Sydney Grammar School

2009 Assessment Examination

FORM VI
MATHEMATICS EXTENSION 1

Friday 29th May 2009

General Instructions

- Writing time — 40 minutes
- Write using black or blue pen.
- Board-approved calculators may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question on a new leaflet.

Structure of the paper

- Total marks — 39
- All three questions may be attempted.
- All three questions are of equal value.

Collection

- Write your name, class and master clearly on each leaflet.
- Hand in the three questions in a single well-ordered pile.
- Hand in a leaflet for each question, even if it has not been attempted.
- If you use a second leaflet for a question, place it inside the first.
- The question papers will be collected separately.

6A: DS	6B: MLS	6C: REP	6D: TCW
6E: PKH	6F: SJE	6G: FMW	6H: BDD

Checklist

- Writing leaflets: 3 per boy.
- Candidature — 111 boys

Examiner
BDD

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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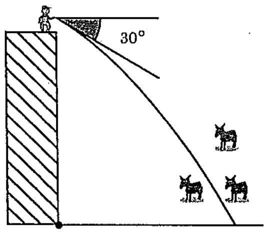
QUESTION ONE (13 marks) Use a separate writing booklet. Marks

(a) Sketch $y = (x-1)(x+3)^2$, showing all intercepts with the axes. (Do not use any calculus to find turning points.) [3]

(b) (i) Use the factor theorem to show that $(x-1)$ is a factor of $P(x) = 6x^3 - 5x^2 - 3x + 2$. [1]

(ii) Factorise $P(x) = 6x^3 - 5x^2 - 3x + 2$ fully as the product of linear factors. [3]

(c)



Borg is harvesting carrots in his clifftop garden and throwing them down to his donkeys in the level field 40 metres below. He launches the carrots at 20 m/s at an angle of 30° below the horizontal, as in the diagram.

Take the base of the cliff as the origin and use the approximation $g \doteqdot 10\,\text{m/s}^2$.

(i) By resolving velocities, find the initial values of $\dot{x}$ and $\dot{y}$. [1]

(ii) Find the horizontal displacement x and the vertical displacement y of the carrots at time t. [3]

(iii) Hence find how far from the base of the cliff the carrots land. [2]

Exam continues next page ...

QUESTION TWO (13 marks) Use a separate writing booklet. Marks

(a) A particle is moving in simple harmonic motion according to the displacement equation $x = 4 - 2\cos 3t$. Assume time t is measured in seconds and displacement x is measured in metres.

(i) State the centre of the motion and its period. [2]

(ii) Between what extremes does the particle move? [1]

(iii) What is the maximum speed of the particle? [1]

(iv) The particle begins from rest. Where is it when it is next stationary? [1]

(v) Find the particle's position at $t = \frac{\pi}{9}$ seconds. [1]

(b) Find the coefficient of x^{13} in the expansion of $\left(2x^3 + \dfrac{3}{x}\right)^7$. [3]

(c) (i) Expand $(1-2x)^5$. [2]

(ii) Hence find the coefficient of x^2 in the expansion of $(3+x)(1-2x)^5$. [2]

QUESTION THREE (13 marks) Use a separate writing booklet. Marks

(a) When $P(x) = 3x^3 + ax^2 + 4x + b$ is divided by $(x-1)$ the remainder is 9, and when divided by $(x+2)$ the remainder is -6. Find a and b. [2]

(b) Suppose that two roots of the cubic equation $x^3 + kx^2 - 36x - 4k^3 = 0$ sum to zero. Assume that $k > 0$. Let the roots be α, $-\alpha$ and β, with $\alpha > 0$.

(i) Write three equations involving the roots and the coefficients of the polynomial. [2]

(ii) Hence find the unknown roots. [2]

(c) Let α, β and γ be the zeroes of the polynomial $2x^3 + 6x^2 + 7x + 10$. [2]

A new monic polynomial $x^3 + bx^2 + cx + d$ is formed with zeroes $\alpha\beta$, $\alpha\gamma$ and $\beta\gamma$.

Find the value of c.

(d) The acceleration of a particle is given by $\ddot{x} = (2 - 3x^2)\,\text{m/s}^2$. The particle is initially at rest at $x = 2$.

(i) Find v^2 as a function of x. [2]

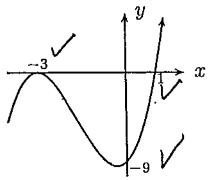
(ii) Factorise v^2 and hence show that the velocity is zero for only one value of x. [2]

(iii) Find an expression for the velocity v. Justify your answer. [1]

END OF EXAMINATION

QUESTION ONE (13 marks)

(a)



(b) (i) $P(1) = 6(1)^3 - 5(1)^2 - 3(1) + 2$
$= 6 - 5 - 3 + 2$
$= 0$

(ii) Hence $(x-1)$ is a factor. By long division:

$$\begin{array}{r|rrrr} & 6x^2 & +\ x & -\ 2 & \\ \hline x-1 & 6x^3 & -\ 5x^2 & -\ 3x & +\ 2 \\ & 6x^3 & -\ 6x^2 & & \\ \hline & & x^2 & -\ 3x & \\ & & x^2 & -\ x & \\ \hline & & & -2x & +\ 2 \\ & & & -2x & +\ 2 \\ \hline & & & & 0 \end{array}$$

Hence $P(x) = (x-1)(6x^2 + x - 2)$
$= (x-1)(3x+2)(2x-1)$

(c) At $t = 0$, $\dot{x} = 20\cos(-30)$
$= 10\sqrt{3}$
$\dot{y} = 20\sin(-30)$
$= -10$

Horizontally	Vertically
$\ddot{x} = 0$	$\ddot{y} = -10$
$\dot{x} = C$	$\dot{y} = -10t + E$
$= 10\sqrt{3}$	$= -10t - 10$
$x = 10\sqrt{3}t + D$	$y = -5t^2 - 10t + F$
$x = 10\sqrt{3}t$	$y = -5t^2 - 10t + 40$

(d) The carrots land when $y = 0$.
Thus $-5t^2 - 10t + 40 = 0$
$t^2 + 2t - 8 = 0$
$(t+4)(t-2) = 0$
$t = 2 \quad (t \geq 0)$

When $t = 2$, $x = 20\sqrt{3}$.
Hence it lands $20\sqrt{3}$ metres from the base of the cliff.

