DS

SYDNEY GRAMMAR SCHOOL TRIAL EXAMINATION 1993

2/3 UNIT MATHEMATICS FORM VI

Time allowed: 3 hours

Exam date: 5th August, 1993

Instructions:

All questions may be attempted.
All questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators may be used.

Collection:

Each question is to be collected separately.
Start each question on a new page.
Attach a cover sheet to each question.
Hand in a cover sheet even when the question has not been attempted.

QUESTION ONE

(a) If $x = 2{\cdot}543$ and $y = 1{\cdot}761$, find the value of $\dfrac{x^2+y^2}{xy}$, correct to 2 decimal places.

(b) Factorise $25a^2 - 16$.

(c) Express $\dfrac{1}{\sqrt{5}-\sqrt{2}}$ with a rational denominator.

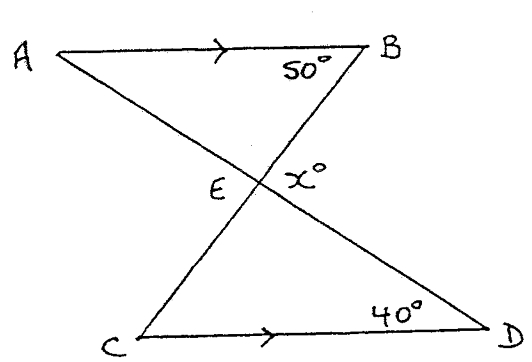
(d) Solve $\dfrac{x}{5} - \dfrac{x+1}{3} = 2$.

(e) If $S = \dfrac{v^2-u^2}{2a}$ and given S, u, v, a are positive integers, find u if $S = 16, v = 31$ and $a = 21$.

(f) Graph the solution set of $|x-1| < 3$ on a number line.

(g) During the winter sales the marked price of a shirt is reduced by 30%. If the sale price is $35 what was the original price?

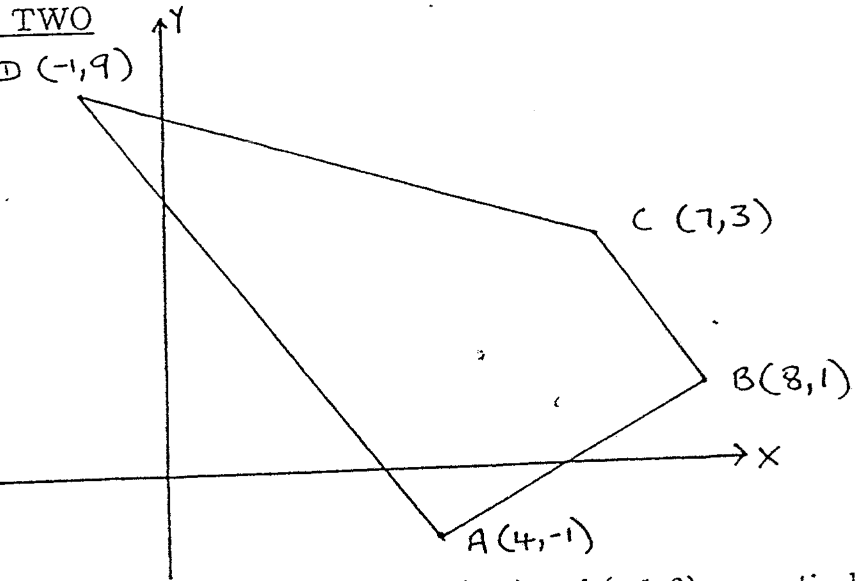
(h)



Find the value of pronumeral stating reasons.

(Exam continues next page ...)

QUESTION TWO



A, B, C and D are the points $(4,-1), (8,1), (7,3)$ and $(-1,9)$ respectively.

(a) Show the equation of AC is $4x - 3y - 19 = 0$.

(b) Show $BC \parallel AD$.

(c) Show $\angle ACD = 90°$.

(d) Show the length of AC is 5 units.

(e) Find the perpendicular distance of B from AC.

(f) Find the area of the trapezium $ABCD$.

(Exam continues overleaf ...)

