SYDNEY GRAMMAR SCHOOL — TRIAL HSC EXAMINATION 1997

2/3 UNIT MATHEMATICS FORM VI

Time allowed: 3 hours

Exam date: 4th August, 1997

Instructions:

All questions may be attempted.
All questions are of equal value.
Part marks are shown in boxes in the left margin.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection:

Each question will be collected separately.
Start each question in a new answer booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your candidate number on each answer booklet.

QUESTION THREE (Start a new answer booklet)

Marks

3 (a) Find $\frac{dy}{dx}$ given:

(i) $y = \tan(3x+5)$,

(ii) $y = \log_e(2x+1)$,

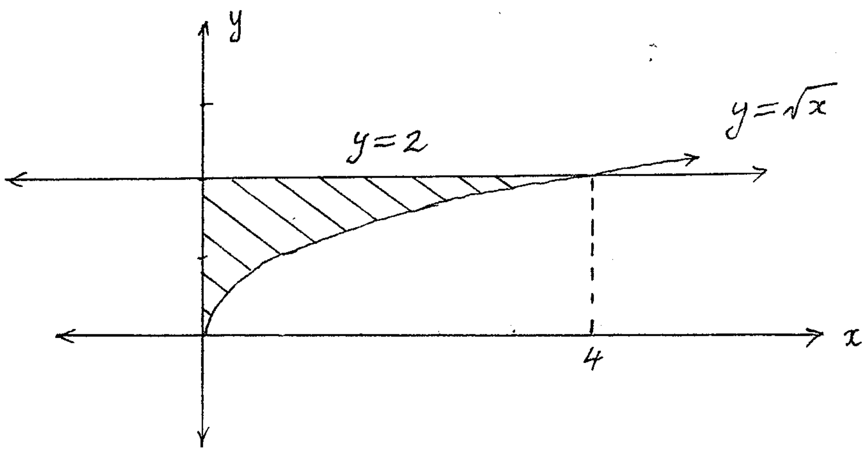
(iii) $y = \frac{e^x}{x}$.

3 (b) (i) Find $\int \cos 2x \, dx$.

(ii) Find $\int_0^1 \frac{2}{x+1} \, dx$.

3 (c) Use the relationship $\tan^2 x + 1 = \sec^2 x$ to evaluate $\int_0^{\frac{\pi}{4}} \tan^2 x \, dx$.

3 (d)



The diagram shows the area bounded by the y axis, the curve $y = \sqrt{x}$ and the line $y = 2$. Find the area of the shaded region.

QUESTION FIVE (Start a new answer booklet)

Marks

$\boxed{4}$ (a) Consider the parabola $4y = x^2 - 4x$.

(i) Show algebraically how the parabola can be expressed in the form

$$(x-2)^2 = 4(y+1).$$

(ii) Write down the co-ordinates of the focus.

(iii) Find the equation of the directrix.

$\boxed{4}$ (b) The nth term of an arithmetic sequence is given by $U_n = 2n - 11$.

(i) Find the first term and the common difference of the sequence.

(ii) Calculate the sum of the series to the fifteenth term.

$\boxed{4}$ (c) If α and β are the roots of the quadratic equation $x^2 - 2x - 1 = 0$ find the value of:

(i) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$,

(ii) $\alpha^2 + \beta^2$.

QUESTION SIX (Start a new answer booklet)

Marks

$\boxed{8}$ (a) Consider the curve $y = 2x^3 + 3x^2 - 12x + 2$.

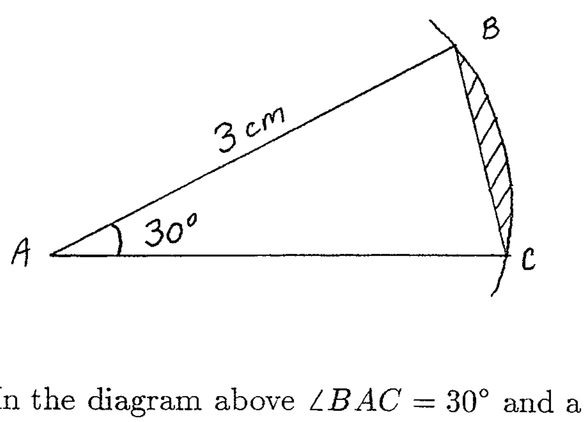
(i) Find all stationary points and determine their nature.

(ii) Find any points of inflexion.

