SYDNEY GRAMMAR SCHOOL

2012 Annual Examination

FORM IV
MATHEMATICS

Wednesday 7th November 2012

General Instructions

- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.

Total — 120 Marks

- All questions may be attempted.
- All necessary working should be shown.
- Start each question on a new page.

Collection

- Write your name, class and master on each page of your answers.
- Staple your answers in a single bundle.
- Write your name and master on this question paper and submit it with your answers.

4A: REP	4B: LRP	4C: MLS	4D: SJE
4E: JMR	4F: SG	4G: PKH	4H: FMW/BR
4I: KWM/DNW	4J: DS		

Checklist

- Writing paper required.
- Candidature — 186 boys

Examiner
DS

QUESTION ONE (12 marks) Start a new page.

(a) Simplify $-3x + 4 - 2x$.

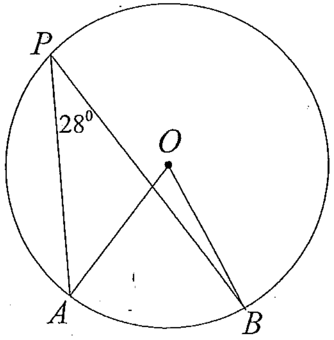
(b) What is the degree of the polynomial $P(x) = 4x^3 + 5x^2 + 6x + 7$?

(c) Factorise $a^2 - 9$.

(d) Write down the solutions of the equation $x(x+1) = 0$.

(e) Expand $\sqrt{3}(5 - \sqrt{3})$.

(f)



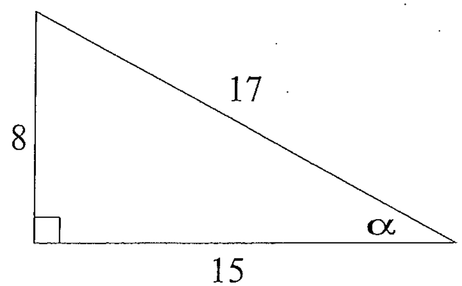
O is the centre of the circle.

Write down the size of $\angle AOB$ in the diagram above, giving a reason for your answer.

(g) Write a^{-3} without a negative index.

(h) How many different car number plates can be made consisting of any 2 letters followed by any 3 digits?

(i)



Write down the value of $\cos\alpha$ in the triangle above.

Exam continues next page ...

QUESTION ONE (Continued)

(j) A square has area $64\,\text{cm}^2$. What is its perimeter?

(k) Write the statement $\log_a x = n$ in exponential form.

(l) In which quadrant does the angle $200°$ lie?

QUESTION TWO (12 marks) Start a new page.

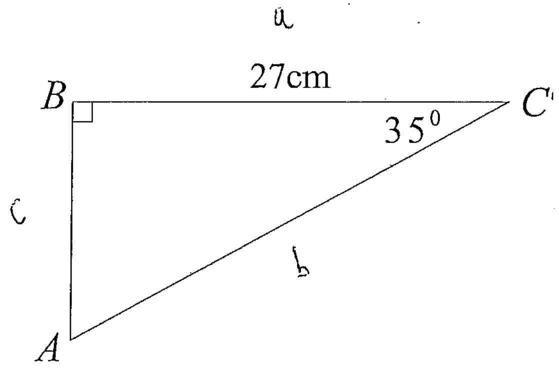
(a) A bag contains 3 yellow buttons, 5 red buttons and 8 blue buttons. A button is chosen at random from the bag. What is the probability that it is NOT red?

(b) Expand $(x+5)^2$.

(c) A pyramid has base area $20\,\text{cm}^2$ and perpendicular height $15\,\text{cm}$. What is its volume?

(d) Find the gradient of the line AB given that A is the point $(9, -2)$ and B is the point $(6, 0)$.

(e)



In the right-angled triangle above, $BC = 27$ cm and $\angle BCA = 35°$. Find the length of side AB correct to the nearest centimetre.

