Sydney Grammar School

TRIAL HSC EXAMINATION 1992

MATHEMATICS

3 UNIT

Time Allowed : 2 hours   13th August 1992

INSTRUCTIONS:

* All questions may be attempted.
* All questions are of equal value.
* All necessary working must be shown.
* Marks may not be awarded for careless or badly arranged work.
* Approved calculators may be used.

SPECIAL INSTRUCTIONS:

* Start each question on a new page.
* Hand in each question separately.
* Write your examination number on each page.

A table of standard integrals is included on the last page.

R.N.

QUESTION ONE (Start each question on a new page)

(a) Solve for x the inequality $\dfrac{1}{x-1} < 5$.

(b) (i) If $f(x) = x^2$ and $g(x) = \dfrac{1}{x}$, show algebraically that the graphs of $y = f(x)$ and $y = g(x)$ intersect at the point $(1,1)$.

(ii) Find $f'(1)$ and $g'(1)$. Hence find the acute angle between the curves $y = x^2$ and $y = \dfrac{1}{x}$ at the point $(1,1)$. (Give your answer correct to the nearest degree).

(c) Use the substitution $u = x + 2$ to evaluate $\displaystyle\int_{-2}^{0} x(x+2)^4\,dx$.

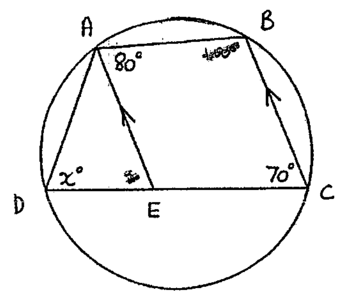
(d) (i) Find the exact value of $\cos\left[\sin^{-1}(-\frac{1}{2})\right]$.

(ii) Evaluate $\displaystyle\int_0^3 \frac{dx}{\sqrt{36-x^2}}$.

(e) Find all value(s) of k for which the line $x - 2y + k = 0$ is a tangent to the circle $x^2 + y^2 = 4$.

QUESTION TWO (Start each question on a new page)

(a)



In the diagram above, $ABCD$ is a cyclic quadrilateral and E is a point on DC such that $AE \parallel BC$. $\angle BAE = 80°$, $\angle BCD = 70°$ and $\angle ADC = x°$.

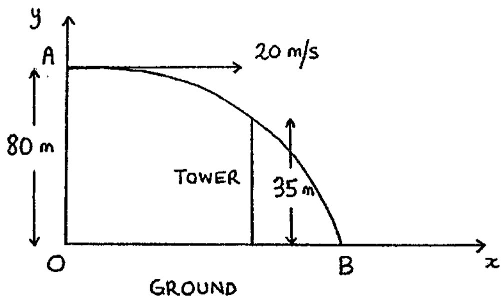
(i) Copy this diagram onto your answer page.

(ii) Find x, stating all reasoning.

(Exam continues next page...)

QUESTION TWO (Continued)

(b)



In the diagram above, an object is projected horizontally with a velocity of 20 m/s from a point A 80 metres above the ground and strikes the ground at point B.

(i) If x and y are respectively the horizontal and vertical components of the object's displacement from O, derive expressions for x and y after t seconds.
(Take O to be the origin and assume the acceleration due to gravity to be $10\,\text{m/s}^2$).

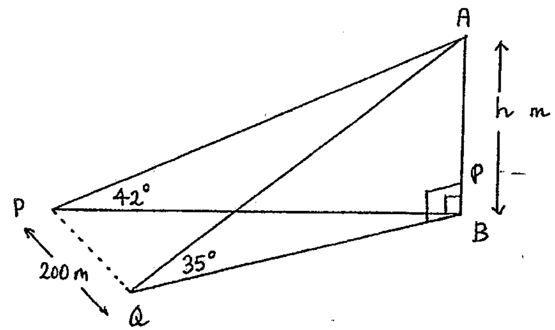
(ii) Find the time taken for the object to reach the point B, and find the distance OB.

(iii) If the object just clears a vertical tower of height 35 metres, find the distance from O to the base of the tower.

(Exam continues overleaf...)

QUESTION TWO (Continued)

(c)



The diagram above shows a point P, due west of B, from which the angle of elevation to the top of a tower AB height h metres, is 42°. From a point Q, bearing 196° from the tower, the angle of elevation to the top of the tower is 35°. The distance from P to Q is 200 metres.