(iii) Sketch the curve for $-3 \le x \le 3$, showing the y-intercept.

(iv) For what values of x is the curve increasing but concave down.

$\boxed{4}$ (b)



In the diagram above $\angle BAC = 30°$ and a circular arc of radius 3 cm and centre A is constructed from B to point C.

(i) Find the area of $\triangle ABC$.

(ii) Calculate the exact area of the shaded segment.

QUESTION NINE (Start a new answer booklet)

Marks

$\boxed{4}$ (a) A particle moves with an acceleration given by $\ddot{x} = \sqrt{t} - \dfrac{1}{\sqrt{t}}$. Initially the velocity is $\frac{4}{3}$ m/s and the displacement is $\frac{4}{3}$ m.

(i) Express the velocity $\dot{x}$ in terms of t.

(ii) Find the displacement x when $t = 1$ sec.

$\boxed{3}$ (b) It costs a manufacturer $\$c$ to make and distribute a calculator. The item sells at $\$x$ each, and the total number sold is given by:

$$n = \frac{a}{x-c} + b(100 - x),$$

where a, b and c are positive constants. Find the selling price that will bring the maximum profit.

$\boxed{5}$ (c) A man borrows \$40 000 from a building society at an interest rate 6% per annum compounded monthly. The loan will be repaid over ten years by equal monthly instalments of $\$Q$. Let $R = 1 + \frac{0 \cdot 06}{12}$.

(i) Show that the total amount owing A after n months is given by:

$$A = 40\,000R^n - Q(1 + R + R^2 + \dots + R^{n-1}).$$

(ii) From this expression, calculate the monthly repayments for the loan to be repaid after ten years.

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QUESTION 1

(a) $\dfrac{2}{\sqrt{3}-2} \times \dfrac{\sqrt{3}+2}{\sqrt{3}-2}$ ✓

$= \dfrac{2(\sqrt{3}+2)}{3-4}$

$= -2\sqrt{3} - 4$ ✓

(b) 30·1 to one decimal place. ✓✓

(c) $(x+1)(x-2) = 0$ ✓✓

OR,

$x^2 - x - 2 = 0$

(d) $|x-1| > 4$

$x - 1 > 4$ or $x - 1 < -4$

$x > 5$ ✓ or $x < -3$ ✓

(e) $\dfrac{x(3-x)}{(3-x)(3+x)}$ ✓ $= \dfrac{x}{3+x}$ ✓

(f) $b = 35° + 76°$ ✓

- supplementary angles.
- external angle of a triangle equals the sum of the two interior opposite angles.

$b = 111°$. ✓

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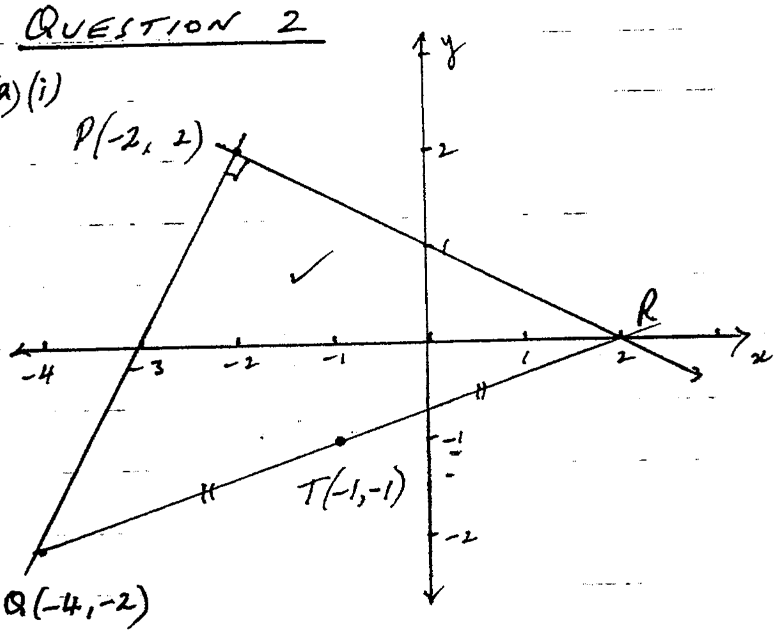
QUESTION 2

(a) (i)