(f) (i) Write down the centre of the circle $(x-3)^2 + (y+4)^2 = 4$.

(ii) Write down the radius of the circle $(x-3)^2 + (y+4)^2 = 4$.

(g) Write the fraction $\dfrac{1}{\sqrt{5}+\sqrt{3}}$ with a rational denominator.

(h) Solve the quadratic equation $(x-1)^2 = 16$.

Exam continues overleaf ...

QUESTION THREE (12 marks) Start a new page.

(a) Evaluate $\log_3 \frac{1}{9}$.

(b) Factorise $x^2 - 4x - 12$.

(c) The line $2x + y = c$ passes through the point $(-1, 7)$. Find the value of c.

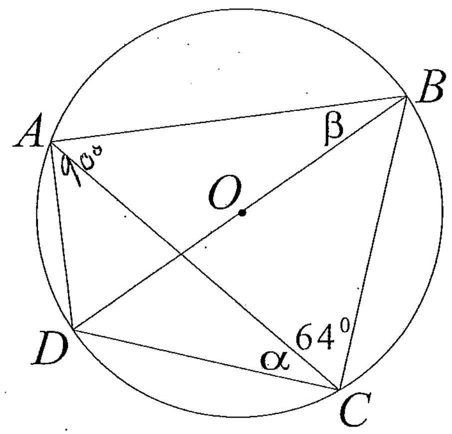
(d) What number must be added to $a^2 - 4a$ to complete the square?

(e) What is the exact value of $\tan 30°$?

(f) Write down the equation of the vertical asymptote of the hyperbola $y = \dfrac{1}{x-2}$.

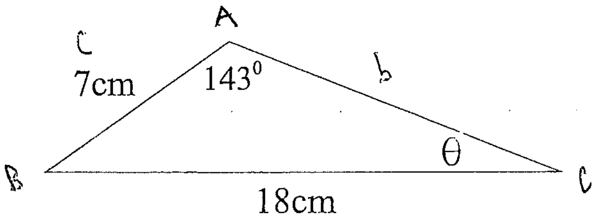
(g) Show that $x - 2$ is a factor of the polynomial $P(x) = x^3 - x^2 + 5x - 14$.

(h)



The diagram above shows a cyclic quadrilateral $ABCD$. The diagonal BD is a diameter of the circle, and $\angle ACB = 64°$. Find the values of α and β, giving reasons for your answers.

(i)



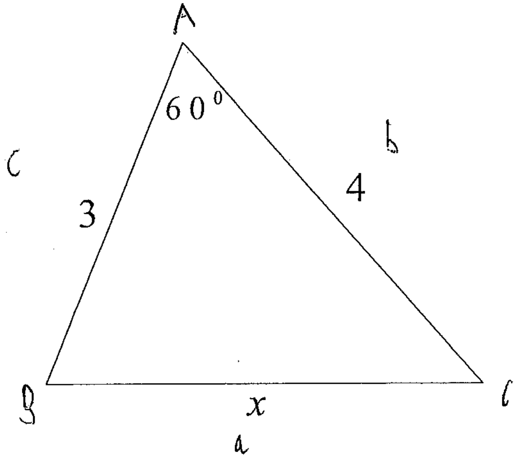
Use the sine rule to find the angle marked θ in the diagram above correct to the nearest tenth of a degree.

Exam continues next page ...

QUESTION FOUR (12 marks) Start a new page.

(a) A line has gradient -2 and passes through the point $(5, -4)$. Find its equation, writing your answer in general form.

(b)



Use the cosine rule to find the exact value of x in the diagram above.

(c) A balloon has volume $1000\,\text{cm}^3$. More air is pumped into the balloon so that its radius is enlarged by a factor of 1·5. What is the volume of the balloon now?

(d) (i) Sketch the graph of $y = \sin x$ for $0° \le x \le 360°$.

