SYDNEY GRAMMAR SCHOOL TRIAL EXAMINATION 2002

# FORM VI MATHEMATICS

**Time allowed:** 3 hours (plus 5 minutes reading time) **Exam date:** 7th August 2002

**Instructions:**

All questions may be attempted.
All questions are of equal value.
Part marks are shown in boxes in the right margin.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

**Collection:**

Each question will be collected separately.
Start each question in a new 4-page examination booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your candidate number on each booklet.

**Checklist:**

SGS Examination booklets required — 10 booklets per boy.
121 boys





QUESTION ONE (Start a new examination booklet)

Marks

(a) Evaluate $\dfrac{2}{20+\log_e 2}$ correct to three significant figures. $\boxed{2}$

(b) Find the exact value of $\tan\dfrac{2\pi}{3}$. $\boxed{1}$

(c) Simplify $(1+\tan^2\theta)$. $\boxed{1}$

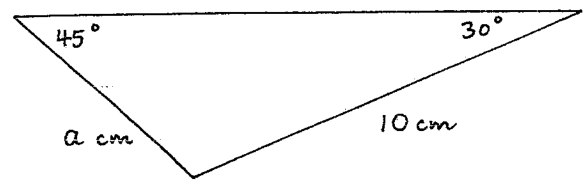
(d) Factorise completely $48x-3x^3$. $\boxed{2}$

(e) Find integers $a$ and $b$ such that $\dfrac{4}{2-\sqrt{3}}=a+b\sqrt{3}$. $\boxed{2}$

(f) Evaluate $\displaystyle\lim_{x\to 0}\frac{x^2+3x}{x}$. $\boxed{1}$

(g) Write down a primitive function of $\dfrac{1}{x+2}$. $\boxed{1}$

(h) $\boxed{2}$

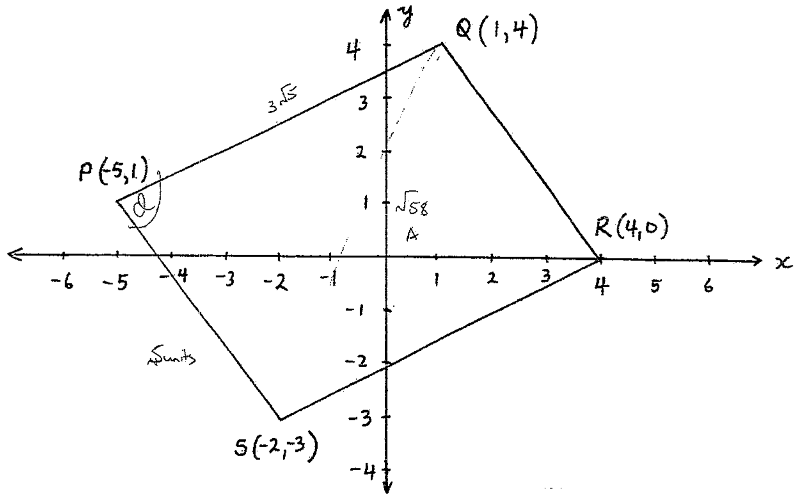



Find the exact value of $a$ in the diagram above.



Exam continues next page ...

QUESTION TWO (Start a new examination booklet)




The quadrilateral $PQRS$ in the diagram above has vertices $P(-5,1)$, $Q(1,4)$, $R(4,0)$ and $S(-2,-3)$.

Marks

(a) Show that the length of the side $PQ$ is $3\sqrt{5}$ units. **1**

(b) Find the gradient of $PQ$ and hence show that its equation is $x - 2y + 7 = 0$. **2**

(c) Show that the perpendicular distance from $S$ to $PQ$ is $\dfrac{11}{\sqrt{5}}$ units. **1**

(d) (i) Show that $PR$ and $QS$ have the same midpoint. **2**

(ii) Hence or otherwise explain why $PQRS$ is a parallelogram. **1**

(e) Find the area of the parallelogram $PQRS$. **1**

(f) Given that $PS = 5$ units and $QS = \sqrt{58}$ units:

(i) find $\angle SPQ$ correct to the nearest degree, **2**

(ii) find the equation of the circle with centre $P(-5,1)$ and radius $PS$. **2**





QUESTION THREE (Start a new examination booklet)

(a) A parabola has equation $x^2 - 4x - 2 = -2y$.

Marks

(i) By completing the square, show that this equation can be written as **1**

$$(x-2)^2 = -2(y-3).$$

(ii) Find the coordinates of the focus. **1**

(iii) Find the coordinates of the point of intersection of the axis of symmetry and the directrix. **1**

(b) Differentiate with respect to $x$:

(i) $\dfrac{1}{2x^2}$, **1**

(ii) $2x\sin x$, **2**

(iii) $\dfrac{\log_e x}{x}$. **2**

(c) Find the equation of the normal to $y = (2x-3)^5$ at the point where $x = 1$. **4**

QUESTION FOUR (Start a new examination booklet)

(a) Evaluate the following definite integrals. Leave your answers in simplest form. **Marks**

