3/4 UNIT MATHEMATICS FORM VI

Time allowed: 2 hours (plus 5 minutes reading) **Exam date:** 15th August, 1997

Instructions:

All questions may be attempted.
All questions are of equal value.
Part marks are shown in boxes in the left margin.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection:

Each question will be collected separately.
Start each question in a new answer booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your candidate number on each answer booklet.

QUESTION ONE (Start a new answer booklet)

Marks

3 (a) Solve the inequality $\dfrac{3x-8}{x} \geq 1$.

2 (b) Solve $\sin 2x = -\frac{1}{2}$, for $0 \leq x \leq 2\pi$.

2 (c) Show that $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan x \, dx = \tfrac{1}{2}\log 3$.

2 (d) Find $\displaystyle\int_0^{0\cdot 4} \frac{3\,dx}{4+25x^2}$.

3 (e) Simplify $\tan\left(2\cos^{-1}\frac{12}{13}\right)$, giving your answer as a fraction reduced to lowest terms.
HINT: Let $\alpha = \cos^{-1}\frac{12}{13}$.

QUESTION TWO (Start a new answer booklet)

Marks

3 (a) Use the substitution $x = 2\sin\theta$ to evaluate $\displaystyle\int_0^1 \sqrt{4-x^2}\, dx$.

9 (b) Consider the function $y = x\sqrt{2-x^2}$, with domain $-\sqrt{2} \leq x \leq \sqrt{2}$.

(i) Show that the function is odd.

(ii) Show that $\dfrac{dy}{dx} = \dfrac{2-2x^2}{\sqrt{2-x^2}}$.

(iii) Find the coordinates of any stationary points and then determine their nature. You may use without proof the fact that $\dfrac{d^2y}{dx^2} = \dfrac{2x^3-6x}{(2-x^2)^{\frac{3}{2}}}$.

(iv) Notice that $\dfrac{dy}{dx} \to -\infty$ as $x \to \left(\sqrt{2}\right)^-$ (that is, as x approaches $\sqrt{2}$ from the left). What does this fact tell you about the shape of the curve near $x = \sqrt{2}$?

(v) Draw a sketch of the curve, showing all x-intercepts, y-intercepts and stationary points.

(vi) Use the substitution $u = 2 - x^2$ to find the area between the curve and the x-axis from $x = 0$ to $x = \sqrt{2}$.

Exam continues next page ...

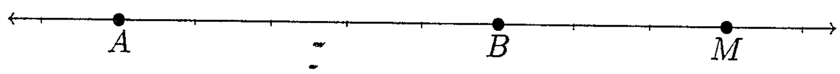
QUESTION THREE (Start a new answer booklet)

Marks

2 (a) Write out the expansion of $\left(3x^3 + \frac{1}{x}\right)^4$, giving the coefficients as integers.

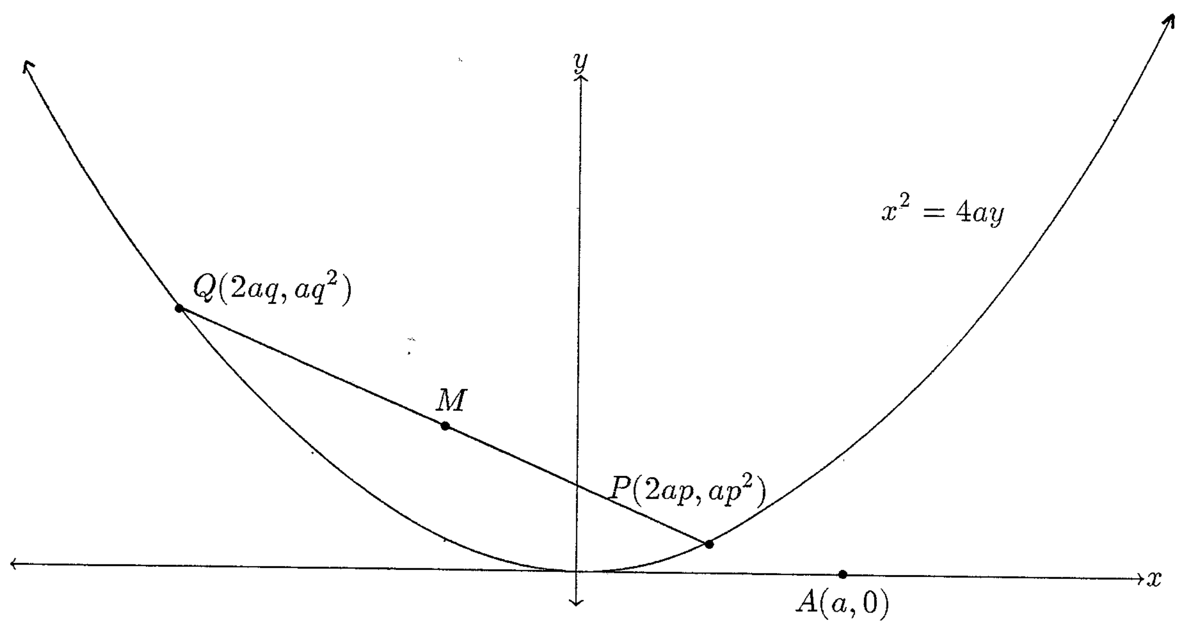
2 (b) Sketch the graph of the function $y = |x - 5| - 2$, showing the coordinates of all significant points, including x-intercepts and y-intercepts.

2 (c)



The point M in the diagram above divides the interval AB externally in the ratio $1 : k$. Find k.

6 (d)



In the diagram above, the points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola with equation $x^2 = 4ay$.

(i) Write down the coordinates of the midpoint M of the chord PQ.

(ii) Show that the equation of the chord PQ is $y = \dfrac{(p+q)x}{2} - apq$.