(i) State the size of $\angle PBQ$.

(ii) Show that $h = \dfrac{200}{\sqrt{\cot^2 42° + \cot^2 35° - 2\cot 35° \cot 42° \cos 74°}}$.

(iii) Find h correct to 3 significant figures.

(Exam continues next page...)

QUESTION THREE (Start each question on a new page)

(a) If α, β, γ are the roots of the equation:

$$2x^3 - 3x^2 + 5x + m = 0,$$

(i) find the value of $\alpha + \beta + \gamma$ and $\alpha^2 + \beta^2 + \gamma^2$,

(ii) find the value of m if $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = -2$.

(b) Use one application of Newton's method to find a three decimal place approximation to a root of the equation $e^x - 2x^2 = 0$. Use $x = 1{\cdot}5$ as a first approximation to this root.

Copy and complete (correct to 2 decimal places) the following table of values for $P(x) = e^x - 2x^2$. Hence (i.e. without further calculation) explain why $x = 2{\cdot}2$ would have been a less suitable first approximation for the root of $P(x) = 0$.

x	2·0	2·1	2·2	2·3	2·4
$P(x)$					

(c) According to Newton's law of cooling, the rate at which a body cools in air is proportional to the difference between its temperature $T°$ and the constant temperature $P°$ of the surrounding air. This is expressed by the equation $T = P + Ae^{kt}$, where A, k are constants and t is the time in hours. The temperature of the air surrounding a heated body is 15°C, and the body cools from 90°C to 65°C in 3 hours.

(i) Find the temperature of the body after a further 2 hours. (Answer to the nearest degree). Also find (in °C per hour) the body's rate of cooling at that time.

(ii) Find (correct to the nearest minute) the time taken for the body to cool to 30°C from its original temperature.

(Exam continues overleaf...)

QUESTION FOUR (Start each question on a new page)

(a) If $\sin\theta = \frac{5}{13}$, $0 < \theta < \frac{\pi}{2}$, find exact values for:

(i) $\sec\theta$,

(ii) $\cos\frac{\theta}{2}$.

(b) P is the point $(4t, 2t^2)$ on the parabola $x^2 = 8y$.

(i) Show that the equation of the normal to the parabola at P is $x + ty = 2t^3 + 4t$.

(ii) If the normal at P cuts the y axis at point A, show that the coordinates of A are $(0, 2t^2 + 4)$.

(iii) If R is the mid point of AP, show that the locus of R is a parabola. Find the vertex and focus of the parabola.

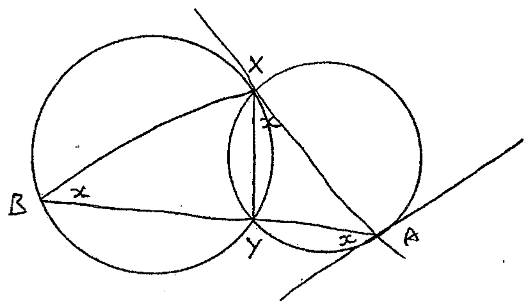
(c) Use mathematical induction to prove that $7^n > 4^n + 5^n$ for all integers $n \geq 2$.

QUESTION FIVE (Start each question on a new page)

(a) Solve the following equation over $0 \leq x \leq 2\pi$:

$$\cos 2x - 3\sin x - 2 = 0.$$

(b)



In the diagram above, two circles intersect at X and Y. The tangent at X to the larger circle cuts the smaller circle at A, and AY produced cuts the larger circle at B.

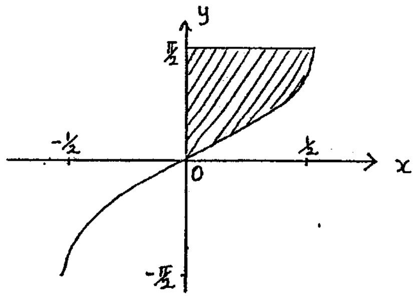
(i) Draw a diagram showing all the given information.

(ii) The tangent to the smaller circle at A is drawn. Prove that this tangent is parallel to BX.

(Exam continues next page...)

QUESTION FIVE (Continued)

(c)



The diagram above shows the graph of $y = \sin^{-1}(2x)$. The shaded area is that bounded by the curve, the y axis and the line $y = \frac{\pi}{2}$.