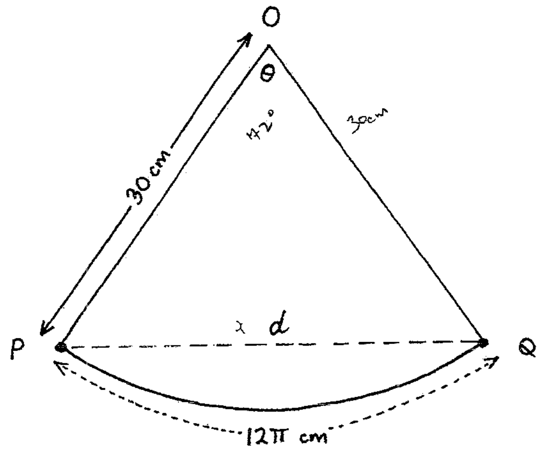
(i) $\displaystyle\int_2^4 \frac{x}{x^2-2}\,dx$, $\boxed{2}$

(ii) $\displaystyle\int_0^{\frac{\pi}{2}} \sec^2\frac{x}{3}\,dx$. $\boxed{3}$

(b) (i) Sketch the graph of $y = e^{-x}$, showing the $y$-intercept, and shade the region bounded by the curve, the $x$-axis and the lines $x = 1$ and $x = -\ln 3$. $\boxed{2}$

(ii) Find the exact area of the shaded region. $\boxed{2}$

(c)




The diagram above shows a pendulum swinging from a fixed point $O$. The pendulum has length 30 cm, and the end of the pendulum swings from $P$ to $Q$ through an arc length of $12\pi$ cm.

(i) Show that the angle $\theta$ through which the pendulum swings is $72°$. $\boxed{1}$

(ii) Find $d$, the shortest distance from $P$ to $Q$, correct to the nearest centimetre. $\boxed{2}$





QUESTION FIVE (Start a new examination booklet)

(a) Let $\alpha$ and $\beta$ be the roots of the equation $2x^2 - 8x - 3 = 0$. Find: **Marks**

(i) $\alpha + \beta$, $\boxed{1}$

(ii) $\dfrac{1}{\alpha\beta}$, $\boxed{1}$

(iii) $\alpha^3\beta^2 + \alpha^2\beta^3$. $\boxed{1}$

(b) Use the substitution $u = x^2 - 3x$ to solve $(x^2-3x)^2 - 2(x^2-3x) - 8 = 0$. $\boxed{3}$

(c) During a busy cricket season, Donald increased his batting score by 4 runs in each successive innings. He was dismissed for 10 runs in his first innings of the season.

(i) How many runs did he score in his 15th innings?

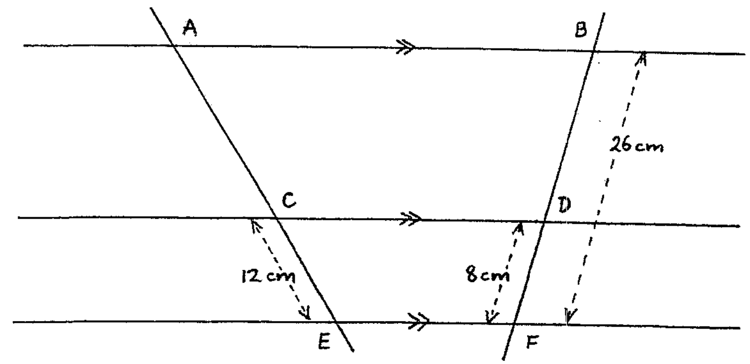
(ii) How many innings did it take for him to score his first century of the season? (NOTE: A century is 100 runs.) $\boxed{1}$

(iii) Show that Donald will score a total of $2n^2 + 8n$ runs in $n$ innings. $\boxed{2}$

(iv) How many innings will it take Donald to pass 1000 runs for the season? $\boxed{2}$

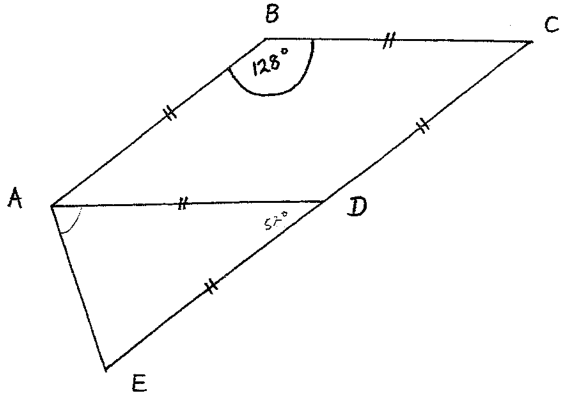
QUESTION SIX (Start a new examination booklet)

(a) **Marks** $\boxed{2}$




In the diagram above, $AB \parallel CD \parallel EF$, $BF = 26$ cm, $DF = 8$ cm and $CE = 12$ cm. Find the length of $AC$, stating your reason.