(ii) By referring to your sketch, explain why the equation $\sin x = 2$ has no solutions.

(e) A parabola has equation $y = (x-2)(x-8) + 7$. The points $(2, 7)$ and $(k, 7)$, where $k \neq 2$, lie on the parabola.

(i) What is the value of k?

(ii) Find the coordinates of the vertex of the parabola.

QUESTION FIVE (12 marks) Start a new page.

(a) Use the quadratic formula to solve the equation $8x^2 - 20x + 11 = 0$.

(b) Simplify $\dfrac{2x^2 - 3x - 2}{x^2 - 3x + 2}$.

(c) Without a calculator, show that $2 \sin 150° \cos 150° = \sin 300°$.

(d) Simplify $\log_6 48 + \log_6 81 - \log_6 18$.

Exam continues overleaf ...

QUESTION SIX (12 marks) Start a new page.

(a) (i) Complete the following table for the equation $y = \log_2 x$.

x					
y	-1	0	1	2	3

(ii) Use the table of values to help sketch the curve $y = \log_2 x$.

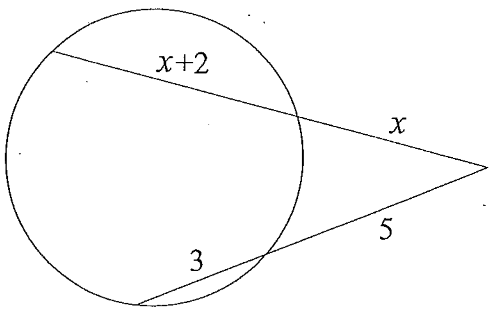
(iii) Explain briefly how the table can be used to sketch the curve $y = 2^x$.

(b) In how many different ways can 5 boys and 2 girls be seated in a row so that the 2 girls are next to each other?

(c) (i) Sketch the graph of the polynomial $y = x(x-3)(x+2)$.

(ii) For what values of x is y positive?

(d)



Find the value of x in the diagram above.

QUESTION SEVEN (12 marks) Start a new page.

(a) Solve $7^x = 700$, writing your solution correct to 4 decimal places.

(b) Solve the equation $\sqrt{3}\tan\alpha + 1 = 0$ for $0^\circ \leq \alpha \leq 360^\circ$.

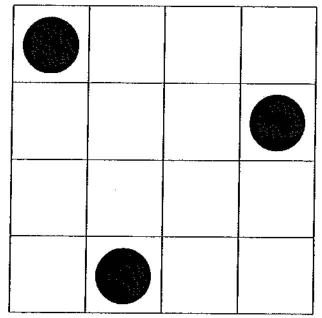
(c) When the polynomial $P(x) = x^3 + kx^2 + 10$ is divided by $x+1$, the remainder is 5. Find the remainder when $P(x)$ is divided by $x - 3$.

(d) Show, by solving the equations simultaneously or otherwise, that the line $y = 2x - 7$ is a tangent to the circle $(x+3)^2 + (y-2)^2 = 45$.

Exam continues next page ...

QUESTION EIGHT (12 marks) Start a new page.

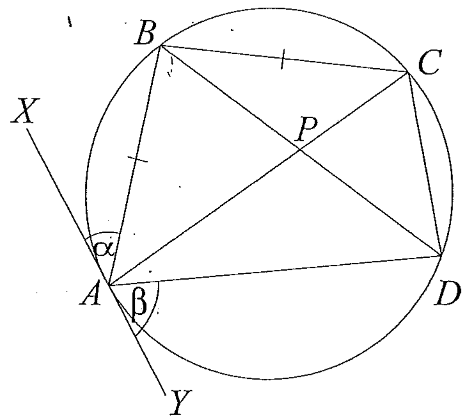
(a)



Three identical coins are randomly placed in different squares of a 4×4 grid. One of the possible outcomes is shown in the diagram above. Determine the probability that no two coins lie in the same row or column.

