1. The displacement x metres of a particle moving in simple harmonic motion is given by
$$x = 3\cos \pi t$$
   (a) What is the period of the oscillation.
   (b) What is the speed v of the particle as it moves through the equilibrium position.
   (c) Show that the acceleration is proportional to the displacement from the equilibrium position.

2. A pebble is projected from the top of a vertical cliff with velocity 20 m/s at an angle of elevation of 30°. The cliff is 40 m high and overlooks a lake. Take the origin O to be the base of the cliff immediately below the point of projection and assume acceleration due to gravity is $10\,\text{m/s}^2$.
   (a) Derive the expressions for for the horizontal and vertical components of the pebbles displacement from O after t seconds.
   (b) Calculate the time which elapses until the pebble hits the lake and the distance of the point of impact from the foot of the cliff.

3. A ball on a spring moves according to $\ddot{x} = -9x$, x is in metres and t is in seconds. Its initial velocity is $\sqrt{27}$ m/s at $x = 1$.
   (a) Find v^2 as a function of x.
   (b) What is the period and amplitude of the motion?
   (c) At what point is the acceleration zero?

4. A steady wind is blowing at 36 km/h. Clouds that are moving horizontally with the wind release raindrops which fall to ground 200 m below. Air resistance may be neglected and let $g = 10\,\text{ms}^{-2}$.
   (a) Find the time taken for a raindrop to reach the ground.
   (b) Find the speed and angle at which a raindrop hits the (horizontal) ground.
   (c) At what angle does a raindrop hit the ground if the windspeed doubles?

5. A The acceleration of a particle P moving in a straight line is given by
$$\frac{d^2x}{dt^2} = 3x(x-4)$$
   where x metres is the displacement from the origin O and t is the time in seconds. Initially the particle is at O and its velocity is $4\sqrt{2}$ m/s.
   (a) Show that $v^2 = 2(x^3 - 6x^2 + 16)$ where v is the velocity of P.
   (b) Calculate the velocity and acceleration of P at $x = 2$.
   (c) In which direction does P move from $x = 2$. give a reason for your answer.
   (d) Briefly describe the motion of P after it moves from $x = 2$.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

6G 2010 MOTION SOLUTIONS

① $x = 3\cos \pi t$ (SHM)

(a) $T = \frac{2\pi}{n} = \frac{2\pi}{\pi} = \underline{2s.}$ ✓

(b) $\dot{x} = \underline{-3\pi \sin \pi t}$ ✓

At equil position $x = 0$

$\therefore \cos \pi t = 0$

$\therefore \pi t = \frac{\pi}{2}, \ldots$

$t = \frac{1}{2}s \ldots$

At $t = \frac{1}{2}$ $V = -3\pi \sin \frac{\pi}{2}$

$= -3\pi$

$\therefore$ speed is $\underline{3\pi \text{ m/s}}$. ✓

(or at equil position ⇒ max speed $= 3\pi$ m/s)

(c) $x = 3\cos \pi t$

$\dot{x} = -3\pi \sin \pi t$

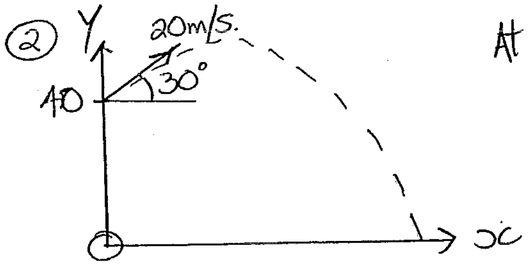
$\ddot{x} = -3\pi^2 \cos \pi t$

$\ddot{x} = -\pi^2 (3\cos \pi t)$ ✓

$\ddot{x} = -\pi^2 x$

$\therefore$ Accelⁿ proportional to displacement with constant $(-\pi^2)$.

②



At $t = 0$ $\dot{x} = 20\cos 30° = \frac{20\sqrt{3}}{2} = 10\sqrt{3}$ m/s

$\dot{y} = 20\sin 30° = 20 \times \frac{1}{2} = 10$ m/s

$x = 0$ m

$y = 40$ m

Horizontal

$\ddot{x} = 0$ ①

$\dot{x} = \int \ddot{x}\, dt$

$= C_1$

sub $t=0$ $\dot{x} = 10\sqrt{3}$ ⇒ $C_1 = 10\sqrt{3}$ ✓

$\underline{\dot{x} = 10\sqrt{3}}$ ②

$x = \int \dot{x}\, dt$

$= 10\sqrt{3}\, t + C_2$

sub $t=0$ $x=0$ ⇒ $C_2 = 0$

$\underline{x = 10\sqrt{3}\, t}$ ③ ✓

Vertical

$\ddot{y} = -10$ ④

$\dot{y} = \int -10\, dt$

$\dot{y} = -10t + C_3$

sub $t=0$ $\dot{y} = 10$ ⇒ $C_3 = 10$ ✓

$\underline{\dot{y} = 10 - 10t}$ ⑤

$y = \int \dot{y}\, dt$

$y = 10t - 5t^2 + C_4$

sub $t=0$ $y = 40$ ⇒ $C_4 = 40$ ✓

$\underline{y = 40 + 10t - 5t^2}$ ⑥

b) Pebble hits lake when $y = 0$

$40 + 10t - 5t^2 = 0$

$8 + 2t - t^2 = 0$

$t^2 - 2t - 8 = 0$

$(t-4)(t+2) = 0$

$t = -2$ OR 4 but $t > 0$ ✓

$\therefore$ hits lake after $\underline{4 \text{ seconds}}$

Distance $x(4) = 10\sqrt{3} \times 4$ ✓

$= \underline{40\sqrt{3} \text{ metres}}$ ✓

③ $\ddot{x} = -9x$ (SHM about origin)

$t = 0,\ x = 1,\ \dot{x} = \sqrt{27}$

$\frac{d}{dx}\left(\frac{1}{2}V^2\right) = -9x$

$\frac{1}{2}V^2 = \int -9x\, dx$

$= \frac{-9x^2}{2} + C$

$V^2 = d - 9x^2$ ✓

sub initial conditions

$27 = d - 9$

$\therefore d = 36$.