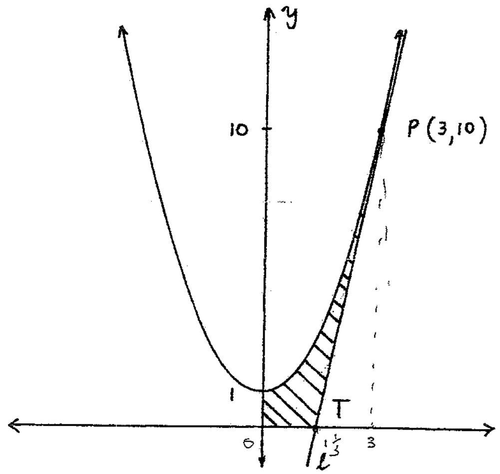
(b)




In the diagram above, $ABCD$ is a rhombus with $\angle ABC = 128°$. The side $CD$ is produced to $E$ so that $DE = CD$. Find $\angle DAE$, giving reasons. $\boxed{3}$

(c) Sketch $y = \dfrac{1}{x-3}$ showing the asymptote and the $y$-intercept. $\boxed{2}$

(d)




The diagram above shows the tangent $\ell$ to the parabola $y = x^2 + 1$ at the point $P(3, 10)$.

(i) Show that the equation of $\ell$ is $y = 6x - 8$. $\boxed{2}$

(ii) $T$ is the point where the tangent crosses the $x$-axis. Show that $T$ has coordinates $(1\frac{1}{3}, 0)$. $\boxed{1}$

(iii) Find the area of the shaded region. $\boxed{2}$





QUESTION SEVEN (Start a new examination booklet)

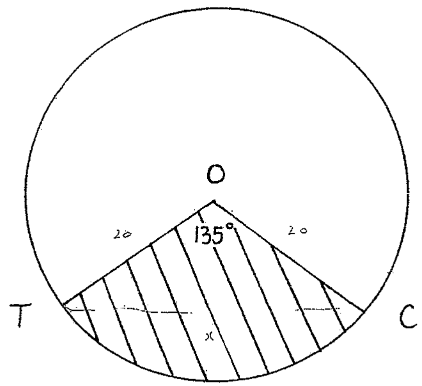
Marks

(a) The curve $y = f(x)$ has a gradient function $f'(x) = 3x^2 - k$, where $k$ is a constant.

(i) Find the value of $k$ if the curve has a stationary point at $N(-1, 3)$. $\boxed{1}$

(ii) Hence find the equation of the curve. $\boxed{2}$

(b)




The diagram above shows a circle with centre $O$. The minor sector $TOC$ has area $150\pi\ \text{cm}^2$, and $\angle TOC = 135°$.

(i) Show that the circle has radius 20 cm. $\boxed{2}$

(ii) Find the area of the minor segment cut off by the chord $TC$, correct to the nearest square centimetre. $\boxed{2}$

(c) Explain why the series $\cos x + \cos^2 x + \cos^3 x + \cdots$ has a limiting sum when $x \neq n\pi$, for some integer $n$. $\boxed{1}$

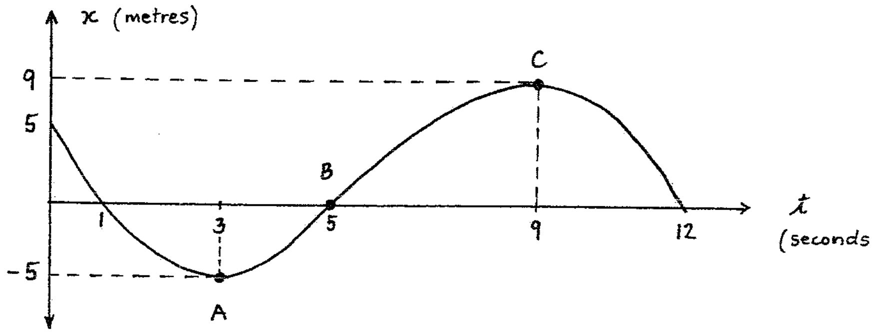
(d) (i) Sketch the graph of $y = 4\cos 2x$ for $-\pi \le x \le \pi$, clearly showing the $x$ and $y$-intercepts. $\boxed{2}$

(ii) On the same set of axes, sketch the graph of $y = |x|$. $\boxed{1}$

(iii) Hence write down the number of solutions of the equation $4\cos 2x - |x| = 0$ for $-\pi \le x \le \pi$. $\boxed{1}$

QUESTION EIGHT (Start a new examination booklet)

(a)




The diagram above shows the displacement–time graph for the first 12 seconds of a particle moving in a straight line. The points $A$ and $C$ are turning points and $B$ is a point of inflexion.

Marks

(i) Where is the particle initially and in what direction is it travelling? [2]

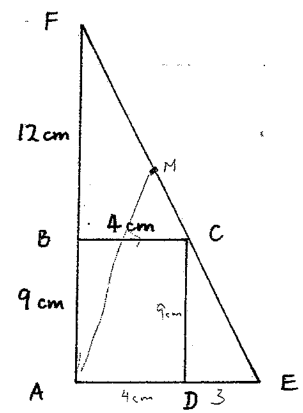
(ii) When does the particle change direction for the first time? [1]

(iii) When does the maximum velocity occur? [1]

(iv) For what values of $t$ is the acceleration negative? [1]

(v) What is the total distance that the particle has travelled in the first 12 seconds? [1]

(b)




$\frac{DE}{9} = \frac{4}{12}$

$DE = \frac{4 \cdot 9}{12}$

$1\frac{1}{3}$

$\frac{4}{3}$

In the diagram above, $ABCD$ is a rectangle, $AB = 9$ cm, $BC = 4$ cm and $BF = 12$ cm.