(iii) Show that the condition for the chord PQ produced to pass through the point $A(a, 0)$ is:

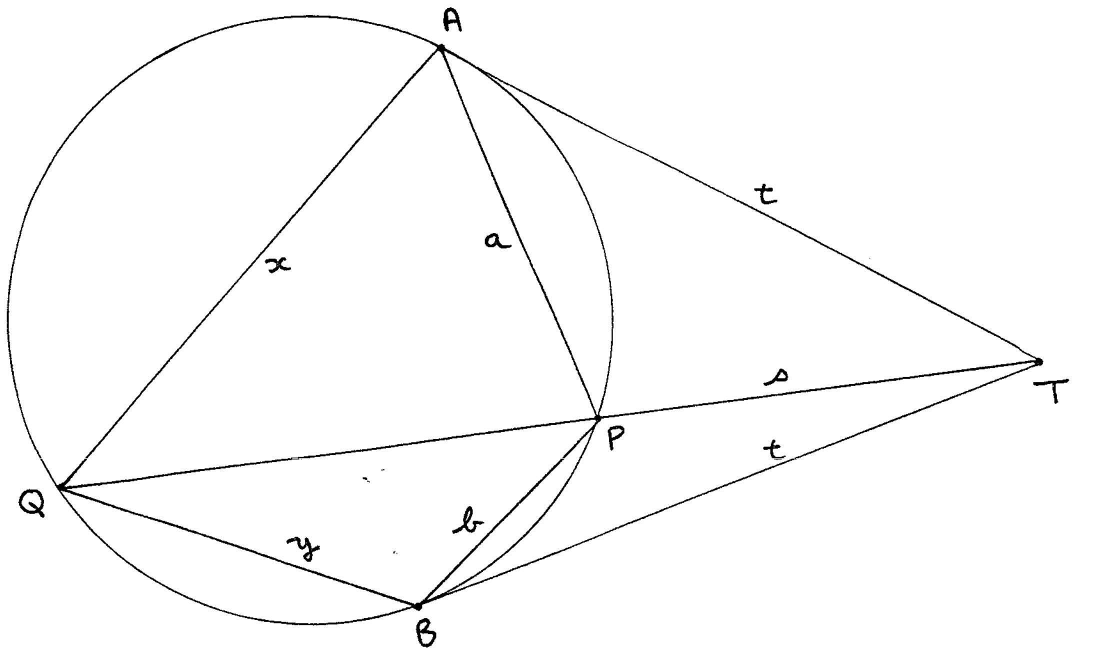
$$p + q = 2pq.$$

(iv) Find the cartesian equation of the locus of M, as the points P and Q move on the parabola subject to the constraint that PQ pass through $A(a, 0)$.

QUESTION FOUR (Start a new answer booklet)

Marks

7 (a)



The diagram above shows two tangents of equal length t drawn from a point T outside a circle, touching the circle at A and B. A secant through T meets the circle first at P and then at Q, where $TP = s$. The sides of the cyclic quadrilateral $APBQ$ are:

$$AP = a, \quad BP = b, \quad AQ = x, \quad BQ = y.$$

(i) Write a careful proof that $\triangle ATP \;|||\; \triangle QTA$.

(ii) Hence express the ratio $\dfrac{a}{x}$ in terms of the lengths s and t.

(iii) Write down the corresponding expression for $\dfrac{b}{y}$, and hence prove that $ay = bx$.

(iv) Prove that $\dfrac{\text{area } \triangle APQ}{\text{area } \triangle BPQ} = \dfrac{a^2}{b^2}$.

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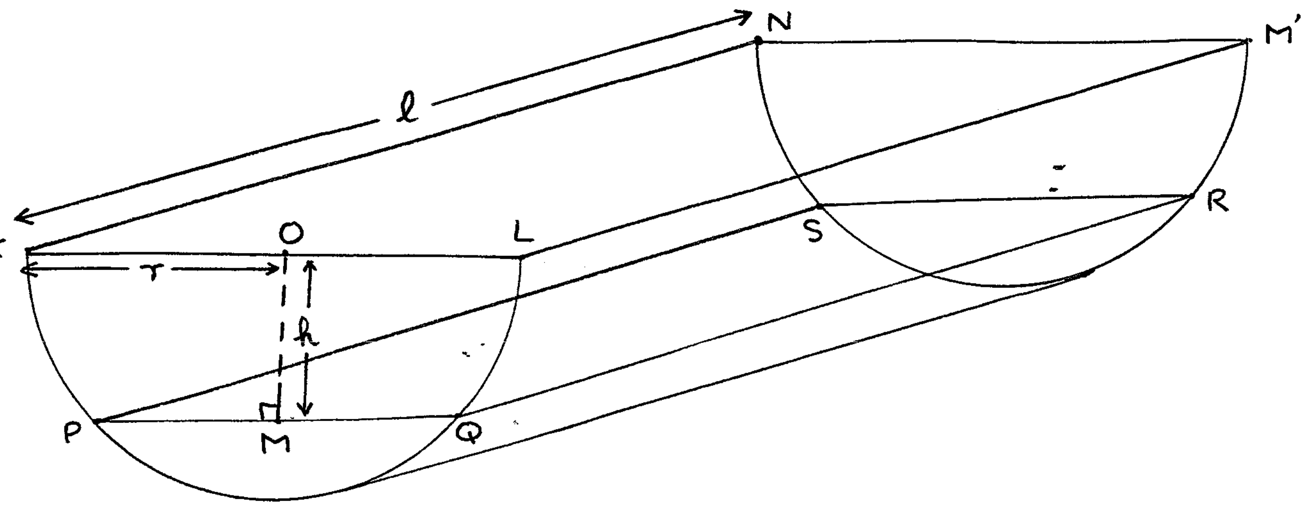
QUESTION FOUR (Continued)

5 (b)



The diagram above shows a horse trough which is in the shape of half a cylinder, with length ℓ and radius r. It is partly filled with water, and the surface $PQRS$ of the water is a distance h below the top $KLM'N$ of the trough. Let M be the midpoint of PQ, and O be the midpoint of KL.