QUESTION THREE

(a) Differentiate:

(i) $(7x+3)^5$,

(ii) e^{-5x},

(iii) $\sin x^2$,

(iv) $x \ln x$.

(b) Find:

(i) $\displaystyle\int (3x+2)\,dx$,

(ii) $\displaystyle\int \frac{1}{3x+2}\,dx$.

(c) Evaluate:

(i) $\displaystyle\int_0^1 (1+e^{-x})\,dx$,

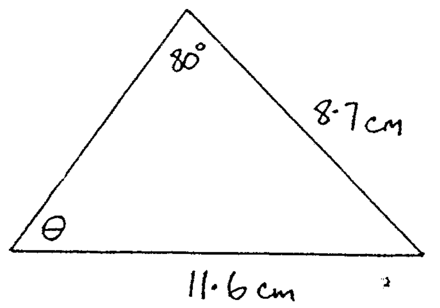
(ii) $\displaystyle\int_0^1 \sin \pi x\,dx$.

(d) Find the equation of the line which passes through the intersection of the lines $x-y=0$ and $2x+y-1=0$ and is parallel to the line whose equation is $4x+y-1=0$. Give your answer in general form.

(Exam continues next page ...)

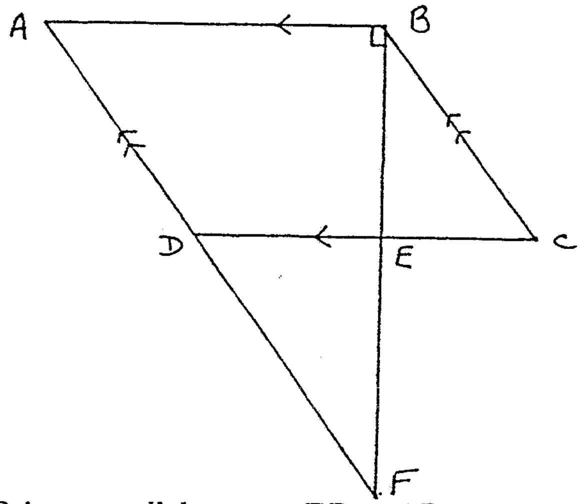
QUESTION FOUR

(a)



Use the Sine Rule to find the value of θ to the nearest minute.

(b)



$ABCD$ is a parallelogram. $FB \perp AB$.

(i) Prove $\triangle CBE \;|||\; \triangle AFB$.

(ii) If $CE = 3$ cm, $BC = 7$ cm and $AF = 15$ cm, find AB.

(c) A total of \$7800 is to be shared among three people. The smallest share is to be \$800. Find the value for each of the remaining shares when:

(i) the values of the three shares are in arithmetic progression,

(ii) the values of the three shares are in geometic progression.

QUESTION FIVE

(a) Find the equation of the tangent to the parabola $y = x^2 - 3x + 2$ at the point $(3, 2)$.

(b) The equation of the parabola is $x^2 - 2x + 25 = 8y$. Express this in the general form $(x-p)^2 = 4a(y-q)$ and hence:

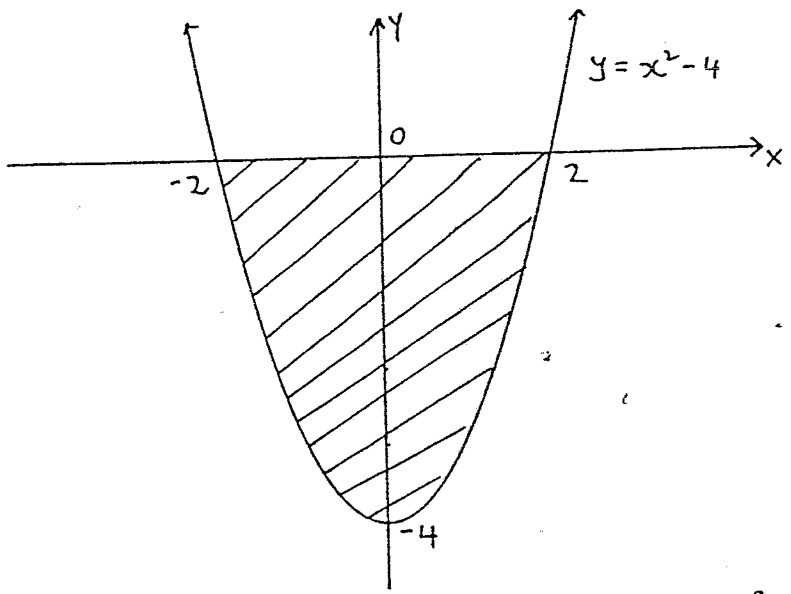
(i) state the coordinates of the vertex,

(ii) find the coordinates of the focus,

(iii) state the equation of the directix.