(i) Prove that $\triangle BFC \parallel\!| \triangle DCE$, giving full reasons. [2]

(ii) Find the area of $\triangle AEF$. [2]

(iii) $M$ lies on $EF$ such that $AM \perp EF$. Find the area of $\triangle AME$. [2]





QUESTION NINE (Start a new examination booklet)

(a) The penguin population $P$ on Paddy Island is decreasing according to the equation $P = Ae^{-kt}$, where $A$ and $k$ are constants and $t$ is time measured in years. On 1st January 1996 there were 13 200 penguins on Paddy Island. By 1st January 2002 the penguins numbered 9900.

Marks

(i) Find the value of $A$ and show that $k = \frac{1}{6}\ln\frac{4}{3}$. [3]

(ii) If the trend continues, how many penguins will be on Paddy Island on 1st January 2010? [2]

(iii) At what rate was the penguin population decreasing on 1st January 2002? [2]

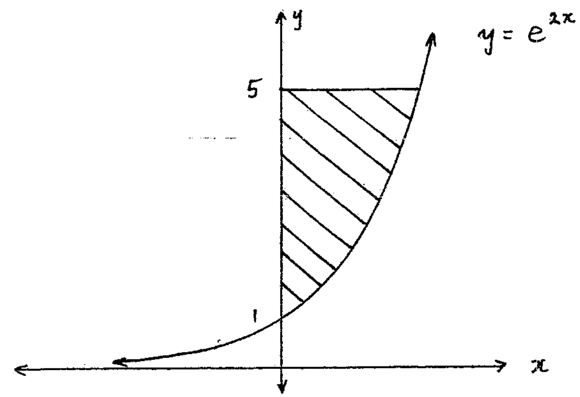
(b) (i) Copy and complete the table below for $f(x) = \left(\log_e \sqrt{x}\right)^2$, calculating each value correct to three decimal places. [1]

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $f(x)$ | 0 | 0·120 | | | |

(ii) Use Simpson's rule with five function values to show that [1]

$$\int_1^5 \left(\log_e \sqrt{x}\right)^2 dx \doteqdot 1{\cdot}22$$

(c)




The diagram above shows the region bounded by the curve $y = e^{2x}$, the $y$-axis and the line $y = 5$.

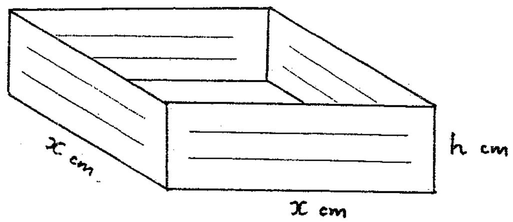
(i) Show that $x = \log_e \sqrt{y}$. [1]

(ii) The shaded area is rotated about the $y$-axis. Write down the definite integral equal to the volume formed. [1]

(iii) Evaluate the volume of the solid of revolution using the approximation in part(b) part(ii), leaving your answer correct to two significant figures. [1]

QUESTION TEN (Start a new examination booklet)

(a) A metal tray, in the shape of a rectangular prism with a square base, is made out of 108 square centimetres of sheet metal. The tray is open at the top.




Let $x$ centimetres be the side length of the base, and let $h$ centimetres be the height.

Marks

(i) Show that $h = \dfrac{108 - x^2}{4x}$. **1**

(ii) Show that the volume $V$ of the tray is given by **1**

$$V = 27x - \frac{x^3}{4}.$$

(iii) Find the maximum volume of the tray. **3**

(b) A particle moves along a straight line so that it is $x$ metres to the right of a fixed point $O$ at time $t$ seconds. The acceleration of the particle is given by

$$\ddot{x} = -\tfrac{2\pi}{3}\sin\tfrac{\pi}{3}t.$$

Initially the particle is travelling with a velocity of 3 m/s.

(i) Find the velocity $v$ as a function of $t$. **1**

(ii) Find the first two times when the particle is stationary. **1**

(iii) How far does the particle travel in the first four seconds? **2**

(c) Consider the quadratic equation in $x$: **3**

$$(p^2 + q^2)x^2 + 2q(p + r)x + (q^2 + r^2) = 0.$$

Find a relation, in simplest form, between $p$, $q$ and $r$ such that the quadratic has real roots.

TCW

(2)

(a) $x^2 - 4x - 2 = -2y$

(i) $x^2 - 4x + 4 = -2y + 2 + 4$

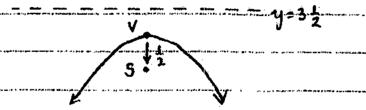
$(x-2)^2 = -2y + 6$ ✓

$(x-2)^2 = -2(y-3)$

(ii) Vertex $= (2,3)$

$a = \frac{1}{2}$

focus $= (2, 2\frac{1}{2})$ ✓




(iii) $(2, 3\frac{1}{2})$ ✓

(b)

(i) $\frac{d}{dx}\left(\frac{1}{2x^2}\right) = \frac{d}{dx}\left(\frac{1}{2}x^{-2}\right)$

$= -x^{-3}$ ✓

$= -\frac{1}{x^3}$

(ii) $\frac{d}{dx}(2x\sin x) = 2\sin x + 2x\cos x$ ✓✓

(iii) $\frac{d}{dx}\left(\frac{\log_e x}{x}\right) = \frac{\frac{1}{x}\times x - 1\times\log_e x}{x^2}$ ✓