(i) Express the length PM in terms of h and r.

(ii) Hence show that the area A of the surface of the water is given by:

$$A = 2\ell\sqrt{r^2 - h^2}\,.$$

(iii) The water in the trough is evaporating in the hot Summer sun in such a way that the distance h from the top of the trough to the surface of the water is increasing at a constant rate.

The trough measurements are $\ell = 250\,\text{cm}$ and $r = 50\,\text{cm}$, and the surface is descending at a constant 0·3 cm per day. Find the rate at which the surface area is decreasing when the surface of the water is 40 cm below the top of the trough (express your answer in units of cm^2 per day).

QUESTION FIVE (Start a new answer booklet)

Marks

4 (a) (i) Use long division to divide the polynomial $f(x) = x^4 - x^3 + x^2 - x + 1$ by the polynomial $d(x) = x^2 + 4$. Express your answer in the form:

$$f(x) = d(x)\,q(x) + r(x).$$

(ii) Hence find the values of the constants a and b so that $x^4 - x^3 + x^2 + ax + b$ is divisible by $x^2 + 4$.

5 (b) Consider the binomial expansion of $(3 + 11x)^{19}$.

(i) Let T_k be the kth term in the expansion (where the terms are written out in increasing powers of x). Show that:

$$\frac{T_{k+1}}{T_k} = \frac{11x(20-k)}{3k}.$$

(ii) Find the greatest coefficient in the expansion. Express your answer by giving the prime factorisation of this coefficient.

3 (c) Consider the binomial expansion:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n.$$

(i) Use a suitable substitution to find the value of:

$$\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \cdots + 2^n\binom{n}{n}.$$

(ii) Differentiate both sides of the identity, and then use a suitable substitution to find the value of:

$$\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \cdots + (-1)^{n-1}n\binom{n}{n}.$$

Exam continues next page ...

QUESTION SIX (Start a new answer booklet)

Marks

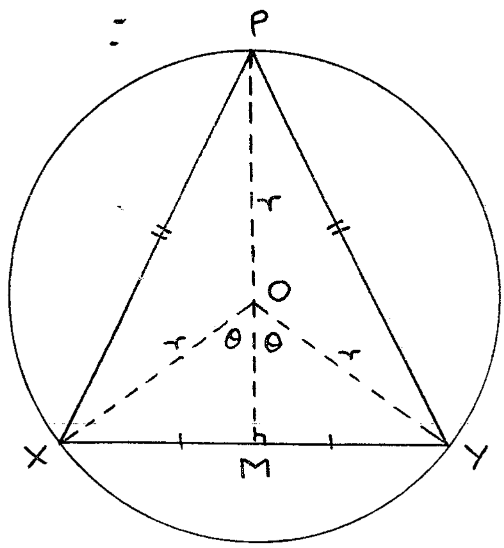
4 (a) The velocity v of a particle moving on the x-axis is given by:

$$v^2 = -3x^2 + 20x + 7.$$

Show that the particle is moving in simple harmonic motion, and find the centre, amplitude and period of that motion.

3 (b) Solve $2\cos^2\theta + \cos\theta = 1$, for $0 \le \theta \le \pi$.

5 (c)



In the diagram above, O is the centre of a circle of constant radius r. A variable chord XY subtends an angle 2θ at the centre O. Let P be the point on the major arc XY so that $\triangle XPY$ is isosceles with $XP = YP$. Let PO meet XY at M, so that PM is the perpendicular bisector of the chord XY.

(i) Prove that the area A of $\triangle XPY$ is given by:

$$A = r^2 \sin\theta(1 + \cos\theta).$$

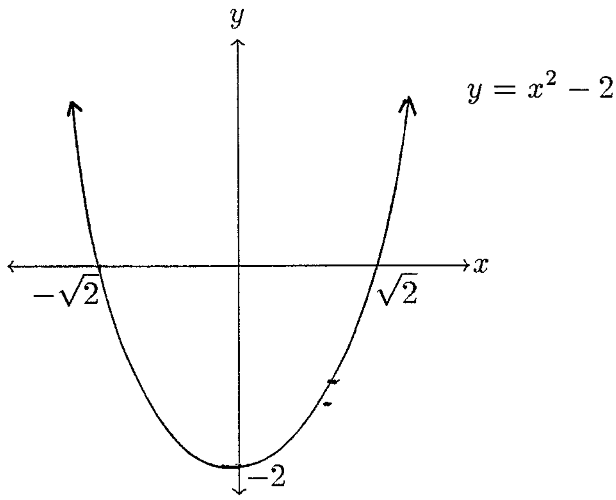
(ii) Show that $\dfrac{dA}{d\theta} = r^2(2\cos^2\theta + \cos\theta - 1)$, and find $\dfrac{d^2A}{d\theta^2}$.

(iii) Use part (b) above to show that $\triangle XPY$ has maximum area when it is an equilateral triangle.

QUESTION SEVEN (Start a new answer booklet)

Marks

5 (a)



It is desired to find approximations to $\sqrt{2}$ by applying Newton's method to find the roots of the function $f(x) = x^2 - 2$ sketched above. Let the initial value be x_1.

(i) State for which initial values x_1 the successive approximations given by Newton's method will eventually get close to $\sqrt{2}$, for which initial values they will eventually get close to $-\sqrt{2}$, and for which initial values the method will fail entirely.

(ii) Show that for the initial value x_1, the next approximation x_2 is:

$$x_2 = \frac{{x_1}^2 + 2}{2x_1}.$$

(iii) Using the initial value $x_1 = 2$, apply Newton's method three times, giving each successive approximation as a rational number without further rounding. Do not use decimal notation at all in this question.

4 (b) Use the method of mathematical induction to prove that for all integers $n \geq 2$, the expression $9^n - 8n - 1$ is divisible by 64.

3 (c) Consider the sum:

$$S_n = a + \frac{a+d}{r} + \frac{a+2d}{r^2} + \frac{a+3d}{r^3} + \frac{a+4d}{r^4} + \cdots + \frac{a+nd}{r^n},$$

where a, d and r are constants, with $|r| > 1$, and n is an integer.