✓

(ii) $m_{PQ} = \dfrac{-2-2}{-4--2}$

$\text{gradient}_{PQ} = 2$. ✓

(iii) $\text{gradient}_{\perp} = -\frac{1}{2}$ ✓ $P(-2, 2)$

$y - y_1 = m(x - x_1)$

$y - 2 = -\frac{1}{2}(x+2)$

$2y - 4 = -x - 2$

$x + 2y - 2 = 0$ ✓

(iv) $x + 2y - 2 = 0$ put $y = 0$

$x = 2$

$R(2, 0)$ ✓

(v) mid-point of the interval joing $Q(-4,-2)$ to $R(2,0)$

$T\left(\frac{-4+2}{2}, \frac{-2+0}{2}\right)$

$T(-1,-1)$ ✓

(d)

$$A = 8 - \int_0^4 x^{1/2}\,.dx \checkmark$$

$$= 8 - \left[\frac{2x^{3/2}}{3}\right]_0^4 \checkmark$$

$$= 8 - \left[\frac{16}{3} - 0\right]$$

$$= 2\tfrac{2}{3} \text{ sq units.} \checkmark$$

OR

$$\text{Area} = \int_0^2 y^2\,.dy$$

$$= \left[\frac{1}{3}y^3\right]_0^2$$

(12)

$$= 2\tfrac{2}{3} \text{ sq units.}$$

QUESTION 4

(a)

(i) $\theta = 27°45'$

$\overline{CL}^2 = 120^2 + 260^2 - 2 \times 120 \times 260 \times \cos 27°45'$ ✓

$CL = 164$ (nearest km) ✓

(ii) let $\alpha = \angle CLA$

$$\frac{\sin\alpha}{120°} = \frac{\sin 27°45'}{164} \checkmark$$

$$\sin\alpha = \frac{120 \sin 27°45'}{164}$$

$$\alpha = 19°56' \checkmark$$

Bearing $= 270° + 19°56'$

$= 289°56'$ (nearest minute) ✓

(b)

(i) In ΔNMO and ΔMLP

$\angle LMP = \angle MON$ (alternate angles of

$\angle LPM = \angle NMP$ (parallel lines ✓ /

$\therefore \Delta MNO \,|||\, \Delta PLM$ (corresponding angles are equal). ✓

(ii) $\dfrac{x+2}{x} = \dfrac{12}{9}$ ✓ (Corresponding sides are in the same ratio.

$12x = 9x + 18$

$3x = 18$

$x = 6$ units. ✓

(c) $2x - 3y + 1 + k(x + y - 3) = 0$

$-2 - 6 + 1 + k(-1 + 2 - 3) = 0$ ✓

$-7 - 2k = 0$

$k = -7/2$ ✓

The equation of line through the pt of intersection:

$$2x - 3y + 1 - \frac{7}{2}(x + y - 3) = 0$$

$$4x - 6y + 2 - 7x - 7y + 21 = 0$$

$$-3x - 13y + 23 = 0$$

$$3x + 13y - 23 = 0 \checkmark$$

(12)

test for a change in concavity.

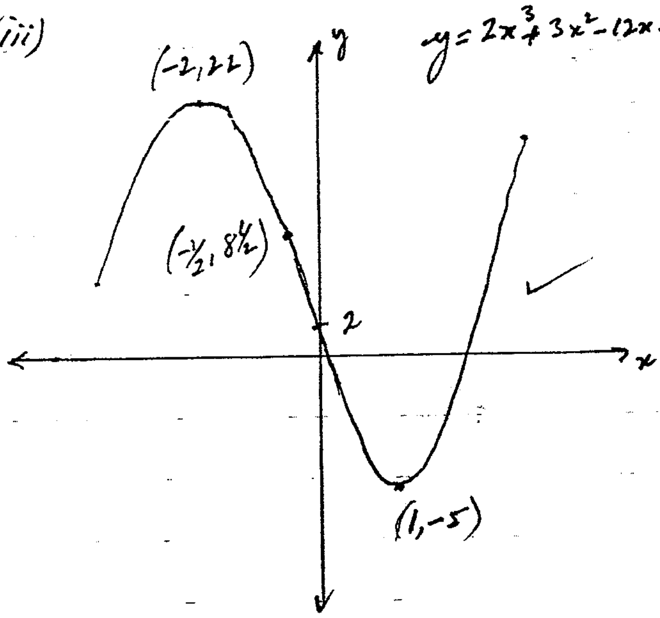
x	-1	$-\frac{1}{2}$	1
$f''(x)$	$-$	0	$+$

R/L $\quad f(-\tfrac{1}{2}+\varepsilon) > 0$

$f(-\tfrac{1}{2}-\varepsilon) < 0$

point of inflexion. $(-\tfrac{1}{2}, 8\tfrac{1}{2})$

(iii)



(iv) Concave down.