(b) Two solid metal spheres of radii 1 cm and 2 cm are melted down and then re-cast to form a solid cone with perpendicular height 4 cm. Find the curved surface area of the cone.

(c)



In the diagram above, $ABCD$ is a cyclic quadrilateral with $AB = BC$. Its diagonals AC and BD intersect at P, and XAY is the tangent at A. Let $\angle XAB = \alpha$ and $\angle YAD = \beta$.

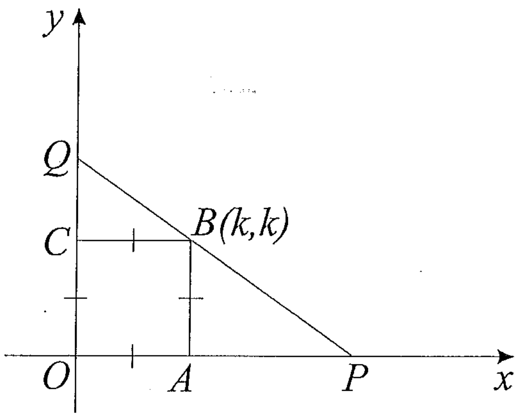
(i) Prove that $\angle BAP = \alpha$.

(ii) Prove that $\angle APD = \angle BCD$.

Exam continues overleaf ...

QUESTION NINE (12 marks) Start a new page.

(a)



In the diagram above, the line QBP passes through the vertex B of the square $OABC$ whose side length is k units. Suppose that the line QBP has gradient m.

(i) Write down the equation of the line QBP in terms of k and m.

(ii) Find the x and y intercepts of the line QBP in terms of k and m.

(iii) Show that $\dfrac{1}{OP}+\dfrac{1}{OQ}=\dfrac{1}{k}$.

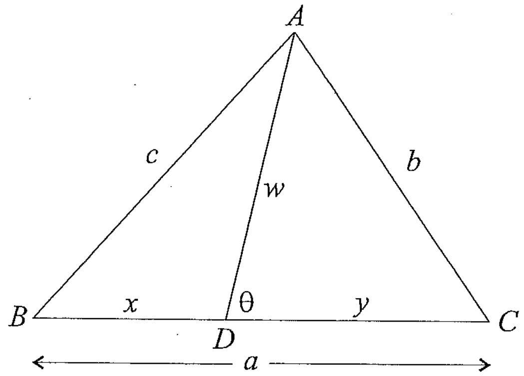
(b) If $x-y=8\sqrt{2}$ and $xy=137$, find the possible values of $x+y$.

(c) Find the points of intersection of the curves $y=\log_{10}\left(x^4\right)$ and $y=\left(\log_{10}x\right)^3$.

Exam continues next page ...

QUESTION TEN (12 marks) Start a new page.

(a)



In the diagram above D is a point on side BC of $\triangle ABC$. Let $BD=x$, $CD=y$, $AD=w$ and $\angle ADC=\theta$. By finding two different expressions for $\cos\theta$, or otherwise, show that:

$$b^2x+c^2y=a(xy+w^2)$$

(b) Consider the cubic equation $x^3-kx+(k+11)=0$.

(i) Use long division to show that $k=x^2+x+1+\dfrac{12}{x-1}$.

(ii) Find all the integer values of k for which the equation has at least one positive integer solution for x.

(c) Bill and Ben have each randomly chosen two of the next n Saturday nights to go to the movies. Find, in terms of n, the probability that there is exactly one of these Saturday nights on which they both go to the movies.

END OF EXAMINATION

Form 4 Annual Exam 2012 **10 × 12 = 120 marks**

(1)(a) $-3x+4-2x = 4-5x$

(b) The degree is 3.

