SAINT IGNATIUS' COLLEGE RIVERVIEW

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2003

MATHEMATICS EXTENSION 2

Time allowed: Three hours
(plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt all questions.
- There are eight questions. All questions are of equal value.
- All necessary working should be shown. Full marks may not be awarded if work is careless or badly arranged.
- The questions are not necessarily arranged in order of difficulty. Candidates are advised to read the whole paper carefully at the start of the examination.
- Approved calculators may be used. A table of standard integrals is provided.
- Each question is to be started in a new booklet. Your number should be written clearly on the cover of each booklet.

This is a trial examination paper only and does not necessarily reflect the content or format of the final Higher School Certificate Examination paper for this subject in the year 2003.

QUESTION 1 (15 marks) Start a new answer booklet.

marks

a) Find $\int \frac{1}{5+4x+x^2}\,dx$ [2]

b) Prove that $\int_4^6 \frac{4\,dt}{(t-1)(t-3)} = 2\log_e\left(\frac{9}{5}\right)$ [3]

c) (i) Use the substitution $x = \frac{2}{3}\sin\alpha$ to prove that $\int_0^{\frac{2}{3}} \sqrt{4-9x^2}\,dx = \frac{\pi}{3}$ [4]

(ii) Hence, or otherwise, find the area enclosed by the ellipse $9x^2 + y^2 = 4$ [2]

d) (i) Given that $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$ prove that $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$, where n is an integer and $n \ge 2$ [3]

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^5 x\,dx$. [1]

QUESTION 2 (15 marks) Start a new answer booklet.

marks

a) Express $\dfrac{2+i}{(1-i)^2}$ in the form $a+bi$. [2]

b) If $z = x+iy$, shade the region represented by $2 < (z+\bar{z}) < 10$ on an Argand diagram. [1]

c) Solve for $z = x+iy$, the equation $z\bar{z} + 2iz = 12+6i$ [4]

d) If $z = x+iy$, show that there are two complex numbers z such that [5]

$|z-2-i| = 1$ and $\arg(z) = \dfrac{\pi}{4}$.

Find the moduli of each of these complex numbers.

e) If the complex number $p = \dfrac{8-2i}{5+3i}$, find $\arg(p)$ in exact form. [3]

QUESTION 3 (15 marks) Start a new answer booklet.

marks

The hyperbola H has the equation $4x^2 - 9y^2 = 36$

a) Write down [4]

(i) Its eccentricity,

(ii) The coordinates of its foci S and S',

(iii) The equation of each directrix

(iv) The equation of each asymptote

b) Sketch the curve for H and include on your diagram the features found in part (a) [2]

c) If $A(x_1, y_1)$ is an arbitrary point on H,

(i) Prove using differential calculus that the equation of the tangent l at A is $4x_1x - 9y_1y = 36$. [3]

(ii) Find the co-ordinates of the point B at which l cuts the x-axis. [1]

(iii) Hence prove that $\dfrac{SA}{S'A} = \dfrac{SB}{S'B}$ [5]

QUESTION 4 (15 marks) Start a new answer booklet.

marks

Consider the function $f(x) = x - 2\sqrt{x}$

a) Determine the domain of f. [1]

b) Find the x-intercepts of the graph of $y = f(x)$. [1]

c) Show that the curve $y = f(x)$ is concave upwards for all positive values of x. [1]

d) Find the co-ordinates of the stationary point and determine its nature. [1]

e) Sketch the graph of $y = f(x)$ clearly showing all essential features. [1]

f) Hence, by considering the graph of $y = f(x)$, sketch the following on separate diagrams, showing the essential features.

(i) $y = |f(x)|$ [2]

(ii) $y = f(x-1)$ [2]

(iii) $y = f(|x|)$ [2]

(iv) $|y| = f(x)$ [2]

(v) $y = \dfrac{1}{f(x)}$ [2]

QUESTION 5 (15 marks) Start a new answer booklet.

marks

a) Show, by the method of mathematical induction that [4]

$1\times 1! + 2\times 2! + 3\times 3! + \ldots\ldots + n\times n! = (n+1)! - 1$, for $n \geq 1$.

b) If (α, β, γ) are the roots of the equation $x^3 + 4x^2 - 3x + 1 = 0$, [3]

find the equation whose roots are α^{-1}, β^{-1}, γ^{-1}.