QUESTION TWO (13 marks)

(a) (i) The centre of motion is $x = 4$. The period is $\frac{2\pi}{3}$.

(ii) It moves between $4 - 2$ and $4 + 2$, i.e. $2 \leq x \leq 6$.

(iii) $\dot{x} = 6\sin 3t$, hence the maximum speed is 6 m/s.

(iv) At the other end of its interval of motion, i.e. $x = 6$.

(v) At $t = \frac{\pi}{9}$,
$x = 4 - 2\cos 3t$
$= 4 - 2\cos\frac{\pi}{3}$
$= 4 - 2 \times \frac{1}{2}$
$= 3$

(b) The general term is
$${}^7C_r(2x^3)^{7-r}\left(\frac{3}{x}\right)^r = {}^7C_r 2^{7-r} 3^r x^{21-3r-r}$$
and thus we want the term with $21 - 4r = 13$, that is with $r = 2$. The coefficient is
$${}^7C_2 2^{7-2} 3^2 = \frac{7 \times 6}{2} \times 32 \times 9$$
$$= 6048$$

(c) (i) Using a binomial expansion,
$(1-2x)^5$
$= 1 - 5(2x) + 10(2x)^2 - 10(2x)^3 + 5(2x)^4 - 1(2x)^5$
$= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$

(ii) When multiplied by $(3+x)$, the x^2 term has coefficient
$$3 \times 40 + 1 \times (-10) = 110$$

QUESTION THREE (13 marks)

(a) We have
$P(1) = 3 + a + 4 + b$
$= 9$ (given)
$P(-2) = 3(-2)^3 + a(-2)^2 + 4(-2) + b$
$= -6$ (given)

Simplifying these equations yields
$a + b = 2 \quad (1)$
$4a + b = 26 \quad (2)$

Hence
$3a = 24 \quad (2) - (1)$
$a = 8$

From (1), $b = -6$.

(b) The three equations are:
(sum) $\alpha + (-\alpha) + \beta = -k$
(sum in pairs) $\alpha(-\alpha) + \alpha\beta + (-\alpha)(\beta) = -36$
(product) $\alpha(-\alpha)\beta = -4k^3$

These simplify to
$\beta = -k$
$\alpha^2 = 36$
$-\alpha^2\beta = 4k^3$

Hence $\alpha = 6$ and by the first and third equation
$-36(-k) = 4k^3$
$9 = k^2$

Thus $k = 3$ (recall $k > 0$ is given). The roots are 6, −6, −3.

(c) The value of c is the sum of the roots taken in pairs,
$(\alpha\beta)(\alpha\gamma) + (\alpha\beta)(\beta\gamma) + (\alpha\gamma)(\beta\gamma)$
$= \alpha\beta\gamma(\alpha + \beta + \gamma)$
$= \frac{-10}{2} \times \frac{-6}{2}$
$= 15$

(d) (i)
$$\frac{d(\frac{1}{2}v^2)}{dx} = 2 - 3x^2$$
$$\frac{1}{2}v^2 = 2x - x^3 + C$$
We are given the initial information $v = 0$ when $x = 2$. Hence
$0 = 2(2) - 2^3 + C$
$C = 4$

So the required function is
$$v^2 = 4x - 2x^3 + 8$$

(ii) A known root of this equation is $x = 2$. We can use long division (or another method) to factorise $v^2 = -2(x^3 - 2x - 4)$.

$$\begin{array}{r|rrrr} & x^2 & +\ 2x & +\ 2 & \\ \hline x-2 & x^3 & +\ 0x^2 & -\ 2x & -\ 4 \\ & x^3 & -\ 2x^2 & & \\ \hline & & 2x^2 & -\ 2x & \\ & & 2x^2 & -\ 4x & \\ \hline & & & 2x & -\ 4 \\ & & & 2x & -\ 4 \\ \hline & & & & 0 \end{array}$$

Hence $v^2 = -2(x-2)(x^2 + 2x + 2)$. This quadratic has discriminant
$b^2 - 4ac = 2^2 - 4 \times 2$
$= -4$

hence it does not factorise further. The velocity is only zero at $x = 2$.

(iii) Since $v^2 = -2(x-2)(x^2+2x+2)$, we have
$$v = \pm\sqrt{2(2-x)(x^2+2x+2)}$$
We must decide whether to take the positive or negative root.
Since the velocity is initially 0 at $x = 2$ and the acceleration is initially negative ($-10\,\text{m/s}^2$), the particle is moving backwards i.e. with negative velocity.
By continuity, v will never become positive without first returning to $v = 0$, but this only happens at $x = 2$. Since the particle is moving backwards, it never gets there and v is always negative. Hence
$$v = -\sqrt{2(2-x)(x^2+2x+2)}$$