(i) Write down the expression for rS_n.

(ii) By subtracting the expressions for S_n and for rS_n, find a formula for S_n.

(iii) Hence show that:

$$\lim_{n \to \infty} S_n = \frac{r(ar - a + d)}{(r-1)^2}.$$

WMP

The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

QUESTION ONE

Marks

3 (a) Given $\dfrac{3x-8}{x} \geq 1,$

$\boxed{\times x^2}$ $3x^2 - 8x \geq x^2$ ✓ and $x \neq 0$

$2x^2 - 8x \geq 0$ and $x \neq 0$

$2x(x-4) \geq 0$ and $x \neq 0$

$x < 0$ or $x \geq 4$. ✓✓ (one mark is for excluding 0)

2 (b) $\sin 2x = -\frac{1}{2}$, where $0 \leq x \leq 2\pi$, that is, $0 \leq 2x \leq 4\pi$,

$2x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}$ or $\frac{23\pi}{6}$ ✓

$x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}$ or $\frac{23\pi}{12}$. ✓

2 (c)
$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan x\,dx = -\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{-\sin x}{\cos x}\,dx$$
$$= -\Big[\log(\cos x)\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \quad ✓$$
$$= -\log\tfrac{1}{2} + \log(\tfrac{1}{2}\sqrt{3})$$
$$= \log 2 - \log 2 + \tfrac{1}{2}\log 3 \quad ✓ \text{ (or some explanation — it's a proof)}$$
$$= \tfrac{1}{2}\log 3.$$

2 (d)
$$\int_0^{0\cdot 4} \frac{3\,dx}{4+25x^2} = \tfrac{3}{10}\Big[\tan^{-1}\tfrac{5}{2}x\Big]_0^{0\cdot 4} \quad ✓$$
$$= \tfrac{3}{10}\left(\tan^{-1}1 - \tan^{-1}0\right)$$
$$= \tfrac{3\pi}{40}. \quad ✓$$

3 (e) To simplify $\tan\left(2\cos^{-1}\frac{12}{13}\right)$:

let $\alpha = \cos^{-1}\frac{12}{13}$,

then $\cos\alpha = \frac{12}{13}$ and $0 \leq \alpha \leq \frac{\pi}{2}$,

and so $\tan\alpha = \frac{5}{12}$. ✓

So
$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$
$$= \tfrac{5}{6} \div \left(1 - \tfrac{25}{144}\right) \quad ✓$$
$$= \tfrac{5}{6} \times \tfrac{144}{119}$$
$$= \tfrac{120}{119}. \quad ✓ \text{ (or } 1\tfrac{1}{119}\text{, but it must be cancelled)}$$

QUESTION TWO

Marks

3 (a) $\displaystyle\int_0^1 \sqrt{4-x^2}\,dx$

$$= \int_0^{\frac{\pi}{6}} 4\cos^2\theta\,d\theta \quad \boxed{\checkmark \text{ (integrand) } \checkmark \text{ (limits)}}$$

$$= \int_0^{\frac{\pi}{6}} (2+2\cos 2\theta)\,d\theta$$

$$= \Big[2\theta + \sin 2\theta\Big]_0^{\frac{\pi}{6}}$$

$$= (\tfrac{\pi}{3} + \sin\tfrac{\pi}{3}) - (0 + \sin 0)$$

$$= \tfrac{\pi}{3} + \tfrac{1}{2}\sqrt{3} \quad \boxed{\checkmark}$$

$$= \frac{2\pi + 3\sqrt{3}}{6}.$$

Let $x = 2\sin\theta$,
then $dx = 2\cos\theta\,d\theta$,
and $4 - x^2 = 4\cos^2\theta$,
so $\sqrt{4-x^2} = 2\cos\theta$.
When $x = 0$, $\theta = 0$,
when $x = 1$, $\theta = \frac{\pi}{6}$.

9 (b) (i) Let $f(x) = x\sqrt{2-x^2}$
then $f(-x) = -x\sqrt{2-(-x)^2}$
$= -x\sqrt{2-x^2}$
$= -f(x)$, $\boxed{\checkmark \text{ (it's a proof)}}$
so the function is odd.

(ii) $y = x\sqrt{2-x^2}$

$$\frac{dy}{dx} = \sqrt{2-x^2} + \frac{x(-2x)}{2\sqrt{2-x^2}}$$

$$= \frac{2-x^2-x^2}{\sqrt{2-x^2}}$$

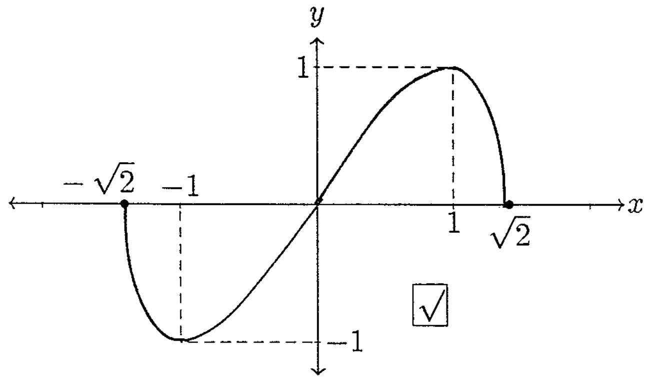
$$= \frac{2-2x^2}{\sqrt{2-x^2}}. \quad \boxed{\checkmark \text{ (it's a proof)}}$$

Let $u = x$
and $v = \sqrt{2-x^2}$,
then $\dfrac{du}{dx} = 1$
and $\dfrac{dv}{dx} = \dfrac{-2x}{2\sqrt{2-x^2}}$.