(Exam continues overleaf ...)

(c)



The area below the x axis bounded by the curve $y = x^2 - 4$ is rotated about the x axis. Find the volume of the solid of revolution thus generated.

(d) Find the values of k for which the quadratic equation $kx^2 - 2kx - (3k + 12) = 0$ has real roots.

QUESTION SIX

(a) The curve $y = x^3 + ax + b$ has a stationary point at $P(1, 5)$. Find the value of the constants a and b.

(b) The position of a particle moving along a straight line is given by $x = 2t - \sin(2t - 6)$.

(i) Show the particle is at rest when $t = 3$.

(ii) Determine when the particle is next at rest.

(iii) Find the acceleration of the particle when $t = 3 + \frac{\pi}{4}$.

(c) Find the approximate value of $\int_1^3 f(x)dx$, using Simpson's Rule and five function values for the table given.

x	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
$f(x)$	2	7	12	13	14

What is the geometrical significance of $\int_1^3 f(x)dx$ given $y = f(x)$ represents a continuous curve drawn on the (x, y) plane and $f(x) > 0$ for $1 \leq x \leq 3$?

(Exam continues next page ...)

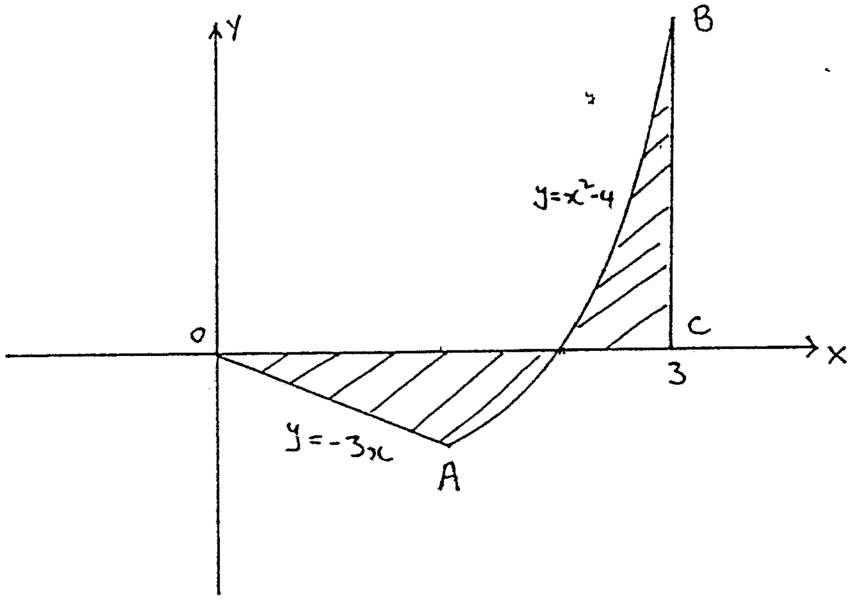
QUESTION SEVEN

(a) A continuous curve $y = f(x)$ has the following properties for the closed interval $a \leq x \leq b$:

$$f(x) > 0, \qquad f'(x) > 0, \qquad f''(x) < 0.$$

Sketch a curve satisfying these conditions.

(b)



The shaded region is bounded by the lines $x = 3$, $y = -3x$ and the x axis and the curve $y = x^2 - 4$.

(i) Show A is the point $(1, -3)$.

(ii) Find the area of the shaded region.

(c) It is assumed that the value, $\$V$, of a car at t months of age decreases at a rate which is proportional to V. For a given car the rate of decrease is given by $\dfrac{dV}{dt} = -\dfrac{1}{100}V$.