(c) $a^2-9=(a+3)(a-3)$

(d) $x=0$ or $x=-1$

(e) $\sqrt{3}(5-\sqrt{3}) = 5\sqrt{3}-3$

(f) $\angle AOB = 2\times 28^\circ$ (angle at centre is twice angle at circumference)

$= 56^\circ$

must have reason

(g) $a^{-3} = \frac{1}{a^3}$

(h) $26^2 \times 10^3 = 676\ 000$

(i) $\cos\alpha = \frac{15}{17}$

(j) The side is 8 cm.

So the perimeter is 32 cm.

(k) $a^n = x$

(l) The 3rd quadrant.

(2)(a) $P(\text{not red}) = \frac{11}{16}$

(b) $(x+5)^2 = x^2+10x+25$

(c) $V = \frac{1}{3}Ah$

$= \frac{1}{3}\times 20\times 15$

$= 100\text{ cm}^3$

(d) $m = \frac{y_2-y_1}{x_2-x_1}$

$= \frac{0--2}{6-9}$

$= -\frac{2}{3}$

(e) $\frac{AB}{27} = \tan 35^\circ$

$AB = 27\tan 35^\circ$

$= 18\cdot 9056\ldots$

$\doteqdot 19\text{ cm}$

(f) The centre is $(3,-4)$ and the radius is 2 units.

(g) $\frac{1}{\sqrt{5}+\sqrt{3}} \times \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}$

$= \frac{\sqrt{5}-\sqrt{3}}{2}$

(h) $(x-1)^2 = 16$

$x-1 = 4$ or -4

$x = 5$ or -3

(3)(a) $\log_3 \frac{1}{9} = -2$

(b) $x^2-4x-12 = (x-6)(x+2)$

(c) $2(-1)+7=c$

$c=5$

(d) $(-2)^2=4$ must be added.

(e) $\tan 30^\circ = \frac{1}{\sqrt{3}}$

(f) $x=2$

(g) $P(2) = 2^3-2^2+5(2)-14$

$= 8-4+10-14$

$= 0$

So $x-2$ is a factor of $P(x)$.

(h) $\angle BCD = 90^\circ$ (angle in a semicircle)

So $\alpha = 26^\circ$ (adjacent complementary angles)

and $\beta = 26^\circ$ (angles at the circumference standing on the same arc)

The 2nd reason is not essential.

(i) By the sine rule,

$\frac{\sin\theta}{7} = \frac{\sin 143^\circ}{18}$

So $\sin\theta = \frac{7\sin 143^\circ}{18}$

So $\theta = \sin^{-1}\left(\frac{7\sin 143^\circ}{18}\right)$

$= 13\cdot 53499\ldots^\circ$

$\doteqdot 13\cdot 5^\circ$

(4)(a) $y-y_1 = m(x-x_1)$

$y+4 = -2(x-5)$

$y+4 = -2x+10$

$2x+y-6=0$

(b) By the cosine rule,

$x^2 = 3^2+4^2-2(3)(4)\cos 60^\circ$

$= 9+16-24\left(\frac{1}{2}\right)$

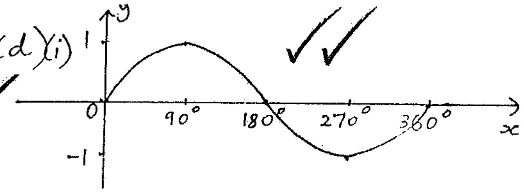
$= 13$

So $x = \sqrt{13}$.

(c) $V = 1000\times 1\cdot 5^3$

$= 3375\text{ cm}^3$

(d)(i)



(ii) The maximum value of $\sin x$ is 1.

So $\sin x$ is never 2.

(e)(i) when $y=7$,

$(x-2)(x-8)=0$.

So $k=8$ since $k\neq 2$.

(ii) The axis of symmetry is the vertical line

$x = \frac{2+8}{2}$

$x = 5$

When $x=5$,

$y = 3\times -3+7$

$= -2$.

So the vertex is $(5,-2)$.