$= \frac{1-\log_e x}{x^2}$ ✓

(c) $y = (2x-3)^5$

$\frac{dy}{dx} = 10(2x-3)^4$ ✓

when $x=1$, $y=-1$ ✓

$\frac{dy}{dx} = 10$

gradient of normal $= -\frac{1}{10}$ ✓

equation of normal:

$y+1 = -\frac{1}{10}(x-1)$

$10y + 10 = -x + 1$ ✓

$x + 10y + 9 = 0$

12

(3)

(a)(i) $\int_2^4 \frac{x}{x^2-2} = \frac{1}{2}\int_2^4 \frac{2x}{x^2-2}\,dx$

$= \frac{1}{2}\left[\log_e(x^2-2)\right]_2^4$ ✓

$= \frac{1}{2}(\log_e 14 - \log_e 2)$

$= \frac{1}{2}\log_e 7$ ✓

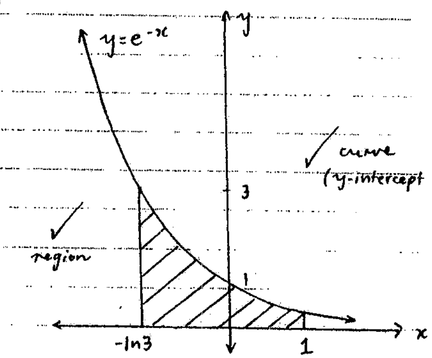
(ii) $\int_0^{\frac{\pi}{2}} \sec^2\frac{x}{3}\,dx = \left[3\tan\frac{x}{3}\right]_0^{\frac{\pi}{2}}$ ✓✓

$= 3\tan\frac{\pi}{6} - 3\tan 0$

$= \frac{3}{\sqrt{3}}$ ✓

$= \sqrt{3}$

(b)(i)




(ii) $A = \int_{-\ln 3}^{1} e^{-x}\,dx$ ✓

$= \left[-e^{-x}\right]_{-\ln 3}^{1}$

$= -e^{-1} + e^{\ln 3}$

$= 3 - \frac{1}{e}$ units² ✓

(c)(i) $l = r\theta$

$\theta = \frac{12\pi}{30}$

$= \frac{2\pi}{5}\times\frac{180^\circ}{\pi}$ ✓

$= 72^\circ$

(ii) $\sin 36^\circ = \frac{\frac{1}{2}d}{30}$ ✓

$d = 60\sin 36^\circ \doteqdot 35$ cm ✓

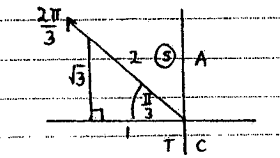
12

(1)

(a) 0·0967 ✓

(3 sig figs) ✓

(b) $\tan\frac{2\pi}{3} = -\sqrt{3}$ ✓




(c) $1 + \tan^2\theta = \sec^2\theta$ ✓

(d) $48x - 3x^3 = 3x(16 - x^2)$ ✓

$= 3x(4-x)(4+x)$ ✓

(e) $\frac{4}{2-\sqrt{3}} = a + b\sqrt{3}$

LHS $= \frac{4}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}}$

$= \frac{8+4\sqrt{3}}{4-3}$ ✓

$= 8 + 4\sqrt{3}$

$\therefore a = 8,\ b = 4$ ✓

(f) $\lim_{x\to 0}\frac{x^2+3x}{x} = \lim_{x\to 0}\frac{x(x+3)}{x}$

$= 3$ ✓

(g) $\log_e(x+2) + C$ ✓

(h) $\frac{a}{\sin 30^\circ} = \frac{10}{\sin 45^\circ}$ ✓

$\frac{a}{\frac{1}{2}} = \frac{10}{\frac{1}{\sqrt{2}}}$

$a = \frac{1}{2}\times 10\times\sqrt{2}$

$a = 5\sqrt{2}$ ✓

12

(2)

(a) $PQ^2 = (1+5)^2 + (4-1)^2$

$= 36 + 9$

$= 45$

$PQ = \sqrt{45}$ ✓

$= 3\sqrt{5}$ units

(b) $m_{PQ} = \frac{4-1}{1+5}$

$= \frac{1}{2}$ ✓

PQ: $y - 1 = \frac{1}{2}(x+5)$

$2y - 2 = x + 5$

$x - 2y + 7 = 0$ ✓

(c) $d = \frac{|-2-2(-3)+7|}{\sqrt{1^2+(-2)^2}}$ ✓

$= \frac{11}{\sqrt{5}}$ units

(d) (i) $M_{PR} = \left(\frac{-5+4}{2}, \frac{1+0}{2}\right)$

$= \left(-\frac{1}{2}, \frac{1}{2}\right)$ ✓

$M_{QS} = \left(\frac{1-2}{2}, \frac{4-3}{2}\right)$

$= \left(-\frac{1}{2}, \frac{1}{2}\right)$ ✓

(ii) The diagonals of PQRS bisect one another. ✓

(e) Area $= bh$

$= 3\sqrt{5}\times\frac{11}{\sqrt{5}}$

$= 33$ units² ✓

(f) (i) $\cos\angle SPQ = \frac{5^2 + (\sqrt{45})^2 - (\sqrt{58})^2}{2\times 3\sqrt{5}\times 5}$