c) Find all the roots of $3x^3 - 26x^2 + 52x - 24 = 0$, given that the roots are in geometric progression. [4]

d) (i) Prove that for any polynomial $P(x)$, if k is a zero of multiplicity 2, then k is also a zero of $P'(x)$. [2]

(ii) Show that $x = 1$ is a double root of the equation [2]

$x^{2n} - nx^{n+1} + nx^{n-1} - 1 = 0$

QUESTION 6 (15 marks) Start a new answer booklet.

marks

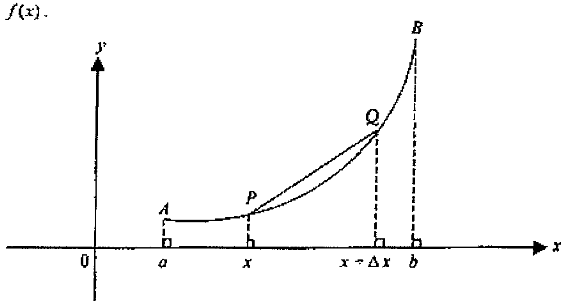
a) The area enclosed by the parabola $y = (x-3)^2$ and the straight line $y = 9$ is rotated about the y–axis. Use the method of cylindrical shells to find the exact volume of the solid formed. [4]

b) The base of a solid is the region bounded by the parabolas $y = 5 - x^2$ and $y = \frac{1}{4}x^2$.
Cross sections by planes perpendicular to the y–axis are semi–circles with their diameters in the base of the solid.

(i) Find the points of intersection of the two parabolas. [1]

(ii) Find (in exact form) the volume of the solid. [4]

c) $P(x, y)$ and $Q(x+\Delta x, y+\Delta y)$ are points on the continuous curve AB, whose equation is $y = f(x)$.



(i) Explain why the length of the arc PQ (ie. Δz) is given by the relation [1]
$\Delta z \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}$

(ii) Hence explain why the length of the arc AB can be expressed as [3]

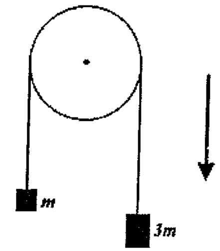
$$AB = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

(iii) Find the length of the arc of the semi–cubical curve $y = x^{\frac{3}{2}}$ between the points $(0,0)$ and $(4,8)$ on the curve. [2]

QUESTION 7 (15 marks) Start a new answer booklet.

marks

a) Particles of mass $3m$ and m are connected by a light inextensible string which passes over a smooth fixed pulley. The string hangs vertically on each side, as shown in the diagram.



The particles are released from rest and move under the influence of gravity. The air resistance on each particle is kv when the speed of the particles is v. The acceleration due to gravity, g, is taken as positive throughout the question and is assumed to be constant. k is a positive constant.

(i) Draw diagram(s) to show the forces acting on each particle. [1]

(ii) Show that the equation of motion is $\dfrac{dv}{dt} = \dfrac{mg - kv}{2m}$ [3]

(iii) Find the terminal or maximum speed of the system, stating your answer in terms of m, g and k. [1]

(iv) Prove that the time elapsed since the beginning of the motion is given by [3]
$t = \dfrac{2m}{k}\ln\left|\dfrac{mg}{mg - kv}\right|$

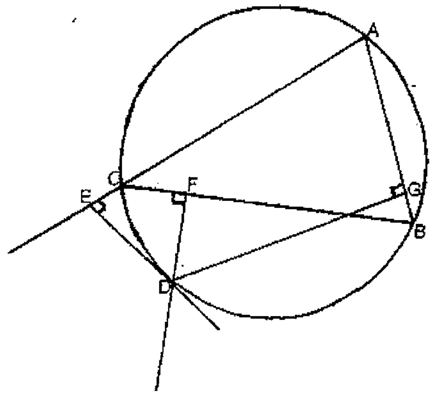
(v) If the bodies have attained a speed equal to half of the terminal speed, show by using the results of (iii) and (iv), that the time elapsed is equal to $\dfrac{V}{g}\ln 4$, where V is the terminal speed. [3]

b) If z_1, z_2, z_3 are the roots of the polynomial equation $P(z) = 0$, where z is a member of the set of complex numbers, solve the system of simultaneous equations [4]

$$z_1 + z_2 + z_3 = 1$$
$$z_1z_2 + z_1z_3 + z_2z_3 = 9$$
$$z_1z_2z_3 = 9$$

QUESTION 8 (15 marks) Start a new answer booklet.

a) marks



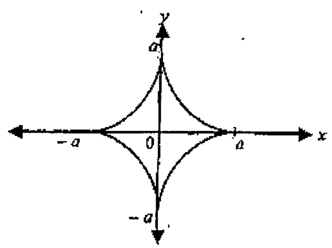
The above diagram shows a triangle ABC inscribed in a circle with D a point on the arc BC. DE is perpendicular to AC produced, DF is perpendicular to BC and DG is perpendicular to AB.