(iii) So dy/dx has zeroes at 1 and at -1, $\boxed{\checkmark}$
and is undefined at $x = \sqrt{2}$ and at $x = -\sqrt{2}$.
When $x = 1$, $y = 1$,
and when $x = 1$, $\dfrac{d^2y}{dx^2} = -4$
< 0,
so $(1, 1)$ is a maximum turning point,
Since the function is odd, $(-1,-1)$ is a minimum turning point. $\boxed{\checkmark\checkmark}$

(iv) The fact that $\dfrac{dy}{dx} \to -\infty$ as $x \to (\sqrt{2})^-$ means that the curve will become vertical as it gets near $x = \sqrt{2}$. $\boxed{\checkmark}$

(v)



(vi) $$\begin{aligned}\text{Area} &= \int_0^{\sqrt{2}} x\sqrt{2-x^2}\,dx \\ &= -\tfrac{1}{2}\int_0^{\sqrt{2}} (-2x)\sqrt{2-x^2}\,dx \\ &= -\tfrac{1}{2}\int_2^{0} u^{\frac{1}{2}}\,du \quad \boxed{\surd \text{ (for integrand)}} \\ &= -\tfrac{1}{2}\times\tfrac{2}{3}\left[u^{\frac{3}{2}}\right]_2^0 \\ &= -\tfrac{1}{3}(0-2\sqrt{2}) \\ &= \tfrac{2}{3}\sqrt{2} \text{ square units.} \quad \boxed{\surd}\end{aligned}$$

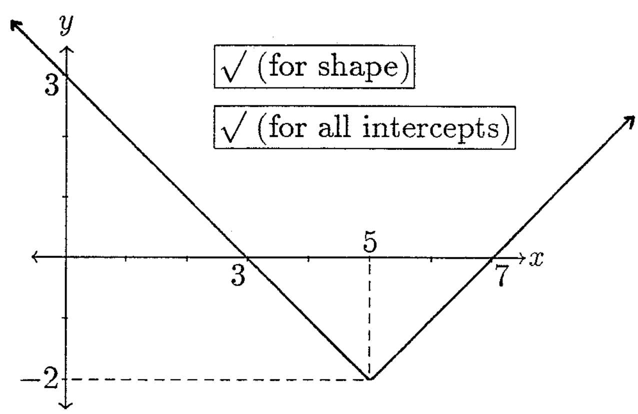
Let $u = 2 - x^2$,
then $du = -2x\,dx$,
and $\sqrt{2-x^2} = u^{\frac{1}{2}}$.
When $x = 0$, $u = 2$,
when $x = \sqrt{2}$, $u = 0$.

QUESTION THREE

Marks

$\boxed{2}$ (a) $$\left(3x^3+\frac{1}{x}\right)^4 = (3x^3)^4 + \left(4\times(3x^3)^3\times\frac{1}{x}\right) + \left(6\times(3x^3)^2\times\frac{1}{x^2}\right) + \left(4\times 3x^3\times\frac{1}{x^3}\right) + \frac{1}{x^4}$$
$$= 81x^{12} + 108x^8 + 54x^4 + 12 + \frac{1}{x^4}. \quad \boxed{\surd \text{ (powers of } x) \ \surd \text{ (coefficients)}}$$

$\boxed{2}$ (b) Given $y = |x-5| - 2$,
$y + 2 = |x - 5|$.
This is $y = |x|$, shifted 2 down, and 5 to the right.



$\boxed{\surd \text{ (for shape)}}$

$\boxed{\surd \text{ (for all intercepts)}}$

$\boxed{2}$ (c) Since $AM : MB = 8 : 3$, the point M divides AB externally in the ratio $8 : 3$. $\boxed{\surd}$
But $8 : 3 = 1 : \frac{3}{8}$, so $k = \frac{3}{8}$. $\boxed{\surd}$

NOTE: Where is M if $k < 0$, if $k = 0$, if $0 < k < 1$, if $k = 1$, if $k > 1$, and if $k = \infty$?

6 (d) (i) The midpoint of PQ is $M = \left(a(p+q), \dfrac{a(p^2+q^2)}{2}\right)$. ☑

(ii) First,

$$\text{gradient of } PQ = \frac{ap^2 - aq^2}{2ap - 2aq} \quad ☑$$
$$= \frac{a(p-q)(p+q)}{2a(p-q)}$$
$$= \frac{p+q}{2},$$

So the chord PQ is

$$y - ap^2 = \frac{p+q}{2}(x - 2ap)$$
$$y - ap^2 = \frac{(p+q)x}{2} - (p+q)ap$$
$$y - ap^2 = \frac{(p+q)x}{2} - ap^2 - apq$$
$$y = \frac{(p+q)x}{2} - apq. \quad \boxed{\surd \text{ (it's a proof)}}$$

(iii) Substituting $A(a, 0)$ into the equation of the chord PA:

$$0 = \frac{(p+q)a}{2} - apq$$
$$0 = \frac{p+q}{2} - pq \quad (\text{since } a > 0)$$
$$p + q = 2pq. \quad \boxed{\surd \text{ (it's a proof)}}$$

(iv) The locus of M is given by the three simultaneous equations:

$$x = a(p+q), \qquad (1)$$
$$y = \frac{a(p^2+q^2)}{2}, \qquad (2)$$
$$p + q = 2pq. \qquad (3)$$

Squaring (1), $x^2 = a^2(p^2 + q^2 + 2pq)$,

and using (3), $x^2 = a^2(p^2+q^2) + a^2(p+q)$.