(i) Show that $V = V_0 e^{-t/100}$ satisfies this equation.

(ii) If the initial value of a car is \$30 000, find, to the nearest dollar, its value when it is 15 months old.

(iii) Find, to the nearest month, how long it would take for a car to be worth 25% of its initial value.

QUESTION EIGHT

(a) A piece of wire is bent to form the complete boundary of a sector of a circle of radius 4 units. The angle of the sector is θ radians. Another piece of wire, the same length as the first, is bent to form the complete boundary of a square. Show that the total combined area of the sector and the square is $\theta^2 + 12\theta + 4$.

(Exam continues overleaf ...)

(b) Given that $f(x) = x(x-2)^2$:

(i) Show that $f'(x) = 3x^2 - 8x + 4$.

(ii) Find 2 values of x for which $f'(x) = 0$, and give the corresponding values of $f(x)$.

(iii) Determine the nature of the turning points of the curve $y = f(x)$.

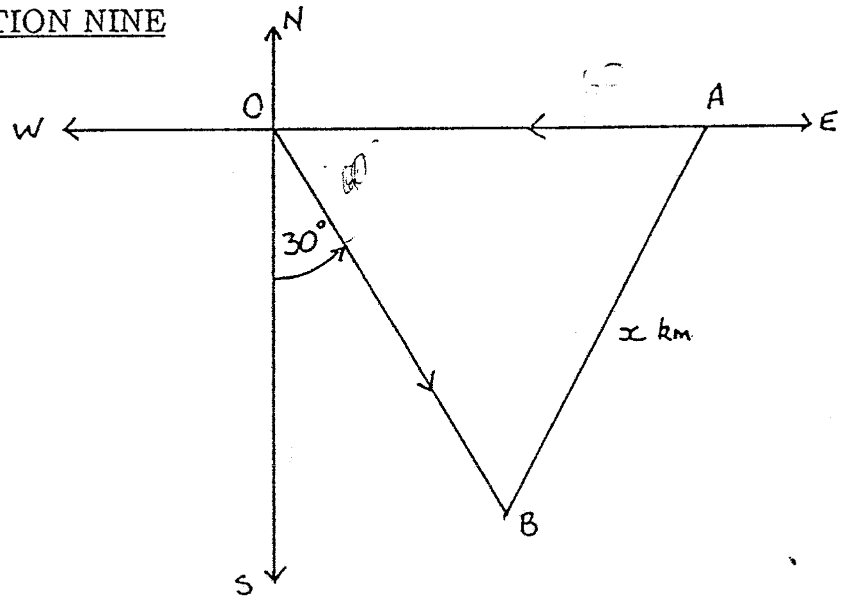
(iv) Find where the curve $y = f(x)$ crosses the x axis.

(v) Sketch the curve $y = f(x)$.

(vi) Use your sketch to solve the inequation $x(x-2)^2 \geq 0$.

QUESTION NINE

(a)



POSITION t HOURS AFTER 7:00AM

Initially, ship A was located 60 km due east of a second ship B whose position was at 0. At 7:00 am ship A began sailing west at a constant 10 km/h, while at the same time ship B began sailing on a bearing of $S30°E$ at a constant speed of 30 km/h.

(i) Show that the distance, x km between the two ships t hours after 7:00 am is given by:

$$x = \sqrt{1300t^2 - 3000t + 3600}.$$

(ii) Hence, find to the nearest minute when the two ships will be closest together.

(b) Consider the geometic series: $\sin^2\theta + \sin^2\theta\cos^2\theta + \sin^2\theta\cos^4\theta + \cdots$ where $0 < \theta < \frac{\pi}{2}$.

(i) Show that the sum, S_n, of the first n terms is given by $S_n = 1 - \cos^{2n}\theta$.

(ii) Explain why this series always has a limiting sum.

(iii) Let S be the limiting sum. Show that $S - S_n = \cos^{2n}\theta$.

(iv) If $\theta = \frac{\pi}{3}$, find the least value of n for which $S - S_n < 10^{-6}$.