$= \frac{12}{30\sqrt{5}}$

$= \frac{2}{5\sqrt{5}}$

$\angle SPQ \doteqdot 80^\circ$ (nearest degr

(ii) $(x+5)^2 + (y-1)^2 = 25$ ✓

12

(6)

(a) $\dfrac{AC}{CE} = \dfrac{BD}{DF}$ (intercepts on // lines) ✓

$\dfrac{AC}{12} = \dfrac{18}{8}$

$AC = 27$ cm ✓

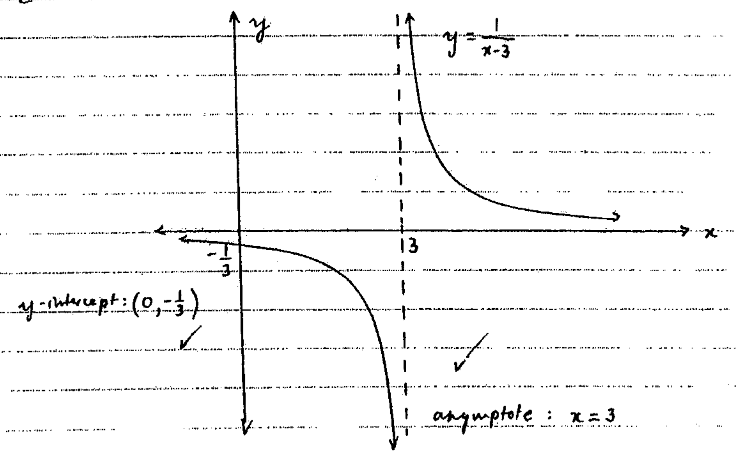
(b) $\angle ADC = 128^\circ$ (opposite ∠'s of rhombus ABCD) ✓

$\angle DAE = \angle DEA$ (base ∠'s of isosceles Δ ADE) ✓

$2\angle DAE = 128^\circ$ (exterior ∠ of Δ ADE)

$\angle DAE = 64^\circ$ ✓

(c)




(d) (i) $y = x^2+1$

$\dfrac{dy}{dx} = 2x$

At $P(3,10)$ $\dfrac{dy}{dx} = 6$ ✓

$l: \quad y - 10 = 6(x-3)$

$y - 10 = 6x - 18$

$y = 6x - 8$ ✓

(ii) when $y = 0$,

$6x - 8 = 0$

$6x = 8$

$x = 1\frac{1}{3}$ ✓

$T = (1\frac{1}{3}, 0)$

(iii) Area $= \int_0^3 (x^2+1)\,dx - \frac{1}{2} \times 10 \times 1\frac{2}{3}$ ✓

$= \left[\dfrac{x^3}{3} + x\right]_0^3 - 8\frac{1}{3}$

$= 12 - 8\frac{1}{3}$

$= 3\frac{2}{3}$ units$^2$ ✓

$\boxed{12}$

(5)

(a) $2x^2 - 8x - 3 = 0$

(i) $\alpha + \beta = \dfrac{8}{2}$

$= 4$ ✓

(ii) $\dfrac{1}{\alpha\beta} = \dfrac{1}{-\frac{3}{2}}$

$= -\dfrac{2}{3}$ ✓

(iii) $\alpha^3\beta^2 + \alpha^2\beta^3 = \alpha^2\beta^2(\alpha+\beta)$

$= (-\frac{3}{2})^2(4)$

$= \frac{9}{4} \times 4$

$= 9$ ✓

(b) $(x^2-3x)^2 - 2(x^2-3x) - 8 = 0$

Let $u = x^2 - 3x$

$u^2 - 2u - 8 = 0$

$(u-4)(u+2) = 0$ ✓

So $x^2 - 3x - 4 = 0$ OR $x^2 - 3x + 2 = 0$

$(x-4)(x+1) = 0$ $\quad (x-1)(x-2) = 0$ ✓ ✓

$x = -1, 1, 2$ or $4$

(c) sequence of scores : $10, 14, 18, \ldots$ [AP: $a = 10$, $d = 4$

(i) score in 15th innings $= a + 14d$

$= 66$ ✓

(ii) when $T_n = 100$, $\quad 10 + 4(n-1) = 100$

$4(n-1) = 90$

$n = 23\frac{1}{2}$ ✓

So it will take Ron 24 innings to score his first centu

(iii) Total runs in $n$ innings $= \frac{n}{2}(2a + (n-1)d)$

$= \frac{n}{2}(20 + 4n - 4)$

$= \frac{n}{2}(16 + 4n)$

$= 2n^2 + 8n$

(iv) when $2n^2 + 8n = 1000$

$n^2 + 4n - 500 = 0$

$n = \dfrac{-4 + \sqrt{2016}}{2}$

$n \doteqdot 20.45$ (2dp)

So Ron will take 21 innings to pass 1000 runs for the season.

**8.**

(a) (i) $x = 5$ metres, travelling in the negative direction. ✓

(ii) $t = 3$ seconds ✓

(iii) $t = 5$ seconds ✓

(iv) $5 < t < 12$ ✓ (or $5 < t \leq 12$, if $t = 12$ is considered part of the domain.)