(i) Copy this diagram into your answer booklet.

(ii) Explain why $DECF$ and $DFGB$ are cyclic quadrilaterals. [2]

(iii) Show that the points E, F and G are collinear. [5]

b) Consider the diagram below which represents the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ [8]



Show that the length of a tangent line to the astroid (at any point (l, m) on it) cut off by the coordinate axes is constant.

Aids To Solutions

Question 1

(a)

$$I = \int \frac{1}{5+4x+x^2}\,dx$$

$$= \int \frac{1}{1+(x+2)^2}\,dx$$

$$= \tan^{-1}(x+2) + c$$

(b)

$$I = \int_4^6 \frac{4\,dt}{(t-1)(t-3)}$$

Let $\frac{4}{(t-1)(t-3)} \equiv \frac{A}{t-1} + \frac{B}{t-3}$

Consider $4 = A(t-3) + B(t-1)$

If $t=1$, $\therefore 4 = A(-2)$ ie $A=-2$

If $t=3$, $\therefore 4 = B(2)$ ie $B=2$

$$\therefore I = \int_4^6 \left(\frac{2}{t-3} - \frac{2}{t-1}\right) dt$$

$$I = \Big[2\ln(t-3) - 2\ln(t-1)\Big]_4^6$$

$$I = 2\left[\ln\left(\frac{t-3}{t-1}\right)\right]_4^6$$

$$I = 2\left[\ln\frac{3}{5} - \ln\frac{1}{3}\right]$$

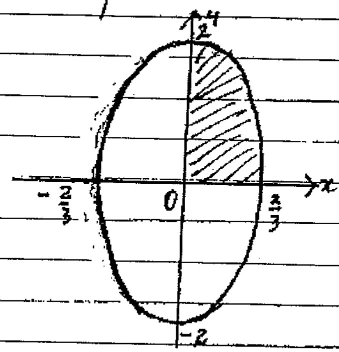
$$I = 2\ln\left(\frac{9}{5}\right)$$

(c) $I = \int_0^{\frac{2}{3}} \sqrt{4-9x^2}\,dx$

$9x^2+y^2=4$

$\frac{x^2}{\left(\frac{2}{3}\right)^2}+\frac{y^2}{2^2}=1$

Ellipse :



* If $x = \frac{2}{3}\sin\alpha$

$\frac{dx}{d\alpha} = \frac{2}{3}\cos\alpha.$

$dx = \frac{2}{3}\cos\alpha\,d\alpha.$

* $\sqrt{4-9x\frac{4}{9}\sin^2\alpha}$

$= 2\sqrt{1-\sin^2\alpha}$

$= 2\sqrt{\cos^2\alpha}$

$= 2\cos\alpha$

consider 4 times the shaded area. The equation for the curve in The first quadrant $y=\sqrt{4-9x^2}$

* $\sin\alpha = \frac{3x}{2}.$

when $x=0$, $\alpha=0$

When $x=\frac{2}{3}$, $\alpha=\frac{\pi}{2}$

Shaded area $A = \int_0^{\frac{2}{3}}\sqrt{4-9x^2}\,dx$

$I = \int_0^{\frac{\pi}{2}} 2\cos\alpha\cdot\frac{2}{3}\cos\alpha\,d\alpha$

$= \frac{4}{3}\int_0^{\pi/2}\cos^2\alpha\,d\alpha$

$= \frac{2}{3}\int_0^{\frac{\pi}{2}}(1+\cos2\alpha)\,d\alpha.$

$= \frac{2}{3}\left[\alpha+\frac{1}{2}\sin2\alpha\right]_0^{\frac{\pi}{2}}$

$= \frac{2}{3}\left(\frac{\pi}{2}\right) = \frac{\pi}{3}$

Required area of Ellipse :

$A = 4I$

$= 4\times\frac{\pi}{3}$

$= \frac{4\pi}{3}$ units^2.