(i) Find this shaded area.

(ii) If this area is rotated about the y axis, find the volume of the solid thus formed.

(Exam continues overleaf...)

QUESTION SIX (Start each question on a new page)

(a) In the expansion of $(5+3x)^{20}$, in ascending powers of x, A_r is the coefficient of x^{r-1} in the term U_r, and A_{r+1} is the coefficient of x^r in the term U_{r+1}.

(i) Use the Binomial Theorem to write expressions for A_{r+1} and A_r. Hence show that:
$$\frac{A_{r+1}}{A_r} = \frac{63-3r}{5r}.$$

(ii) Hence find the greatest coefficient in the expansion of $(5+3x)^{20}$. (Leave your answer in the form ${}^{20}C_r 3^a 5^b$.)

(b) On Wednesday morning at 5 a.m. an aeroplane crashes into a harbour. The rescue team and its equipment are most effective when the depth of water in the harbour is no more than 7 metres. At low tide the water is 5 metres deep, and at high tide the depth is 10 metres. Low tide occurs at 4 a.m. and high tide at 10.15 a.m. Assume that the movement of the tides is *SHM*.

(i) State the period and amplitude of the motion.

(ii) If the deadline for the rescue operation is 6 p.m. on Wednesday evening, find the periods of time between 5 a.m. and 6 p.m. during which the rescue team can work most effectively.

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Excellent!

1.(a). $\frac{1}{x-1} < 5.$

$(x-1) < 5(x-1)^2$

$5(x-1)^2 - (x-1) > 0.$

$[x-1](5x-5-x+1) > 0.$

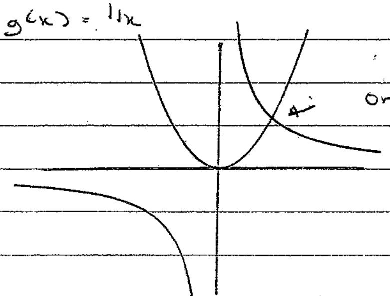
$(x-1)(4x-4) > 0.$

$4(x-1)^2 > 0.$

$\therefore \quad x<1 \cup x>1$ All real x $x \neq 1.$

(b) (i) $f(x) = x^2$

$g(x) = 1/x$



$x^2 = 1/x$

$x^3 = 1$

$x = 1$

$(1,1).$

(ii). $f'(x) = 2x$ $\quad g'(x) = -x^{-2}$

$f'(1) = 2$ $\quad g'(1) = -1/1$

$= -1$

$\therefore \tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$

$= \left|\frac{2-(-1)}{1-2}\right|$

$\theta = \tan^{-1}(3)$

$= 71°34'$

(c). $\int_{-2}^{0} x(x+2)^4 \cdot dx$

Let $u = x+2$ $\quad$ At $x=0, u=2$

$du = dx$ $\quad u=-2, u=0.$

$I = \int_0^2 (u-2)u^4 \cdot du$

$= \int_0^2 u^5 - 2u^4 \cdot du$

$= \left[\frac{u^6}{6} - \frac{2u^5}{4}\right]_0^2$

$= \frac{-16}{3}.$

(d)(i) $\cos(\sin^{-1}(-1/2))$

$= \cos(-\pi/6)$

$= \sqrt{3}/2$

(ii) $\int_0^3 \frac{dx}{\sqrt{36-x^2}}$

$= \sin^{-1}(x/6)]_0^3.$

$= \sin^{-1}(1/2)$

$= \pi/6.$

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(e) $x - 2y + k = 0.$

$x = 2y - k$

$x^2 = (2y-k)^2$

$\therefore (2y-k)^2 + y^2 = 4.$

$4y^2 - 4ky + k^2 + y^2 - 4 = 0.$

$5y^2 - 4ky + (k^2 - 4) = 0.$

$\Delta = 0.$

$16k^2 - 20(k^2-4) = 0.$

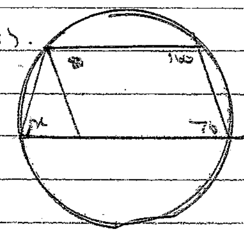
$16k^2 - 20k^2 + 80 = 0.$

$-4k^2 + 80 = 0.$

$20 = k^2$

$k = \pm 2\sqrt{5}$

2.(a).