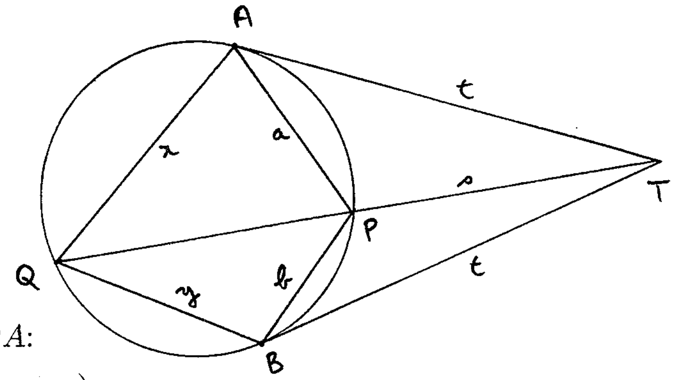
Now using (1) and (2), $x^2 = 2ay + ax$. ☑☑

NOTE: Rearranging, $\left(x - \frac{1}{2}a\right)^2 = 2a\left(y + \frac{1}{8}a\right)$, so the locus is a parabola pointing upwards, with vertex $\left(\frac{1}{2}a, -\frac{1}{8}a\right)$ and focal length $\frac{1}{2}a$. But the locus of M is not the whole parabola $x^2 = 2ay + ax$ — what restriction on x and y is needed (I don't think this is easy, as lines through A with large positive gradient cut through the parabola again)?

For 4 unit people only, is a restriction needed if complex values of all the variables are allowed? This answer will need an algebraic and a geometric interpretation.

QUESTION FOUR

Marks



[7] (a)

(i) In the triangles TAP and TQA:

1. $\angle ATP = \angle QTA$ (common),
2. $\angle PAT = \angle AQT$ (alternate segment theorem), [√ (for using it)]

so $\triangle TAP \,|||\, \triangle TQA$ (AA). [√ (it's a proof)]

(ii) Hence $\dfrac{AP}{QA} = \dfrac{TP}{TA}$ (matching sides of similar triangles),

that is, $\dfrac{a}{x} = \dfrac{s}{t}$. [√]

(iii) Similarly, $\dfrac{b}{y} = \dfrac{s}{t}$,

and so $\dfrac{a}{x} = \dfrac{b}{y}$

$ay = bx$, [√ (it's a proof)]

that is, the products of the pairs of opposite sides are equal.

NOTE: It's not hard now to prove the converse result, and so show that the property proven in (iii) is in fact a characterisation of a cyclic quadrilateral in which the intersection of the tangents at one pair of opposite vertices is collinear with the other two vertices. It follows then that if this happens with one pair of opposite vertices, it happens with both. Can you complete the statement and proof of all this?

For 4 unit people only, how do this theorem and its proof relate to the geometric interpretations of the 4 unit Trial's last question Q8(b), which defined the cross ratio of four complex numbers and applied it to whether or not the four points or their four images were concyclic? Can you prove the theorem above using complex numbers, or the 4 unit Trial theorem using Euclidean geometry? How do both of these theorems relate to the two parts of Q7 of last year's 4 unit SGS Trial? Can you provide a Euclidean proof of Sylvester's theorem in last year's Half Yearly?

(iv) Now area $\triangle APQ = \frac{1}{2}ax\sin A$, (sine rule), [√]

and area $\triangle BPQ = \frac{1}{2}by\sin B$.

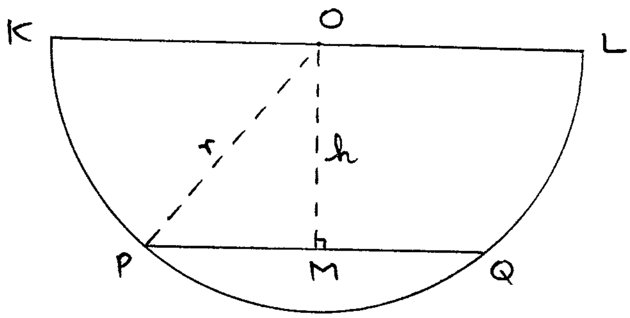
But $A + B = 180^\circ$, (opposite angles of cyclic quadrilateral),

so $\sin A = \sin B$, (the sines of supplementary angles are equal), [√]

also $\dfrac{x}{y} = \dfrac{a}{b}$, (since $ay = bx$). [√]

So $\dfrac{\text{area } \triangle APQ}{\text{area } \triangle BPQ} = \dfrac{a^2}{b^2}$. [(it's a proof)]

5 (b)



(i) By Pythagoras, $PM^2 = r^2 - h^2$, ✓

(ii) So $PQ = 2\sqrt{r^2 - h^2}$,

and $A = 2\ell\sqrt{r^2 - h^2}$. ✓ (it's a proof)

(iii) Hence $\dfrac{dA}{dt} = \dfrac{dA}{dh} \times \dfrac{dh}{dt}$ (chain rule)

$$= \frac{2\ell(-2h)}{2\sqrt{r^2 - h^2}}\,\frac{dh}{dt} \quad \text{(chain rule)}$$

$$= -\frac{2\ell h}{\sqrt{r^2 - h^2}}\,\frac{dh}{dt}. \quad \text{✓ (the signs must be correct)}$$

In units of centimetres and days, $\ell = 250$, $r = 50$, $dh/dt = 0{\cdot}3$ and $h = 40$,

so $$\frac{dA}{dt} = -\frac{20\,000}{\sqrt{2500 - 1600}} \times 0{\cdot}3 \quad ✓$$

$$= -\frac{20\,000}{30} \times 0{\cdot}3$$

$$= -200.$$

So the surface area is decreasing at 200 cm^2 per day. ✓

QUESTION FIVE

Marks

4 (a) (i)

$$\begin{array}{r|l}
 & x^2 - x - 3 \\
\hline
x^2 + 4 & x^4 - x^3 + x^2 - x + 1 \\
 & x^4 \qquad + 4x^2 \quad ✓ \\
\hline
 & -x^3 - 3x^2 - x + 1 \\
 & -x^3 \qquad - 4x \\
\hline
 & -3x^2 + 3x + 1 \\
 & -3x^2 \qquad - 12 \\
\hline
 & 3x + 13 \quad ✓
\end{array}$$