(d)(i) $I_n = \int_0^{\frac{\pi}{2}}\cos^n x\,dx$

$I_n = \int_0^{\frac{\pi}{2}}\cos x\cos^{n-1}x\,dx$

Now using integration by parts:

If $u = \cos^{n-1}x$

then $u' = (n-1)(\cos^{n-2}x)\times -\sin x$

$= -(n-1)\sin x\cos^{n-2}x$

and $v' = \cos x$

Then $v = \sin x$

$\therefore I_n = \left[\sin x\cos^{n-1}x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\sin x\times-(n-1)\sin x\cos^{n-2}x\,dx$

$I_n = 0 + (n-1)\int_0^{\frac{\pi}{2}}\sin^2x\cos^{n-2}x\,dx$

$I_n = (n-1)\int_0^{\frac{\pi}{2}}(1-\cos^2x)\cos^{n-2}x\,dx$

$I_n = (n-1)\left[\int_0^{\frac{\pi}{2}}\cos^{n-2}x\,dx - \int_0^{\frac{\pi}{2}}\cos^n x\,dx\right]$

$I_n = (n-1)I_{n-2} - (n-1)I_n$

$I_n + (n-1)I_n = (n-1)I_{n-2}$

$nI_n = (n-1)I_{n-2}$

$I_n = \left(\frac{n-1}{n}\right)I_{n-2}$

(ii) $I_5 = \int_0^{\frac{\pi}{2}}\cos^5x\,dx$

$= \frac{4}{5}I_3$

$= \frac{4}{5}\times\frac{2}{3}I_1$

$= \frac{8}{15}\int_0^{\frac{\pi}{2}}\cos x\,dx$

$= \frac{8}{15}\left[\sin x\right]_0^{\frac{\pi}{2}}$

$= \frac{8}{15}(1-0)$

$= \frac{8}{15}$

Question 2

(a) $\frac{2+i}{(1-i)^2}$

$= \frac{2+i}{-2i}$

$= \frac{2+i}{-2i} \times \frac{i}{i}$

$= \frac{i^2+2i}{-2i^2}$

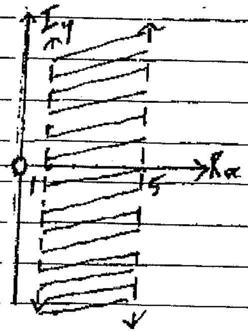
$= \frac{-1+2i}{2}$

$= -\frac{1}{2} + i$

(b) $2 < (z+\bar{z}) \le 10$

Now $z + \bar{z} = 2x$

so $1 < x \le 5$



(c) $z\bar{z} + 2iz = 12+6i$

$x^2+y^2+2i(x+iy) = 12+6i$

$x^2+y^2-2y+2xi = 12+6i$

~~Equate real & imaginary parts~~

$x^2+y^2-2y = 12$ --- (1)

$2x = 6$ --- (2)

$\therefore x = 3$

Sub in (1)

$9 + y^2 - 2y = 12$

$y^2 - 2y - 3 = 0$

$(y-3)(y+1) = 0$

$y = 3$ or -1

$\therefore$ Solution is:

$z = 3+3i$

or

$z = 3-i$

(d) For $|z-2-i| = 1$

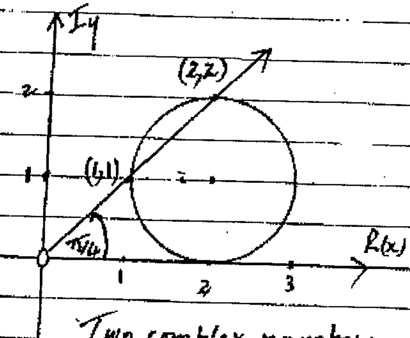
$|z-(2+i)| = 1$

circle centre $(2+i)$

radius $r = 1$

and $\arg(z) = \frac{\pi}{4}$

a ray excluding origin in 1st quadrant at an angle of $\frac{\pi}{4}$ to +ve real axis



Two complex numbers $(1+i)$ and $(2+2i)$

$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$

$|2+2i| = \sqrt{2^2+2^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

Note cartesian equations are

$(x-2)^2 + (y-1)^2 = 1$

and $y = x$

(e) $p = \frac{8-2i}{5+3i}$

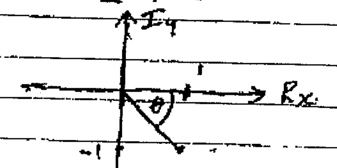
$p = \frac{8-2i}{5+3i} \times \frac{5-3i}{5-3i}$