JCM

Solutions
1993 2UNIT TRIAL

Suggested marking scheme

Question 1 — 14 marks

(a) $\dfrac{x^2+y^2}{xy} = \dfrac{2.543^2 + 1.761^2}{2.543 + 1.761}$

$= 2.\cancel{22}\,14$ ✓

(b) $25a^2 - 16 = (5a-4)(5a+4)$ ✓

(c) $\dfrac{1}{\sqrt5 - \sqrt2} = \dfrac{1}{\sqrt5-\sqrt2} \times \dfrac{\sqrt5+\sqrt2}{\sqrt5+\sqrt2}$ ✓

$= \dfrac{\sqrt5+\sqrt2}{3}$ ✓

(d) $\dfrac{x}{5} - \dfrac{x+1}{3} = 2$

$3x - 5(x+1) = 30$ ✓

$3x - 5x - 5 = 30$

$-2x = 35$

$x = -17\tfrac12$ ✓

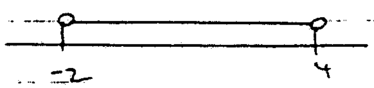
(e) $S = \dfrac{v^2-u^2}{2a}$

$16 = \dfrac{31^2-u^2}{42}$ ✓

$u^2 = -42\times16+31^2$

$u^2 = 289$

$u = 17$ ✓

(f) 

✓

(g) let $\$x$ be the marked price

$70\%\, x = \$35$ ✓

$x = \$35 \div \tfrac{7}{10}$

$= \$50$ ✓

(h) $x = 50+40$ ✓

i.e. $x = 90$

(alt. ∠'s, AB∥CD) ✓ or equivalent.

... equal to ...

QUESTION 2

(a) $G_{AC} = \frac{4}{3}$ ✓

equn. AC : $y-3 = \frac{4}{3}(x-7)$

$3y-9 = 4x-28$

i.e. $4x-3y-19=0$ ✓✓

(b) $G_{BC} = \frac{2}{-1} = -2$

$G_{AD} = \frac{9-1}{-1-4} = -2$ ✓

since $G_{BC} = G_{AD}$ then $BC \parallel AD$ ✓

(c) $G_{DC} = \frac{9-3}{-1-7}$

$= -\frac{3}{4}$ ✓

since $G_{AC} \times G_{DC} = -1$ then $AC \perp DC$ ✓

(d) $AC = \sqrt{(7-4)^2+(3-(-1))^2}$

$= \sqrt{9+16}$

$= 5$ ✓✓

(e) $d = \left| \frac{4\times 8 - 3\times 1 - 19}{\sqrt{4^2+(-3)^2}} \right|$ ✓

$= 2$ ✓

(f) $DC = \sqrt{8^2+6^2}$

$= 10$ ✓

Area of Trapezium = Area $\triangle$ ACD + Area $\triangle$ ACB

$= \frac{1}{2} AC \times DC + \frac{1}{2} AC \times d$ ✓

$= \frac{5}{2}(10+2)$

$= 30u^2$ ✓

~~OR A = $\frac{1}{2}$~~

Allow 1 mark if area formula correctly stated

i.e. $A = \frac{1}{2}(AD+BC) \times$ distance between AD & BC

QUESTION 3 — 14 marks.