$\underline{V^2 = 36 - 9x^2}$ ✓

$V^2 = 9(4 - x^2)$

$V^2 = -9(x^2 - 4)$

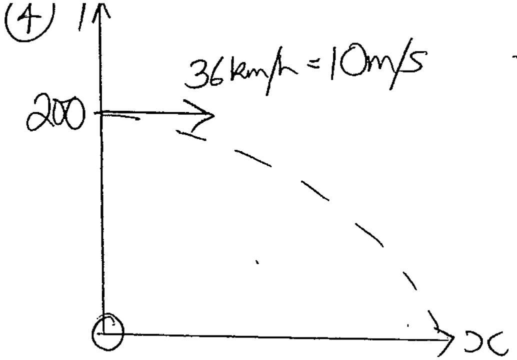
b) $T = \frac{2\pi}{n} = \underline{\frac{2\pi}{3}}$ seconds. ✓ (since $\ddot{x} = -n^2 x$, $n = 3$)

Amp. $V = 0$ ⇒ $x = \pm 2$ ✓

Amplitude $= \underline{2m}$ ✓

c) Accel is zero at centre of motion ie $\underline{x = 0}$ ✓

(4)



Horizontal

$\ddot{x} = 0$

$\dot{x} = \int \ddot{x}\,dt$

$= C_1$ $\quad t=0\ \dot{x}=10 \quad \therefore C_1 = 10$

$\dot{x} = 10$

$x = \int \dot{x}\,dt$

$x = 10t + C_2$ $\quad t=0\ x=0 \quad \therefore C_2 = 0$

$x = 10t$ ✓

Vertical

$\ddot{y} = -10$

$\dot{y} = \int -10\,dt$

$\dot{y} = C_3 - 10t$

$t=0\ \dot{y}=0$

$C_3 = 0$

$\dot{y} = -10t$

$y = \int \dot{y}\,dt$

$y = C_4 - 5t^2$

$t=0\ y=200$

$C_4 = 200$ ✓

$y = 200 - 5t^2$

$= 5(40 - t^2)$

a) Raindrop reaches the ground $y = 0$

$5(40 - t^2) = 0$

$t = \pm\sqrt{40} \quad (t > 0)$ ✓

$= 2\sqrt{10}$ s.

Raindrop hits ground after $\underline{2\sqrt{10}\text{ seconds}}$

b) Need resultant speed on impact

$\dot{x}(2\sqrt{10}) = 10 \qquad \dot{y}(2\sqrt{10}) = -10 \times 2\sqrt{10}$

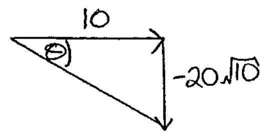
$= -20\sqrt{10}$

$\therefore V^2 = \dot{x}^2 + \dot{y}^2$

$= 100 + 4000$

$= 4100$

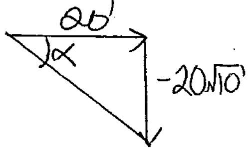
speed of impact $\sqrt{4100} = \underline{10\sqrt{41}\text{ m/s}}$ ✓



$\tan\theta = \dfrac{20\sqrt{10}}{10} = 2\sqrt{10} \qquad \theta = \tan^{-1}(2\sqrt{10})$

angle of impact to horizontal $\underline{\doteqdot 81°1'}$ ✓

c) If windspeed doubles $\quad \dot{x} = 20$ m/s



$\tan\alpha = \dfrac{20\sqrt{10}}{20} = \sqrt{10} \qquad \alpha = \tan^{-1}(\sqrt{10})$

new angle of impact $\underline{\doteqdot 72°27'}$ ✓

(5) $\dfrac{d^2x}{dt^2} = 3x(x-4) \qquad$ (ie NOT SHM)

$t=0 \quad x=0 \quad V = 4\sqrt{2}$

a) $\dfrac{d}{dx}\left(\tfrac{1}{2}V^2\right) = 3x^2 - 12x$

$\tfrac{1}{2}V^2 = \dfrac{3x^3}{3} - 6x^2 + C$ ✓

$V^2 = 2x^3 - 12x^2 + d$

sub initial conds.

$32 = d$

$\therefore V^2 = 2x^3 - 12x^2 + 32$

$\underline{V^2 = 2(x^3 - 6x^2 + 16)}$ ✓ "SHOW"

b) at $x = 2 \quad V^2 = 2(8 - 24 + 16)$

$= 0$ ✓

$\therefore$ $\underline{\text{velocity is zero at } x = 2}$

$\dfrac{d^2x}{dt^2} = 3 \times 2(2-4)$

$= -12$ m/s² ✓

$\therefore$ acceln $\underline{\text{is } -12\text{ m/s}^2 \text{ at } x = 2}$

c) from $x = 2$ velocity zero & acceln negative hence particle moves to the $\underline{\text{left (negative direction)}}$ ✓

d) After $x = 2$ particle moves towards origin accelerating (ie increasing speed to left). At $x = 0$ the acceln is zero but particle continues to move to left. As particle moves to left of origin acceln becomes positive and so slows particle until bringing it to instantaneous rest then back towards O. ✓