(5)(a) $8x^2 - 20x + 11 = 0$

$\Delta = b^2 - 4ac$
$= (-20)^2 - 4(8)(11)$
$= 400 - 352$
$= 48$

$\therefore x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$
$= \dfrac{20 \pm \sqrt{48}}{16}$
$= \dfrac{20 \pm 4\sqrt{3}}{16}$
$= \dfrac{5 \pm \sqrt{3}}{4}$

(b) $\dfrac{2x^2 - 3x - 2}{x^2 - 3x + 2}$
$= \dfrac{(2x+1)(x-2)}{(x-1)(x-2)}$
$= \dfrac{2x+1}{x-1}$

(c) LHS $= 2\sin 150^\circ \cos 150^\circ$
$= 2 . \sin 30^\circ . -\cos 30^\circ$
$= 2 \times \frac{1}{2} \times -\frac{\sqrt{3}}{2}$
$= -\frac{\sqrt{3}}{2}$

RHS $= \sin 300^\circ$
$= -\sin 60^\circ$
$= -\frac{\sqrt{3}}{2}$

So $2\sin 150^\circ \cos 150^\circ = \sin 300^\circ$.

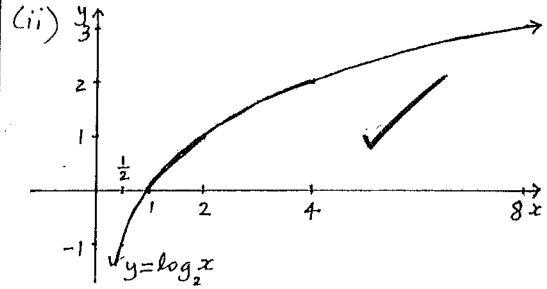
(d) $\log_6 48 + \log_6 81 - \log_6 18$
$= \log_6 \dfrac{48 \times 81}{18}$
$= \log_6 216$
$= \log_6 (6^3)$
$= 3$

(6)(a)(i)

x	$\frac{1}{2}$	1	2	4	8
y	-1	0	1	2	3

(-1 per error)

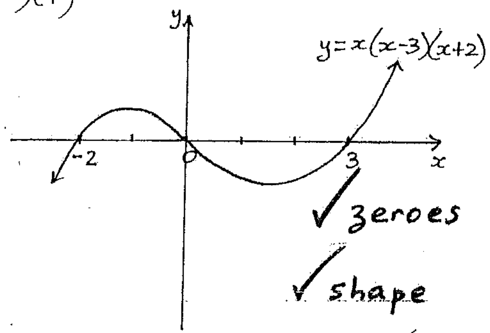
(ii)



(iii) $y = 2^x$ is the same as $x = \log_2 y$.
So just swap the x and y values in (i).

(b) $2! \times 6! = 1440$

(c)(i)



zeroes
shape

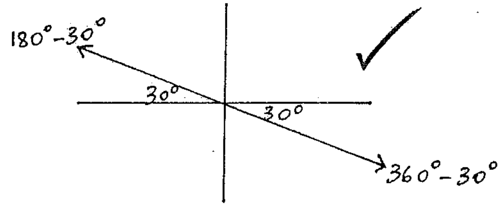
(ii) $-2 < x < 0$ or $x > 3$
(or in words)

(d) $x(2x+2) = 5(8)$ (intersecting secants theorem)
$2x^2 + 2x = 40$
$x^2 + x - 20 = 0$
$(x+5)(x-4) = 0$
$x = 4$ ($x \neq -5$ since $x > 0$)

not essential

(7)(a) $7^x = 700$
$\log_{10}(7^x) = \log_{10} 700$
$x \log_{10} 7 = \log_{10} 700$
$x = \dfrac{\log_{10} 700}{\log_{10} 7}$
$= 3.366589\ldots$
$\doteqdot 3.3666$

(b) $\sqrt{3} \tan \alpha + 1 = 0$
$\tan \alpha = \dfrac{-1}{\sqrt{3}}$



$\alpha = 150^\circ$ or 330°

(c) $P(x) = x^3 + kx^2 + 10$
$P(-1) = 5$
So $(-1)^3 + k(-1)^2 + 10 = 5$
$-1 + k + 10 = 5$
$k = -4$.