(a) (i) $35(7x+3)^4$ ✓

(ii) $-5e^{-5x}$ ✓

(iii) $2x\cos x^2$ ✓

(iv) $1+\ln x$ ✓

(b) (i) $\int(3x+2)\,dx = \frac{3}{2}x^2+2x+c$ ✓ do not penalise omission $+c$

(ii) $\int \frac{1}{3x+2}\,dx = \frac{1}{3}\ln(3x+2)+c$ ✓

(c) (i) $\int_0^1 (1+e^{-x})\,dx = \left[x-e^{-x}\right]_0^1$ ✓

$= (1-e^{-1})-(0-e^0)$

$= 2-e^{-1}$ ✓

(ii) $\int_0^1 \sin\pi x\,dx = -\frac{1}{\pi}\left[\cos\pi x\right]_0^1$ ✓

$= -\frac{1}{\pi}\{\cos\pi-\cos 0\}$

$= -\frac{1}{\pi}(-1-1)$

$= \frac{2}{\pi}$ ✓

(d) ~~let the line be~~ For pt. of intersection

$x-y=0$

$2x+y-1=0$

$3x-1=0$ ✓✓

$x=\frac{1}{3}$

$y=\frac{1}{3}$

equn $y-\frac{1}{3}=-4(x-\frac{1}{3})$ ✓

$y=-4x+\frac{5}{3}$

$12x+3y-5=0$ ✓

By k method: let the line be $(x-y)+k(2x+y-1)=0$ ✓

$\therefore -\frac{(1+2k)}{k-1}=-4$ ✓

$k=\frac{5}{2}$ ✓

equn is $(x-y)+\frac{5}{2}(2x+y-1)=0$

i.e. $12x+3y-5=0$ ✓

QUESTION 4 14 marks

(a) $\frac{\sin\theta}{8.7} = \frac{\sin 80^\circ}{11.6}$ ✓

$\sin\theta = \frac{8.7 \times \sin 80^\circ}{11.6}$

$\theta \doteqdot 47^\circ 37'$ ✓

(b) (i) In Δ's CBE & AFB

$\angle CBE = \angle AFB$ (alt. $\angle$'s BC // AF)

$\angle BCE = \angle FAB$ (opp. $\angle$'s of parm) two only ✓✓✓

$\angle CEB = \angle ABF$ (alt. $\angle$'s, AB // CD)

$\therefore \Delta CBE \,|||\, \Delta AFB$ (AA)

(ii) $\frac{AB}{AF} = \frac{CE}{CB}$ (ratio of matching sides of similar Δ's equal) ✓