(ii). $x = 80°$ (opp. angles of cyclic quad add to π)

(b). $\ddot{x} = 0.$

$\dot{x} = \int \ddot{x} \cdot dt$

$= c_1$

At $t=0$, $\dot{x} = V\cos\theta$

$\therefore \dot{x} = V\cos\theta$

$x = \int \dot{x} \cdot dt$

$= Vt\cos\theta + c_2$

At $t=0, x=0$

$x = Vt\cos\theta$

$x = 20t$

$\ddot{y} = -g$

$\dot{y} = \int \ddot{y} \cdot dt$

$= -gt + c_3$

At $t=0$, $\dot{y} = V\sin\theta$

$\dot{y} = -gt + V\sin\theta$

$y = \int \dot{y} \cdot dt$

$= -\frac{gt^2}{2} + Vt\sin\theta + c_4$

At $t=0$, $y = 80.$

$\therefore y = \frac{-gt^2}{2} + Vt\sin\theta + 80.$

$y = -5t^2 + 80.$

(ii). $y = -5t^2 + 80.$

$y = 0, \quad 80 = 5t^2$

$16 = t^2$

$\underline{t = 4 \text{ secs.}}$

$t = 4 \therefore \quad x = 80.$

$\therefore \quad \underline{OB = 80 \text{ metres.}}$

(iii). $x = 20t$

$t = x/20$

$y = -5t^2 + 80.$

$y = \frac{-5x^2}{400} + 80$

$y = 35.$

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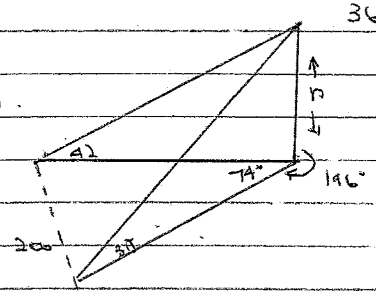
$35 = \frac{-5x^2}{400} + 80$

$14000 = -5x^2 + 32000$

$3600 = x^2$

$\underline{x = 60. \text{ metres.}}$

(c).



(i) $P\hat{B}Q = 74°$

(ii). $\frac{h}{OB} = \tan 35$

$OB = h \cot 35$

$PB = h \cot 42$

$\therefore \Delta PQB$

$PQ^2 = PB^2 + BQ^2 - 2\, PB \cdot BQ \cos P\hat{B}Q$

$200^2 = h^2\cot^2 35 + h^2\cot^2 42 - 2h^2 \cot 35 \cot 42 \cos 74$

$200^2 = h^2(\cot^2 35 + \cot^2 42 - 2\cot 35 \cot 42 \cos 74)$

$h^2 = \frac{200^2}{\cot^2 35 + \cot^2 42 - 2\cot 35 \cot 42 \cos 74}$

$h = \frac{200}{\sqrt{\cot^2 35 + \cot^2 42 - 2\cot 35 \cot 42 \cos 74}}$

(iii). $\underline{h = 129 \text{ m.}}$

3.(a). $2x^3 - 3x^2 + 5x + m = 0.$

$\underline{\alpha + \beta + \gamma = 3/2.}$

$(\alpha^2 + \beta^2 + \gamma^2) = (\Sigma\alpha)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$

$= \frac{9}{4} - 2(5/2)$

$= 9/4 - 6$

$= \underline{-11/4.}$

(ii). $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = -2$

$\frac{\alpha\beta + \beta\gamma + \alpha\gamma}{\alpha\beta\gamma} = -2$

$\frac{5/2}{-m/2} = -2$

$5/2 = m$

$\underline{m = 5/2.}$

(b). $e^x - 2x^2 = 0.$

$f'(x) = e^x - 4x$

$x_1 = 1.5 - \frac{f(1.5)}{f'(1.5)}$

$= 1.5 - (-0.018)$

-1.518

$= \underline{1.49 \text{ (2dp)}}$

$P(x) = e^x - 2x^2$

x	2	2.1	2.2	2.3	2.4
$P(x)$	-0.61	-0.65	0.65	0.6[illegible]	-0.[illegible]

P(2.2) = greatest value

$\therefore$ T.P at $x \doteq 2.2$

$\therefore$ Less suitable at tangent

(c). $T = P + Ae^{kt}$

$T = 15 + Ae^{kt}$

At $t = 0$, $T = 90$ At $t = 3$, $T = 65$

$90 = 15 + A$ $65 = 15 + 75e^{3k}$

$\underline{A = 75.}$ $2/3 = e^{3k}$

$\ln(2/3) = k.$

(i). $t = 5$.