So $P(3) = 3^3 - 4(3)^2 + 10$
$= 27 - 36 + 10$
$= 1$.

(d) $2x - y - 7 = 0$ (1)
$(x+3)^2 + (y-2)^2 = 45$ (2)

From (1), $y = 2x - 7$.

Substitute into (2):
$(x+3)^2 + (2x-9)^2 = 45$
$x^2 + 6x + 9 + 4x^2 - 36x + 81 = 45$
$5x^2 - 30x + 45 = 0$
$x^2 - 6x + 9 = 0$
$(x-3)^2 = 0$

$x = 3$ is the only solution
So there is only one point of intersection.
So the line is a tangent.

(8)(a) After the first coin is placed, 15 squares remain of which 9 can be chosen for the second coin. Then 14 squares remain of which 4 can be chosen for the third coin.

So P(no 2 coins in same row or column)

$= \frac{9}{15} \times \frac{4}{14}$

$= \frac{3}{5} \times \frac{2}{7}$

$= \frac{6}{35}$.

(b) $V = \frac{4\pi}{3}(1^3 + 2^3)$

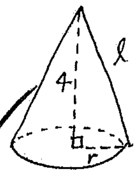
$= \frac{4\pi}{3}(9)$

$= 12\pi \text{ cm}^3$

So $12\pi = \frac{1}{3}\pi r^2 (4)$

$r^2 = 9$

$r = 3$ cm.



So by Pythagoras' theorem,

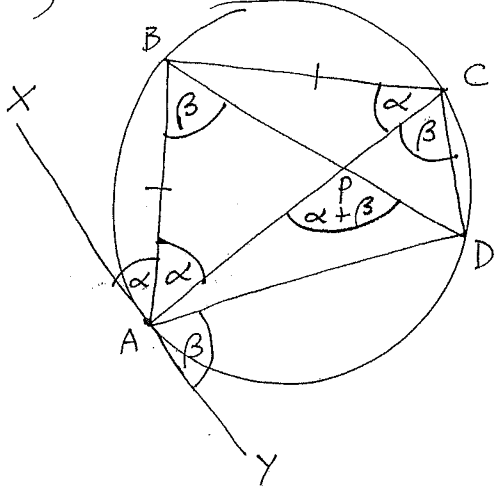
$l = 5$ cm.

So the curved surface has area

$\pi r l = \pi(3)(5)$

$= 15\pi \text{ cm}^2$.

(c)



(i) $\angle BCA = \alpha$ (alternate segment theorem)

So $\angle BAP = \alpha$ (base angles of isosceles ΔABC)

So $\angle BAP = \alpha = \angle XAB$.

(ii) $\angle ACD = \beta$ (alternate segment theorem)

So $\angle ABD = \beta$ (angles at the circumference standing on the same arc)

So $\angle APD = \alpha + \beta$ (exterior angle of ΔABP)

So $\angle APD = \alpha + \beta = \angle BCD$.

(9)(a)(i) $y - k = m(x - k)$

(ii) When $y = 0$,

$-k = m(x - k)$

$x - k = \frac{-k}{m}$

$x = k - \frac{k}{m}$ (x-intercept)

When $x = 0$,

$y - k = -mk$

$y = k - km$ (y-intercept)

(iii) $OP = k - \frac{k}{m}$

$= \frac{km - k}{m}$

and $OQ = k - km$.

So $\frac{1}{OP} + \frac{1}{OQ} = \frac{m}{km-k} + \frac{1}{k-km}$

$= \frac{m}{km-k} - \frac{1}{km-k}$

$= \frac{(m-1)}{k(m-1)}$

$= \frac{1}{k}$.