(v) distance $= 15 + 18 = 33$ metres ✓

(b) (i) In $\Delta$s BFC and DCE

✓ $\angle BFC = \angle DCE$ (corresponding $\angle$'s, $AF \parallel CD$)

✓ $\angle FBC = \angle CDE = 90°$ (given ABCD is a rectangle)

$\therefore \Delta BFC \; ||| \; \Delta DCE$ (AA)

(ii) $\dfrac{DE}{4} = \dfrac{CD}{FB}$ (matching sides of similar $\Delta$s in the same ratio)

$DE = 4 \times \dfrac{9}{12}$

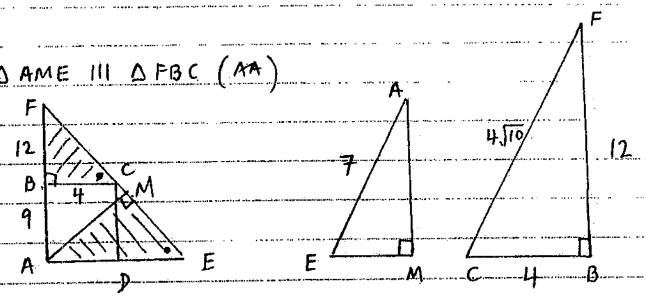
$DE = 3$ cm ✓

Area $\Delta AEF = \frac{1}{2} \times 7 \times 21$

$= 73\frac{1}{2}$ cm$^2$ ✓

(iii) NO REASONS REQUIRED

(iii) $\Delta AME \; ||| \; \Delta FBC$ (AA)




$CF^2 = 16 + 144$

$CF = \sqrt{160} = 4\sqrt{10}$ cm

$\dfrac{AM}{12} = \dfrac{7}{4\sqrt{10}}$ (matching sides of similar $\Delta$s in same ratio)

$AM = \dfrac{21}{\sqrt{10}}$ ✓

and

$\dfrac{EM}{4} = \dfrac{7}{4\sqrt{10}}$

$EM = \dfrac{7}{\sqrt{10}}$

Area of $\Delta AME = \frac{1}{2} \times \dfrac{21}{\sqrt{10}} \times \dfrac{7}{\sqrt{10}}$

$= \dfrac{147}{20}$ ✓

$= 7\dfrac{7}{20}$ cm$^2$

NOTE: THERE ARE A NUMBER OF CORRECT METHODS HERE.

$\boxed{12}$

**7.**

(a) $f'(x) = 3x^2 - k$

(i) $N(-1, 3)$ is a stationary point

$f'(-1) = 0$

$3 - k = 0$

$k = 3$ ✓

(ii) $f'(x) = 3x^2 - 3$

$f(x) = x^3 - 3x + c$ ✓

substitute $N(-1,3)$: $3 = -1 + 3 + c$

$c = 1$

equation: $y = x^3 - 3x + 1$ ✓

(b) (i) sector: $A = \frac{1}{2} r^2 \theta$

$150\pi = \frac{1}{2} r^2 \frac{3}{4}\pi$ ✓

$r^2 = 400$

$r = 20$ cm ✓

(ii) segment: $A = \frac{1}{2} r^2 (\theta - \sin\theta)$

$A = \frac{1}{2} \times 400 \left(\frac{3}{4}\pi - \sin\frac{3\pi}{4}\right)$ ✓

$= 150\pi - 100\sqrt{2}$

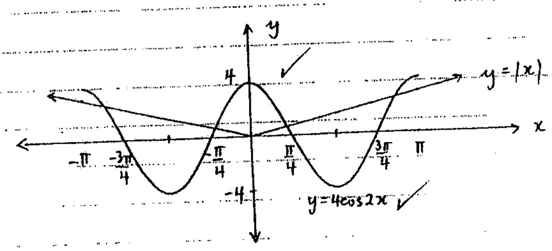
$\doteq 330$ cm$^2$ (nearest square centimetre)

(c) $\cos x + \cos^2 x + \cos^3 x + \cdots$

GP: $r = \cos x$ and $-1 < \cos x < 1$ for $x \neq n\pi$

so the series has a limiting sum since $-1 < r <$

(d) (i) (ii)




(iii) $4\cos 2x - |x| = 0$, $-\pi \leq x \leq \pi$, has 4 solutio

$\boxed{12}$

(10)

(a) (i) $SA = x^2 + 4xh$

$108 = x^2 + 4xh$

$108 - x^2 = 4xh$

$h = \dfrac{108 - x^2}{4x}$ ✓

(ii) $V = lbh$

$V = x^2 h$

$= x^2\left(\dfrac{108 - x^2}{4x}\right)$

$= \dfrac{108x}{4} - \dfrac{x^3}{4}$

$= 27x - \dfrac{x^3}{4}$ cm³

(iii) $\dfrac{dV}{dh} = 27 - \dfrac{3x^2}{4}$ ✓

when $\dfrac{dV}{dh} = 0$, $\dfrac{3x^2}{4} = 27$

$x^2 = 36$

$x = 6$ ✓

$h = \dfrac{108 - 36}{24}$

$h = 3$

$\dfrac{d^2V}{dh^2} = -\dfrac{3x}{2}$

when $x = 6$, $\dfrac{d^2V}{dh^2} = -9$

$< 0$ ✓

so the volume is a maximum when $x = 6$

maximum volume $= 27(6) - \dfrac{216}{4}$

$= 108$ cm³ ✓

(b) (i) $\ddot{x} = -\dfrac{2\pi}{3}\sin\dfrac{\pi}{3}t$, $t \geq 0$

$v = 2\cos\dfrac{\pi}{3}t + c_1$

substitute $v = 3$ when $t = 0$:

