1 = \frac{1}{2}(x - \frac{\pi}{4})$ ✓

$y = \frac{1}{2}x - \frac{\pi}{8} + 1$ ✓

(c) (i) $\int \sec^2(2-3x)\,dx = -\frac{1}{3}\tan(2-3x) + c$ ✓

(ii) $\int (3\sin x - 4\sin 4x)\,dx = -3\cos x + \cos 4x + c$ ✓

(d) $\int_0^{\frac{\pi}{3}} 3\cos\frac{1}{2}x\,dx = 6\left[\sin\frac{1}{2}x\right]_0^{\frac{\pi}{3}}$ ✓

$= 6\left[\sin\frac{\pi}{6} - 0\right]$

$= 3.$ ✓

(e) Area $= \int_0^{\pi} \sin x\,dx + \left|\int_{\pi}^{\frac{5\pi}{4}} \sin x\,dx\right|$ ✓

$= -\left[\cos x\right]_0^{\pi} + \left|-\left[\cos x\right]_{\pi}^{\frac{5\pi}{4}}\right|$ ✓

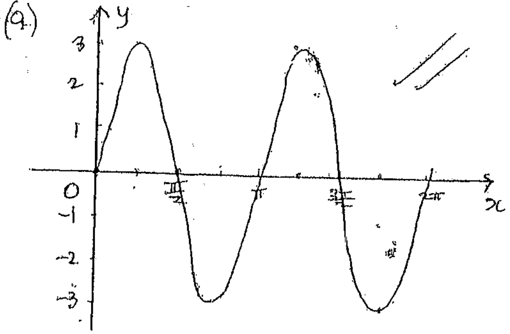
$= -(\cos\pi - \cos 0) + \left|-(\cos\frac{5\pi}{4} - \cos\pi)\right|$ ✓

$= 2 + \left|\frac{1}{\sqrt{2}} - 1\right|$

$= \left(3 - \frac{\sqrt{2}}{2}\right)$ units^2 ✓

Question 5

(a)



(b) $\text{Volume} = \pi\int_0^{\frac{\pi}{3}} \sec^2 x \, dx$

$= \pi\left[\tan x\right]_0^{\frac{\pi}{3}}$

$= \pi\left[\tan\frac{\pi}{3} - 0\right]$

$= \pi\sqrt{3}$ units^3

(c) (i) $\tan\theta = \frac{3\sqrt{3}}{3}$

$= \sqrt{3}$

$\theta = \frac{\pi}{3}$

(ii) $OP^2 = 3^2 + (3\sqrt{3})^2$

$= 36$

$OP = 6$

$OQ = 6$

Coords of Q : $(6, 0)$

(iii) Shaded area $= \frac{1}{2}\times 6^2 \times \frac{\pi}{3} - \frac{3\times 3\sqrt{3}}{2}$

$= \left(6\pi - \frac{9\sqrt{3}}{2}\right)$ units^2

(d) (i) $y = \int \frac{1}{4x-3}\, dx$

$y = \frac{1}{4}\log(4x-3) + c$

when $x = 1$, $y = -2$

$-2 = \frac{1}{4}\log 1 + c$

$c = -2$

$y = \frac{1}{4}\log(4x-3) - 2$

(ii) when $x = 2$,

$y = \frac{1}{4}\log 5 - 2$

(iii) $\frac{dy}{dx} = (4x-3)^{-1}$

$\frac{d^2y}{dx^2} = -(4x-3)^{-2}\times 4$

$= -\frac{4}{(4x-3)^2}$

Since $(4x-3)^2 > 0$ for all x $(x \neq \frac{3}{4})$, $-\frac{4}{(4x-3)^2} < 0$ for all x in domain.

Curve always concave down $\left(\frac{d^2y}{dx^2} < 0\right)$.