So $x^4 - x^3 + x^2 - x + 1 = (x^2 + 4)(x^2 - x - 3) + (3x + 13)$. ✓

(ii) We know $x^4 - x^3 + x^2 - x + 1 = (x^2 + 4)(x^2 - x - 3) + (3x + 13)$,

so $x^4 - x^3 + x^2 - 4x - 12 = (x^2 + 4)(x^2 - x - 3)$,

and so $a = -4$ and $b = -12$. ✓

5 (b) (i) In the binomial expansion of $(3+11x)^{19}$:

$$T_{k+1} = {}^{19}C_k\, 3^{19-k}\,(11x)^k, \quad \boxed{\surd}$$

and $$T_k = {}^{19}C_{k-1}\, 3^{20-k}\,(11x)^{k-1},$$

so $$\frac{T_{k+1}}{T_k} = \frac{19! \times (20-k)! \times (k-1)!}{(19-k)! \times k! \times 19!} \times \frac{3^{19-k}\,(11x)^k}{3^{20-k}\,(11x)^{k-1}}$$

$$= \frac{11x(20-k)}{3k}, \text{ as required. } \boxed{\surd \text{ (it's a proof)}}$$

(ii) Put $\dfrac{\text{coeff } T_{k+1}}{\text{coeff } T_k} > 1$,

then $$\frac{11(20-k)}{3k} > 1 \quad \boxed{\surd}$$

$$220 - 11k > 3k$$

$$220 > 14k$$

$$k < 15\tfrac{5}{7}$$

$$k \le 15.$$

So coeff $T_{16} >$ coeff T_{15}, but coeff $T_{17} <$ coeff T_{16},

so T_{16} has the greatest coefficient, and: $\boxed{\surd}$

$$\text{coeff } T_{16} = {}^{19}C_{15} \times 3^4 \times 11^{15}$$

$$= \frac{19 \times 18 \times 17 \times 16}{1 \times 2 \times 3 \times 4} \times 3^4 \times 11^{15}$$

$$= \frac{19 \times 3 \times 17 \times 4}{1} \times 3^4 \times 11^{15}$$

$$= 2^2 \times 3^5 \times 11^{15} \times 17 \times 19. \quad \boxed{\surd \text{ (prime factorisation required)}}$$

3 (c) (i) The expansion is:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n.$$

Putting $x = 2$ makes the LHS equal to 3^n, and so:

$$\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \cdots + 2^n\binom{n}{n} = 3^n. \quad \boxed{\surd}$$

(ii) Differentiating both sides of the original identity:

$$n(1+x)^{n-1} = \binom{n}{1} + \binom{n}{2}2x + \binom{n}{3}3x^2 + \cdots + \binom{n}{n}nx^{n-1}. \quad \boxed{\surd}$$

Putting $x = -1$ makes the LHS equal to 0, and so:

$$\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \cdots + (-1)^{n-1}n\binom{n}{n} = 0. \quad \boxed{\surd}$$

QUESTION SIX

Marks

4 (a) Since

$$v^2 = -3x^2 + 20x + 7,$$
$$\ddot{x} = \frac{d}{dx}\left(\tfrac{1}{2}v^2\right)$$
$$= -3x + 10 \quad \boxed{\surd}$$
$$= -3(x - 3\tfrac{1}{3}),$$

so the particle is moving in simple harmonic motion about the centre $x = 3\frac{1}{3}$. $\boxed{\surd}$

Also $n^2 = 3$, so period $= \dfrac{2\pi}{\sqrt{3}}$ $\boxed{\surd}$

$$= \tfrac{2}{3}\pi\sqrt{3}.$$

Factoring, $v^2 = -(3x+1)(x-7)$,

so the particle is at rest at $x = -\frac{1}{3}$ and $x = 7$,

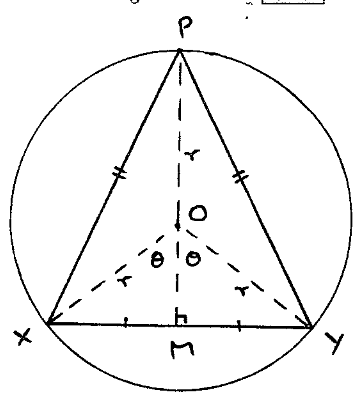
so amplitude $= \frac{1}{2}\left(7 + \frac{1}{3}\right)$

$$= 3\tfrac{2}{3}. \quad \boxed{\surd}$$

3 (b) Given $2\cos^2\theta + \cos\theta = 1$, where $0 \le \theta \le \pi$,

$$2\cos^2\theta + \cos\theta - 1 = 0$$
$$(2\cos\theta - 1)(\cos\theta + 1) = 0 \quad \boxed{\surd}$$
$$\cos\theta = \tfrac{1}{2} \ \text{or} \ -1$$
$$\theta = \tfrac{\pi}{3} \ \text{or} \ \pi. \quad \boxed{\surd\surd}$$

5 (c)



(i) From the diagram, $XM = r\sin\theta$,

and $PM = PO + OM$

$$= r + r\cos\theta, \quad \boxed{\surd}$$

and $A = \frac{1}{2} \times 2r\sin\theta \times r(1 + \cos\theta)$

$$A = r^2\sin\theta(1 + \cos\theta). \quad \boxed{\surd \text{ (it's a proof)}}$$

OR $A = \text{area } \triangle POX + \text{area } \triangle POY + \text{area } \triangle XOY$

$$= \tfrac{1}{2}r^2\sin(\pi - \theta) + \tfrac{1}{2}r^2\sin(\pi - \theta) + \tfrac{1}{2}r^2\sin 2\theta \quad \text{(sine rule)} \quad \boxed{\surd}$$
$$= r^2\sin\theta + r^2\sin\theta\cos\theta$$
$$= r^2\sin\theta(1 + \cos\theta). \quad \boxed{\surd \text{ (it's a proof)}}$$

(ii) Hence $\dfrac{dA}{d\theta} = r^2\cos\theta(1+\cos\theta) + r^2\sin\theta(-\sin\theta)$ (product rule)

$$= r^2\left(\cos\theta + \cos^2\theta - \sin^2\theta\right)$$
$$= r^2\left(\cos\theta + \cos^2\theta - 1 + \cos^2\theta\right)$$
$$= r^2\left(2\cos^2\theta + \cos\theta - 1\right).\quad \boxed{\surd\text{ (it's a proof)}}$$

So $\dfrac{d^2A}{d\theta^2} = r^2\left(-4\cos\theta\sin\theta - \sin\theta\right)$ $\boxed{\surd}$

$$= -r^2\sin\theta\left(4\cos\theta + 1\right).$$

(iii) So by part (b), since θ is acute, $dA/d\theta$ is zero when $\theta = \frac{\pi}{3}$.