$T = \cancel{TM} +$

$15 + 75\, e^{5\ln(2/3)/3}$

$= 53°$

(ii). $30 = 15 + 75 e^{\frac{\ln(2/3)}{3} t}$

$15/75 = e^{\frac{\ln(2/3)t}{3}}$

$\ln(1/5) = \frac{\ln(2/3)t}{3}$

$\frac{3\ln(1/5)}{\ln(2/3)} = t$

$t = 11.9$

$\therefore$ 12 hrs

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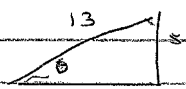
4.(a). $\sin\theta = 5/13$.



(i) $\sec\theta$

$= 13/12$.

(ii). $\cos\theta/2$.

$\cancel{2\sin\theta/2\cos\theta/2 = 5/13}$.

Let $t = \tan\theta/2$

$\sin\theta = \frac{2t}{1+t^2} = 5/13$.

$26t = 5 + 5t^2$

$5t^2 - 26t + 5 = 0$.

$t = 5,\ 1/5$.

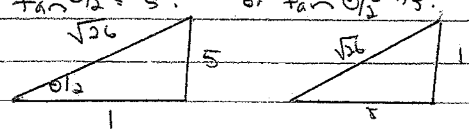
Quicker to use
$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$

$\frac{12}{13} = 2\cos^2\frac{\theta}{2} - 1$

$\frac{25}{26} = \cos^2\frac{\theta}{2}$

$\therefore \cos\frac{\theta}{2} = \frac{5}{\sqrt{26}}$ since $0 < \frac{\theta}{2} < \frac{\pi}{2}$

$\therefore \tan\theta/2 = 5$. or $\tan\theta/2 = 1/5$.



$\therefore \cos\theta/2 = 1/\sqrt{26}$ or $5/\sqrt{26}$

But $0 < \theta < \pi/2$

$\therefore \cos\theta/2 = \frac{5}{\sqrt{26}}$

(b)(i) $P(4t, 2t^2)$

$\frac{x^2}{8} = y$

$\frac{dy}{dx} = \frac{2x}{8}$

at $P(4t, 2t^2)$

$m = \frac{8t}{8}$

$= t$

$\therefore m_2 = \frac{-1}{t}$

$\therefore\ y - 2t^2 = \frac{-1}{t}(x - 4t)$

$ty - 2t^3 = -x + 4t$

$x + ty = 2t^3 + 4t$.

(ii). A, when $x = 0$.

$ty = 2t^3 + 4t$

$y = 2t^2 + 4$

$\therefore A(0, 2t^2 + 4)$

(iii) $P(4t, 2t^2)$ $A(0, 2t^2+4)$

$\therefore x = 2t,\ y = 2t^2 + 2$.

$t = x/2$ $y = 2(\frac{x}{2})^2 + 2$

$y = \frac{x^2}{2} + 2$

$2y = x^2 + 4$.

$2y - 4 = x^2$

$4(y/2 - 1) = x^2$.

Vertex: $(0, 2)$ $a = 1$

Focus: $(0, 1)$

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(c). $7^n > 4^n + 5^n$

Let $n = 2$.

LHS: 7^2 RHS: 41

$= 49$ $\therefore$ LHS > RHS

Assume true for $n = k$.

$7^k > 4^k + 5^k$

Prove true for $n = k+1$

LHS $= 7^{k+1} > 7(4^k + 5^k)$

$7^k - 4^k - 5^k > 0.$

RTP, true $n = k+1$

$7^{k+1} - 4^{k+1} - 5^{k+1} > 0.$

LHS $= 7(7^k - 4^k - 5^k) + 3 \cdot 4^k + 2 \cdot 5^k$

$> 0 \qquad > 0.$

$\therefore 7^{k+1} - 5^{k+1} - 4^{k+1} > 0.$

Statement...