$3 = 2 + c_1$

$c_1 = 1$

$\therefore v = 2\cos\dfrac{\pi}{3}t + 1$ ✓

(ii) when $v = 0$, $2\cos\dfrac{\pi}{3}t + 1 = 0$

$\cos\dfrac{\pi}{3}t = -\dfrac{1}{2}$

$\dfrac{\pi}{3}t = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \ldots$

$t = 2, 4, \ldots$ ✓

so the particle is stationary after 2 seconds then 4 seconds.

(9)

(a) (i) $P = Ae^{-kt}$

1/1/1996: when $t = 0$,

$13\,200 = Ae^0$

$A = 13200$ ✓

1/1/2002: when $t = 6$,

$9900 = 13200\,e^{-6k}$ ✓

$e^{-6k} = \dfrac{3}{4}$

$-6k = \log_e \dfrac{3}{4}$

$k = -\dfrac{1}{6}\log_e \dfrac{3}{4}$ ✓

$k = \dfrac{1}{6}\log_e \dfrac{4}{3}$

(ii) 1/1/2010: when $t = 14$, ✓

$P = 13\,200\,e^{-14k}$

$\doteqdot 6746$ ✓

(iii) Rate of decrease: $\dfrac{dP}{dt} = -kP$ ✓

1/1/2002, when $t = 6$, $\dfrac{dP}{dt} = -\dfrac{1}{6}\log_e\dfrac{4}{3} \times 13200 \times e^{-6k}$

$= -\log_e\dfrac{4}{3} \times 2200 \times \dfrac{3}{4}$

$\doteqdot -474.68$ (2dp)

So the penguin population was decreasing at a rate of approximately 475 penguins / year. ✓

(b) (i)

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $f(x)$ | 0 | 0.120 | 0.302 | 0.480 | 0.648 |

✓

(ii) $\displaystyle\int_1^5 \left(\log_e\sqrt{x}\right)^2 dx \doteqdot \frac{1}{3}\left(0 + 4(0.120) + 0.302\right) + \frac{1}{3}\left(0.302 + 4(0.\ldots\right.$ $+ 0.64\ldots$

$\doteqdot 1.22$

(c) (i) $y = e^{2x}$

$\log_e y = 2x$

$x = \dfrac{1}{2}\log_e y$ ✓

$x = \log_e\sqrt{y}$

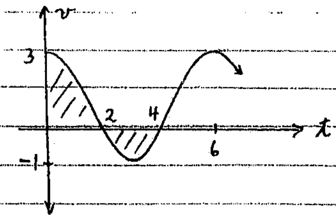
(ii) $V = \pi\displaystyle\int_1^5 \left(\log_e\sqrt{y}\right)^2 dy$ ✓

(iii) $V \doteqdot \pi \times 1.22$

$\doteqdot 3.8$ units (2 sig figs) ✓

12

(iii)




$v = 2\cos\frac{\pi}{3}t + 1$

PERIOD = 6 s

$$\text{Distance} = \int_0^2 v\,dt - \int_2^4 v\,dt$$

$$= \left[\frac{6}{\pi}\sin\frac{\pi}{3}t + t\right]_0^2 - \left[\frac{6}{\pi}\sin\frac{\pi}{3}t + t\right]_2^4 \quad ✓$$

$$= \frac{6}{\pi}\sin\frac{2\pi}{3} + 2 - \frac{6}{\pi}\sin\frac{4\pi}{3} - 4 + \frac{6}{\pi}\sin\frac{2\pi}{3} + 2$$

$$= \frac{6}{\pi}\left(2\sin\frac{2\pi}{3} - \sin\frac{4\pi}{3}\right)$$

$$= \frac{6}{\pi}\left(2 \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right)$$

$$= \frac{9\sqrt{3}}{\pi}\ \text{m} \quad ✓$$

(c) $(p^2+q^2)x^2 + 2q(p+r)x + (q^2+r^2) = 0$

For real roots $\Delta \geq 0$

$$(2q(p+r))^2 - 4(p^2+q^2)(q^2+r^2) \geq 0 \quad ✓$$

$$\cancel{4}q^2(p^2+2pr+r^2) - \cancel{4}(p^2q^2+p^2r^2+q^4+q^2r^2) \geq 0$$

$$\cancel{q^2p^2} + 2q^2pr + \cancel{q^2r^2} - \cancel{p^2q^2} - p^2r^2 - q^4 - \cancel{q^2r^2} \geq 0$$

$$-q^4 + 2q^2pr - p^2r^2 \geq 0 \quad ✓$$

$$q^4 - 2q^2pr + (pr)^2 \leq 0$$

$$(q^2 - pr)^2 \leq 0$$

For real roots $\underline{q^2 = pr}$ ✓

( The roots are real and equal )

$\boxed{12}$

TOTAL $\frac{}{120}$