$\{x : x < -\tfrac{1}{2}\}$

(b)

(i) Area $\Delta ABC = \tfrac{1}{2} \times 3 \times 3 \sin 30^\circ$

$= \tfrac{9}{4}$ cm².

(ii)

Segment Area = Area Sector − Area ΔABC

$= \tfrac{1}{2} r^2 \theta - \tfrac{9}{4}$

$= \tfrac{1}{2} \times 9 \times \tfrac{\pi}{6} - \tfrac{9}{4}$

(12)

$A = \tfrac{3}{4}(\pi - 3)$ cm²

QUESTION 7

(a)

(i) $x^2 + 6x + k + 8$

$\Delta = b^2 - 4ac$

$\Delta = 36 - 4(k+8)$

$\Delta = 4 - 4k$

(ii) $y = 4x + k$ is a tangent to the parabola $y = -8 - 2x - x^2$ if there is only one point of intersection. $4x + k = -8 - 2x - x^2$

$x^2 + 6x + k + 8 = 0$

one root. $\Delta = 0$

$4 - 4k = 0$

$k = 1$

(b) $27, 18, 12, \ldots$

(i) $a = 27$

$r = \tfrac{2}{3}$

$S_n = \dfrac{a(1-r^n)}{(1-r)}$

$S_8 = \dfrac{27\left(1-\left(\frac{2}{3}\right)^8\right)}{\frac{1}{3}}$

$= 81\left(1 - \dfrac{256}{38}\right)$

$= 77\tfrac{68}{81}$ cm.

(ii) $S_\infty = \dfrac{a}{1-r}$

$= \dfrac{27}{\frac{1}{3}}$

$= 81$ cm.

QUESTION 9

a) $\ddot{x} = t^{1/2} - t^{-1/2}$

(i) $\dot{x} = \frac{2}{3}t^{3/2} - 2t^{1/2} + c_1$ ✓

when $t=0$, $\dot{x} = \frac{4}{3}$

$\frac{4}{3} = c$

$\dot{x} = \frac{2}{3}t^{3/2} - 2t^{1/2} + \frac{4}{3}$ ✓

(ii) $x = \frac{4}{15}t^{5/2} - \frac{4}{3}t^{3/2} + \frac{4}{3}t + c_2$

$x = \frac{4}{3}$ when $t=0$

$c_2 = 4/3$

$x = \frac{4}{15}t^{5/2} - \frac{4}{3}t^{3/2} + \frac{4}{3}t + \frac{4}{3}$ ✓

when $t=1$.

$x = \frac{4}{15} - \frac{4}{3} + \frac{4}{3} + \frac{4}{3}$

$x = 1\frac{3}{5}$ m ✓

b)

Profit / calculator $= x - c$

Total Profit $= (x-c)n$

$P = (x-c)\left\{\frac{a}{(x-c)} + b(100-x)\right\}$

$P = a + b(100-x)(x-c)$ ✓

$P = a + b(100x - 100c - x^2 + xc)$

$P = a + 100bx - 100bc - bx^2 + bcx$

$\frac{dP}{dx} = 100b - 2bx + bc$

$100b - 2bx + bc = 0$

$2bx = 100b + bc$

$2x = 100 + c$

✓ $x = \frac{100+c}{2}$ //

$\frac{d^2P}{dx^2} = -2b.$

$-2b < 0$ local max.

(c)

(i) A_n - the amount owing after n months.

$A_1 = 40000R - Q$ ✓

$A_2 = (40000R - Q)R - Q$

$= 40000R^2 - QR - Q$

$A_3 = (40000R^2 - QR - Q)R - Q$

$= 40,000R^3 - QR^2 - QR - Q$

$= 40,000R^3 - Q(1+R+R^3)$ ✓

$A_n = 40000R^n - Q(1+R+R^2+\dots+R^{n-1})$ ✓

(ii) $A_{120} = 0$

$Q = \frac{40000R^{120}}{(1+R+R^2+\cdots R^{n-1})}$ G.P.

$Q = \frac{40000R^{120}(R-1)}{R^n-1}$ ✓

$Q = \frac{40000(1.005)^{120} \times 0.005}{(1.005)^{120}-1}$

$Q = \$444.08$ / month ✓

(nearest cent)

(12)