Question 6

(a) (i) $f(0) = 0 \times e^{2\times 0} = 0$

x intercept : $(0, 0)$

(ii) $f'(x) = x \times 2e^{2x} + 1 \times e^{2x}$

$= e^{2x}(2x+1)$

(iii) $f'(-\frac{1}{2}) = e^{-1}\times 0 = 0$

$f'(x) = 0$ at $x = -\frac{1}{2}$

$f(-\frac{1}{2}) = -\frac{1}{2}\times e^{-1} = -\frac{1}{2e}$

Stationary point at $\left(-\frac{1}{2}, -\frac{1}{2e}\right)$

$f''(-\frac{1}{2}) = 4e^{-1}(-\frac{1}{2}+1)$

$= \frac{4}{e}\times\frac{1}{2} > 0$

$\left(-\frac{1}{2}, -\frac{1}{2e}\right)$ is a relative minimum

(iv) $4e^{2x}(x+1) = 0$

$x = -1$

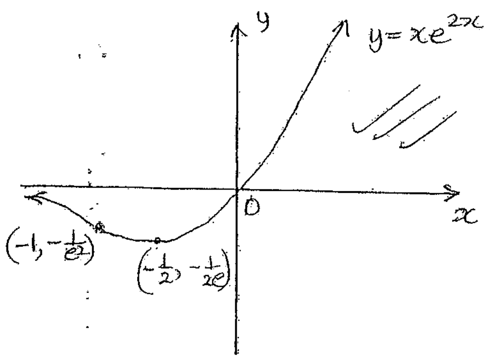
$f(-1) = -1\times e^{-2} = -\frac{1}{e^2}$

$f''(x) = 0$ at $\left(-1, -\frac{1}{e^2}\right)$

x	-2	-1	$-\frac{1}{2}$
$f''(x)$	$-\frac{4}{e^4}$	0	$\frac{4}{2e}$

$f''(x)$ changes sign, so $\left(-1, -\frac{1}{e^2}\right)$ a pt. of inflexion.

(v) As $x \to -\infty$, $xe^{2x} \to 0$



(b) (i) $x = e^{\frac{1}{y}}$

$\frac{1}{y} = \ln x$

$y = \frac{1}{\ln x}$

$\text{Volume} = \pi\int_2^5 y^2\, dx$

$= \pi\int_2^5 \frac{1}{(\ln x)^2}\, dx$

(ii)

x	2	3	4	5
$f(x)$	$\frac{1}{(\ln 2)^2}$	$\frac{1}{(\ln 3)^2}$	$\frac{1}{(\ln 4)^2}$	$\frac{1}{(\ln 5)^2}$

$\pi\int_2^5 \frac{1}{(\ln x)^2}\, dx$

$\doteqdot \frac{\pi}{2}\left[f(2) + 2f(3) + 2f(4) + f(5)\right]$

$= \frac{\pi}{2}\left[\frac{1}{(\ln 2)^2} + \frac{2}{(\ln 3)^2} + \frac{2}{(\ln 4)^2} + \frac{1}{(\ln 5)^2}\right]$

$\doteqdot 8.1$

Volume $\doteqdot 8.1$ units^3

Question 7

(a) $\int_{\pi}^{\frac{4\pi}{3}} \frac{\sin x\,dx}{1-\cos x} = \left[\ln(1-\cos x)\right]_{\pi}^{\frac{4\pi}{3}}$

$= \ln\left(1-\cos\frac{4\pi}{3}\right) - \ln(1-\cos\pi)$

$= \ln\frac{3}{2} - \ln 2$

$= \ln\left(\frac{3}{4}\right)$

(b) (i) $S = 2x^2 + 3xy$

Also, $x^2y = 36$

$y = \frac{36}{x^2}$

$S = 2x^2 + 3x \times \frac{36}{x^2}$

$S = 2x^2 + \frac{108}{x}$

(ii) $\frac{ds}{dx} = 4x - \frac{108}{x^2}$

At min, $4x - \frac{108}{x^2} = 0$

$4x = \frac{108}{x^2}$

$4x^3 = 108$

$x^3 = 27$

$x = 3$

when $x = 3$, $y = \frac{36}{3^2} = 4$

$\frac{d^2s}{dx^2} = 4 + \frac{216}{x^3}$

when $x=3$, $\frac{d^2s}{dx^2} = 4 + \frac{216}{3^3} > 0$

There is a minimum value

(iii) when $x=3$, $S = 2\times3^2 + \frac{108}{3}$

$= 54$

Minimum value of $S = 54\ m^2$

(c) (i) $y = x^{\frac{3}{2}}$

$y' = \frac{3}{2}x^{\frac{1}{2}}$

(ii) Length of OA

$= \int_0^{\frac{4}{3}} \sqrt{1+\left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\,dx$

$= \int_0^{\frac{4}{3}} \sqrt{1+\frac{9}{4}x}\,dx$

$= \frac{4}{9}\times\frac{2}{3}\left[\left(1+\frac{9}{4}x\right)^{\frac{3}{2}}\right]_0^{\frac{4}{3}}$

$= \frac{8}{27}\left[\left(1+\frac{9}{4}\times\frac{4}{3}\right)^{\frac{3}{2}} - 1\right]$

$= \frac{8}{27}\left(4^{\frac{3}{2}} - 1\right)$

$= \frac{56}{27}$ units