$\frac{AB}{15} = \frac{3}{7}$

$AB = 6\frac{3}{7}$ cm ✓

(c) (i) let c.d. be d

$800 + (800 + d) + (800 + 2d) = 7800$ ✓

$3d = 5400$

$d = 1800$ ✓

Three shares are \$800, \$2600, \$4400 ✓

(ii) let common ratio be r

$800 + 800r + 800r^2 = 7800$ ✓

$800r^2 + 800r - 7000 = 0$

$8r^2 + 8r - 70 = 0$

$4r^2 + 4r - 35 = 0$ ✓

$(2r + 7)(2r - 5) = 0$

since $r > 0$ $r = 2\frac{1}{2}$ ✓

shares are \$800, \$2000, \$5000 ✓

QUESTION 5

14 marks

(a) $y = x^2 - 3x + 2$

$\frac{dy}{dx} = 2x - 3$ ✓

when $x = 3$ $\frac{dy}{dx} = 3$ ✓

equn. of tangent is

$y - 2 = 3(x - 3)$ ✓

$y = 3x - 7$

(b) $x^2 - 2x + 25 = 8y$

$x^2 - 2x + 1 = 8y - 24$

$(x-1)^2 = 4 \times 2(y-3)$ ✓

(i) $V(1,3)$ ✓

(ii) $S(1,5)$ ✓

(iii) d: $y = 1$ ✓

(c) $V = \pi \int_{-2}^{2} y^2 \, dx$

$= 2\pi \int_0^2 (x^2-4)^2 \, dx$ ✓

$= 2\pi \int_0^2 x^4 - 8x^2 + 16 \, dx$ ✓

$= 2\pi \left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x \right]_0^2$

$= 34\frac{2}{15}\pi$ ✓

(d) $kx^2 - 2kx - (3k+12) = 0$

$\Delta = (-2k)^2 - 4 \cdot k^-(3k+12)$ ✓

$= 4k^2 + 12k^2 + 48k$

$= 16k^2 + 48k$

$= 16k(k+3)$

For real roots $\Delta \geq 0$ ✓

ie. $16k(k+3) \geq 0$

$k \leq -3$ or $k \geq 0$ ✓

since $k \neq 0$

solution is $k \leq -3$ or $k > 0$ ✓

QUESTION 6

(a) $y = x^3 + ax + b$ 14 marks

$P(1,5)$ is on the curve

$\therefore \quad 5 = 1 + a + b$

$a + b = 4 \quad — ①$

$\frac{dy}{dx} = 3x^2 + a$

since stat. pt. at $x = 1$

$\frac{dy}{dx} = 0$ at $x = 1$

i.e. $0 = 3 + a$

$a = -3$

sub in ① $b = 7$

(b) $x = 2t - \sin(2t-6)$

(i) $v = \frac{dx}{dt} = 2 - 2\cos(2t-6)$

when $t = 3$: $v = 2 - 2\cos 0$

$= 0$

$\therefore$ particle at rest at $t = 3$

(ii) particle at rest when $v = 0$

i.e. $2 - 2\cos(2t-6) = 0$

$\cos(2t-6) = 1$

$2t - 6 = 0, 2\pi, \ldots$

$\therefore$ next at rest when $t = 3 + \pi$

(iii) $a = \frac{dv}{dt} = +4\sin(2t-6)$

when $t = 3 + \frac{\pi}{4}$: $a = 4\sin(6 + \frac{\pi}{2} - 6)$

$= 4\sin\frac{\pi}{2}$

$= 4$

$\int_1^3 f(x)\,dx \doteq \frac{1/2}{3}\left[2 + 4\times 7 + 2\times 12 + 4\times 13 + 14\right]$

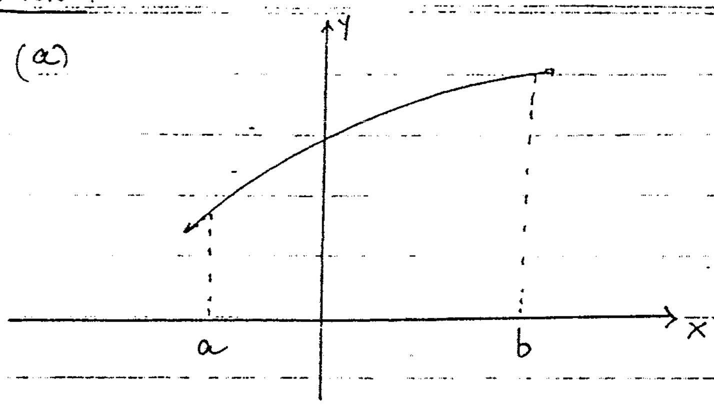
$= 20$

$\int_1^3 f(x)\,dx$ represents the area under the curve $y = f(x)$ bounded by $x = 1$, $x = 3$ and the X axis

QUESTION 7

14 marks

(a)



✓✓

(b) (i) For A: $x^2-4=-3x$ ✓

$x^2+3x-4=0$

$(x+4)(x-1)=0$

$x=-4$ or 1 ✓

when $x=1$, $y=-3$

$\therefore$ A is $(1,-3)$

(ii) $\text{Area} = \left|\int_0^1 -3x\,dx\right| + \left|\int_1^2 x^2-4\,dx\right| + \int_2^3 x^2-4\,dx$ ✓✓

$= \left|\left[\frac{-3x^2}{2}\right]_0^1\right| + \left|\left[\frac{x^3}{3}-4x\right]_1^2\right| + \left[\frac{x^3}{3}-4x\right]_2^3$ ✓

$= \frac{3}{2} + \frac{5}{3} + \frac{7}{3}$

$= 5\frac{1}{2}\,u^2$ ✓

(c) (i) $V = V_0 e^{-t/100}$

$\frac{dV}{dt} = -\frac{1}{100} V_0 e^{-t/100}$

$= -\frac{1}{100} V$ ✓

(ii) when $t=0$ $V=30\,000$ $\therefore V_0 = 30\,000$ ✓

$V = 30\,000 e^{-t/100}$

when $t=15$: $V = 30\,000 e^{-15/100}$ ✓

$= \$25\,821$ ✓

(iii) for $V = 25\% V_0$: $\cdot 25 = e^{-t/100}$ ✓

$\frac{-t}{100} = \ln \cdot 25$

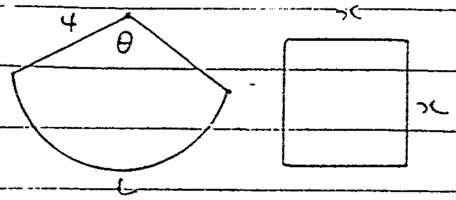
$t = -100 \ln \cdot 25$

$= 139$ months ✓

QUESTION 8

14 marks.