When $\theta = \frac{\pi}{3}$, $\dfrac{d^2A}{d\theta^2}$ is negative, by part (ii).

So the area of $\triangle APB$ is maximum when $\theta = \frac{\pi}{3}$,

in which case the triangle is equilateral. $\boxed{\surd\text{ (it's a proof)}}$

NOTE: Surely there is a less complicated proof for such an obvious result. Are there alternative calculus approaches? Are there purely geometric proofs?

QUESTION SEVEN

Marks

$\boxed{5}$ (a) (i) When $x_1 > 0$, the successive values converge to $\sqrt{2}$. When $x_1 < 0$, they converge to $-\sqrt{2}$. When $x_1 = 0$, the method fails, because x_2 will be undefined. $\boxed{\surd\surd}$

(ii) Using the formula, $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$$= x_1 - \frac{{x_1}^2 - 2}{2x_1} \quad (\text{since } f'(x) = 2x)$$
$$= \frac{2{x_1}^2 - {x_1}^2 + 2}{2x_1}$$
$$= \frac{{x_1}^2 + 2}{2x_1}.\quad \boxed{\surd\text{ (it's a proof)}}$$

(iii) Using the initial value $x_1 = 2$ and the formula above:

$x_2 = 1\frac{1}{2}$, $\boxed{\surd}$

$x_3 = 1\frac{5}{12}$,

$x_4 = 1\frac{169}{408}$. $\boxed{\surd\text{ (or } \tfrac{577}{408}\text{, but it must be cancelled)}}$

NOTE: Squaring, $\left(\frac{577}{408}\right)^2 = \frac{332929}{166464} = 2\frac{1}{166464}$, which means that $\frac{577}{408}$ is an excellent rational approximation to use for $\sqrt{2}$. Using the calculator:

$$\tfrac{577}{408} - \sqrt{2} \doteqdot 0{\cdot}000\,002.$$

$\boxed{4}$ (b) THEOREM: *For all integers $n \ge 0$, the expression $9^n - 8n - 1$ is divisible by 64.*

PROOF: A. When $n = 2$, then $9^n - 8n - 1 = 81 - 16 - 1$

$$= 64, \text{ which is divisible by 64,}$$

so the result is true when $n = 2$. $\boxed{\surd}$

B. Suppose $k \geq 2$ is an integer for which the result is true.

That is, $9^k - 8k - 1 = 64\ell$, where $\ell \in \mathbf{Z}$. (**)

We now prove the result for $n = k + 1$.

That is, we prove $9^{k+1} - 8k - 9$ is divisible by 64. [√ (setting up part B)]

$$\begin{aligned} 9^{k+1} - 8k - 9 &= 9 \times 9^k - 8k - 9 \\ &= 9(64\ell + 8k + 1) - 8k - 9, \text{ by the induction hypothesis (**)} \quad [\surd] \\ &= 9 \times 64\ell + 64k, \text{ which is divisible by 64, as required.} \quad [\surd] \end{aligned}$$

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 2$. [(maximum 3/4 if missing)]

NOTE: Modulo arithmetic is not part of the 2/3/4 unit Syllabuses, but the alternative proof using modulo 16 arithmetic is very quick and makes the result somewhat more obvious.

3 (c) (i) Given $S_n = a + \dfrac{a+d}{r} + \dfrac{a+2d}{r^2} + \dfrac{a+3d}{r^3} + \dfrac{a+4d}{r^4} + \cdots + \dfrac{a+nd}{r^n}$,

[$\times r$] $rS_n = ar + (a+d) + \dfrac{a+2d}{r} + \dfrac{a+3d}{r^2} + \dfrac{a+4d}{r^3} + \cdots + \dfrac{a+nd}{r^{n-1}}$.

(ii) Subtracting the previous two lines:

$$\begin{aligned} (r-1)S_n &= ar + d + \frac{d}{r} + \frac{d}{r^2} + \cdots + \frac{d}{r^{n-1}} - \frac{a+nd}{r^n} \quad [\surd] \\ &= ar + d\left(\frac{1 - \frac{1}{r^n}}{1 - \frac{1}{r}}\right) - \frac{a+nd}{r^n}, \text{ since the middle terms form a GP,} \end{aligned}$$

so $$S_n = \frac{1}{r-1}\left(ar + d\left(\frac{1 - \frac{1}{r^n}}{1 - \frac{1}{r}}\right) - \frac{a+nd}{r^n}\right). \quad [\surd \text{ (unsimplified)}]$$

(iii) $$\begin{aligned} \lim_{n\to\infty} S_n &= \frac{1}{r-1}\left(ar + d\left(\frac{1-0}{1-\frac{1}{r}}\right) + 0\right), \text{ since } \frac{1}{r^n} \to 0 \text{ as } n \to \infty. \\ &= \frac{1}{r-1}\left(ar + \frac{dr}{r-1}\right) \\ &= \frac{1}{(r-1)^2}\left(ar^2 - ar + dr\right) \\ &= \frac{r(ar - a + d)}{(r-1)^2}. \quad [\surd \text{ (it's a proof)}] \end{aligned}$$

NOTE: The original series is the ratio of the terms of any AP over the terms of any GP, so it's nice to be able to find its sum, and to find also the limit of its sum. What physical situations could be modelled by such an object? What can be done with the ratio of the terms of any GP over the terms of any AP?

WMP