5.(a). $\cos 2x - 3\sin x - 2 = 0.$

$(1 - 2\sin^2 x) - 3\sin x - 2 = 0.$

$-2\sin^2 x - 3\sin x - 1 = 0.$

$2\sin^2 x + 3\sin x + 1 = 0.$

$\sin x = \frac{-3 \pm \sqrt{9-8}}{4}$

$= \frac{-3 \pm 1}{4}$

$= -1/2$ or -1

$x = 11\pi/6, \ \frac{7\pi}{6}, \ 3\pi/2$

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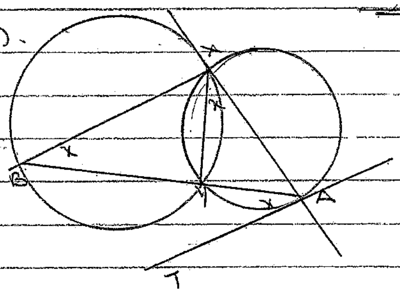
(b).



Let $Y\hat{A}T = x.$

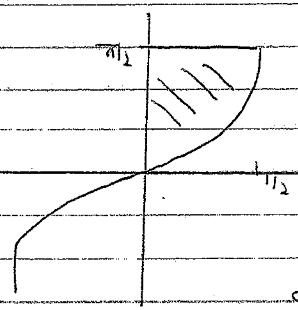
$Y\hat{X}A = x$ ($\angle$s in the alternate segment)

$X\hat{B}A = x$ (" ")

$\therefore X\hat{B}A = T\hat{A}B$

$\therefore BX \parallel TA$ (alternate angles are equal).

(c).



$y = \sin^{-1}(2x)$ $\quad \frac{\sin y}{2} = x$

Area $= \ldots - \int_0^{\pi/2} x \, dy$

$= \frac{1}{2}\int \sin y \, dy$

$= -\frac{1}{2}\left[-\cos y\right]_0^{\pi/2}$

$= 1/2.$

(ii). Volume $= \pi\int_0^{\pi/2} x^2 \, dy$

$\cos 2x = 1 - 2\sin^2 x$

$= \frac{1 - \cos 2x}{2}$

$= \frac{\pi}{4}\int_0^{\pi/2} \sin^2 y \, dy$

$= \frac{\pi}{8}\int_0^{\pi/2} 1 - \cos 2y \, dy$

$= \frac{\pi}{8}\left[y - \frac{\sin 2y}{2}\right]_0^{\pi/2}$

$= \frac{\pi^2}{16}$

6.(a). $(5+3x)^{20}.$

$\frac{A_{r+1}}{A_r} = \frac{\binom{20}{r}(5)^{20-r}(3x)^r}{\binom{20}{r-1}(5)^{21-r}(3x)^{r-1}}$

$= \frac{20!}{(20-r)!\, r!} \times (5)^{20-r} \times (3)^r$

$\frac{20!}{(21-r)!\,(r-1)!} \times (5)^{21-r} (3)^{r-1}$

$= \frac{(21-r)!\,(r-1)! \times 3}{(20-r)!\, r! \times 5.}$

$= \frac{63-3r}{5r}.$

(ii). $\frac{63-3r}{5r} > 1$

$63 - 3r > 5r$

$63 > 8r$

$r = 7$

$\therefore \binom{20}{7}(3)^{13}(5)^7$

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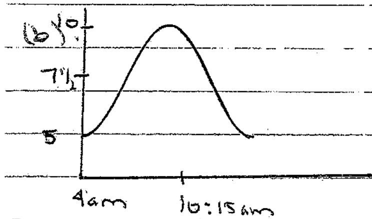
(b)



(i) $T = 25/2 = \frac{2\pi}{n}$

$25n = 4\pi$

$n = 4\pi/25$

Amp: $2\frac{1}{2}.$

(ii). $x = 7\frac{1}{2} - 2\frac{1}{2}\cos(4\pi/25\, t)$

$2x = 15 - 5\cos(\frac{4\pi}{25}t)$

$x = 7$

$14 = 15 - 5\cos(\frac{4\pi}{25}t)$

$1/5 = \cos(\frac{4\pi}{25}t)$

$1.369 = 4\pi/25\, t.$ $\quad$ or $4.9137 = 4\pi/25\, t.$

$t = 2.72$ $\quad$ $t = 9.775$

$\therefore$ 8:06am. $\quad$ 2:47pm

$\therefore$ Between: 5am – 8:06am

2:47pm – 6pm.