(a)



Sector: $l = r\theta$

$= 4\theta$

Perimeter $= 4\theta + 8$ ✓

Area $= \frac{1}{2}r^2\theta$

$= 8\theta$ ✓

Square: Perimeter $= 4\theta + 8$

side $= \theta + 2$ ✓

area $= (\theta+2)^2$

$= \theta^2 + 4\theta + 4$ ✓

$\therefore$ total area $= \theta^2 + 12\theta + 4$

(b) $f(x) = x(x-2)^2$

(i) $f(x) = x^3 - 4x^2 + 4x$

$f'(x) = 3x^2 - 8x + 4$ ✓

(ii) when $f'(x) = 0$: $3x^2 - 8x + 4 = 0$

$(3x-2)(x-2) = 0$ ✓

$x = \frac{2}{3}$ or 2 ✓

when $x = \frac{2}{3}$ $f(x) = \frac{32}{27}$

when $x = 2$ $f(x) = 0$ ✓

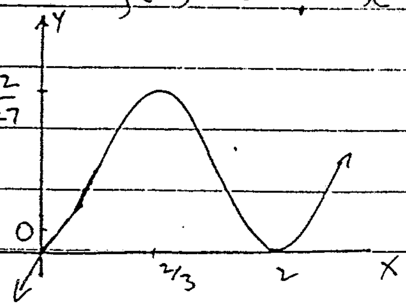
(iii) $f''(x) = 6x - 8$

$f''(\frac{2}{3}) < 0$ $\therefore$ max t.pt at $(\frac{2}{3}, \frac{32}{27})$ ✓✓

$f''(2) > 0$ $\therefore$ min t.pt at $(2, 0)$

(iv) when $f(x) = 0$; $x = 0, 2$ ✓

(v)



✓✓

(vi) for $x(x-2)^2 \geq 0$

$x \geq 0$ ✓

QUESTION 9

14 marks

(a) at t hours

$OA = 60 - 10t$

$OB = 30t$

$\angle AOB = 60°$

(i) By cosine rule: $x^2 = (60-10t)^2 + (30t)^2 - 2.(60-10t).30t\cos 60°$ ✓

$\therefore x^2 = 3600 - 1200t + 100t^2 + 900t^2 - 1800t + 300t^2$

$= 1300t^2 - 3000t + 3600$ ✓

$x = \sqrt{1300t^2 - 3000t + 3600}$

(ii) closest together when x^2 is a minimum ie $\frac{d(x^2)}{dt} = 0$

$\frac{d(x^2)}{dt} = 2600t - 3000$

for min. $2600t - 3000 = 0$ ✓✓

$t = \frac{3000}{2600}$

$= \frac{15}{13}$ hrs

ie when t = 1hr 9min.

since $\frac{d^2(x^2)}{dt^2} > 0$ then minimum at 8:09 a.m. ✓

(b) (i) $S_n = \frac{a(1-r^n)}{1-r} = \frac{\sin^2\theta\left(1-(\cos^2\theta)^n\right)}{1-\cos^2\theta}$ ✓

$= \frac{\sin^2\theta\,(1-\cos^{2n}\theta)}{\sin^2\theta}$

$= 1 - \cos^{2n}\theta$ ✓

(ii) Since $0 < \theta < \frac{\pi}{2}$ then $0 < \cos^2\theta < 1$ ✓

a limiting sum exists if $|r| < 1$

(iii) $S = \frac{a}{1-r} = \frac{\sin^2\theta}{1-\cos^2\theta}$ ✓

$= 1$

$\therefore S - S_n = 1 - (1 - \cos^{2n}\theta)$ ✓

$= \cos^{2n}\theta$

(iv) when $\theta = \pi/3$: $S - S_n = \left(\cos\frac{\pi}{3}\right)^{2n}$

$= \left(\frac{1}{2}\right)^{2n}$ ✓

for $S - S_n < 10^{-6}$: $\left(\frac{1}{2}\right)^{2n} < 10^{-6}$ ✓

$2n\ln\left(\frac{1}{2}\right) < \ln 10^{-6}$

$2n > \frac{\ln 10^{-6}}{\ln \frac{1}{2}}$ ✓✓

$2n > 19.9$—