(b) $x - y = 8\sqrt{2}$

So $(x-y)^2 = 128$.

But $(x+y)^2 = (x-y)^2 + 4xy$

$= 128 + 4(137)$

$= 676$.

So $x + y = 26$ or -26.

(c) $y = \log_{10}(x^4)$ ①

$y = (\log_{10} x)^3$ ②

① is equivalent to

$y = 4\log_{10} x$ (for $x > 0$)

Let $\log_{10} x = u$.

We have $y = u^3$ and $y = 4u$.

So $u^3 = 4u$

$u^3 - 4u = 0$

$u(u^2 - 4) = 0$

$u(u+2)(u-2) = 0$

$u = -2$ or 0 or 2.

So $\log_{10} x = -2$ or 0 or 2.

So $x = \frac{1}{100}$ or 1 or 100.

When $x = 10^{-2}$, $y = (-2)^3$

$= -8$.

When $x = 1$, $y = 0$.

When $x = 10^2$, $y = 2^3$

$= 8$.

So the three points of intersection are $\left(\frac{1}{100}, -8\right)$, $(1, 0)$ and $(100, 8)$.

(10)(a) By the cosine rule in ΔACD,

$$\cos\theta = \frac{w^2+y^2-b^2}{2wy}$$

$\angle ADB = 180° - \theta$ (adjacent angles on a line)

So by the cosine rule in ΔABD,

$$\cos(180°-\theta) = \frac{w^2+x^2-c^2}{2wx},$$

$$\text{so } -\cos\theta = \frac{w^2+x^2-c^2}{2wx}.$$

$$\text{So } \frac{w^2+y^2-b^2}{2wy} = \frac{-w^2-x^2+c^2}{2wx}$$

$$\text{So } w^2x+y^2x-b^2x = -w^2y-x^2y+c^2y$$

$$b^2x+c^2y = w^2x+w^2y+x^2y+y^2x$$

$$b^2x+c^2y = w^2(x+y)+xy(x+y)$$

$$b^2x+c^2y = (x+y)(xy+w^2)$$

$$b^2x+c^2y = a(xy+w^2)$$

(b)(i) $x^3-kx+(k+11)=0$

$$kx-k = x^3+11$$

$$k(x-1) = x^3+11$$

$$k = \frac{x^3+11}{x-1}$$

$$\begin{array}{r} x^2+x+1 \\ x-1\,\big)\,\overline{x^3+0x^2+0x+11} \\ \underline{x^3-x^2} \\ x^2+0x \\ \underline{x^2-x} \\ x+11 \\ \underline{x-1} \\ 12 \end{array}$$

So $x^3+11 = (x-1)(x^2+x+1)+12$.

So $k = \dfrac{x^3+11}{x-1}$

$$= x^2+x+1+\frac{12}{x-1}.$$

(ii) For k to be an integer, we require $\dfrac{12}{x-1}$ to be an integer.

The positive integer values of x for which this happens are

$x = 2, 3, 4, 5, 7, 13$.

When $x=2$, $k = 7+12 = 19$.

When $x=3$, $k = 13+6 = 19$.

When $x=4$, $k = 21+4 = 25$.

When $x=5$, $k = 31+3 = 34$.

When $x=7$, $k = 57+2 = 59$.

When $x=13$, $k = 183+1 = 184$.

So the possible values of k are 19, 25, 34, 59 and 184.

(−1 per error or omission)

(c) There are $\frac{1}{2}n(n-1)$ ways of choosing 2 Saturday nights from n.

Suppose that Bill has chosen his 2 Saturday nights.

Then Ben has 2 choices for the Saturday night that they both go out, and $(n-2)$ choices for his other Saturday night.

So P(one night in common)

$$= \frac{2(n-2)}{\frac{1}{2}n(n-1)}$$

$$= \frac{4(n-2)}{n(n-1)}.$$