$p = \frac{40-24i-10i+6i^2}{25-9i^2}$

$= \frac{34-34i}{25+9}$

$= \frac{34(1-i)}{34}$

$= 1-i$



$\arg(1-i) = -\tan^{-1}\frac{1}{1}$

$= -\frac{\pi}{4}$

Question 3 — Aids To Solutions

(a) $4x^2-9y^2=36$

$\frac{x^2}{9}-\frac{y^2}{4}=1$

$a=3,\ b=2$

(a)(i) for a hyperbola

$b^2=a^2(e^2-1)$

$\frac{4}{9}=e^2-1$

$e^2=\frac{4}{9}+\frac{9}{9}$

$e^2=\frac{13}{9}$

$e=\frac{\sqrt{13}}{3}$ $(e>1)$

(ii) Foci S and S' are $(\pm ae,0)$

$\therefore$ S is $(\sqrt{13},0)$

and S' is $(-\sqrt{13},0)$

(iii) Directrices $x=\pm\frac{a}{e}$

$x=\pm\frac{3}{\frac{\sqrt{13}}{3}}$

$x=\pm\frac{9}{\sqrt{13}}$

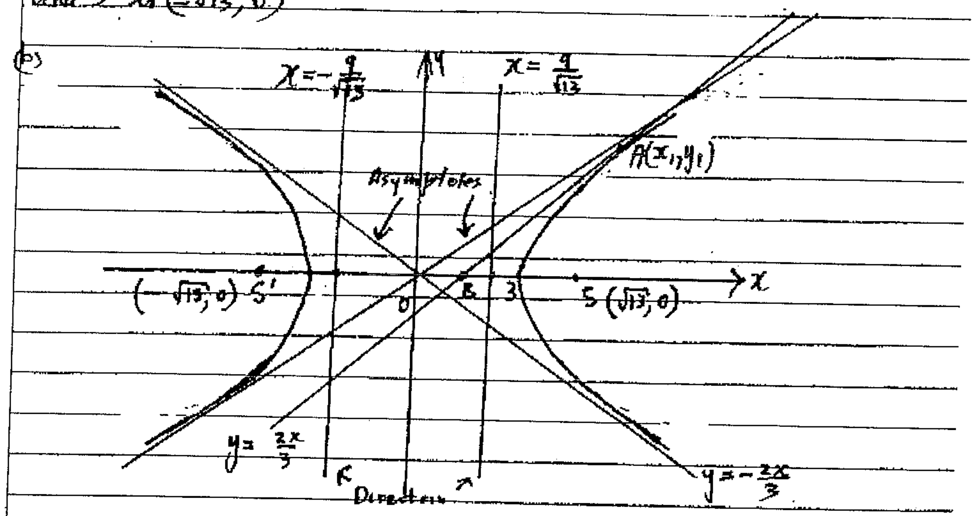
(iv) Asymptotes: $\frac{x^2}{9}-\frac{y^2}{4}=0$

$\frac{y^2}{4}=\frac{x^2}{9}$

$y^2=\frac{4x^2}{9}$

$y=\pm\frac{2x}{3}$

(b)



Aids To Solutions

(c)(i) Equation of tangent at $A(x_1,y_1)$

$4x^2-9y^2=36$ --- (1)

$8x-18y\frac{dy}{dx}=0$

$9y\frac{dy}{dx}=4x$

$\frac{dy}{dx}=\frac{4x}{9y}$

at (x_1,y_1) $\frac{dy}{dx}=\frac{4x_1}{9y_1}$

Gradient of tangent $m=\frac{4x_1}{9y_1}$

Equation of tangent:

$y-y_1=m(x-x_1)$

$y-y_1=\frac{4x_1}{9y_1}(x-x_1)$

$9y_1y-9y_1^2=4x_1x-4x_1^2$

$4x_1x-9y_1y=4x_1^2-9y_1^2$

But $4x_1^2-9y_1^2=36$ because (x_1,y_1) lies on the hyperbola. (2)

$\therefore 4x_1x-9y_1y=36$ --- (3)

(ii) Co-ordinates B: Let $y=0$ in (3)

$4x_1x=36$

$x=\frac{9}{x_1}$

$\therefore$ B is $\left(\frac{9}{x_1},0\right)$

(iii) The Distance AS

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$AS^2=(\sqrt{13}-x_1)^2+(0-y_1)^2$

$AS^2=(\sqrt{13}-x_1)^2+y_1^2$

$AS^2=(\sqrt{13}-x_1)^2+\frac{4x_1^2-36}{9}$ using (2)

$9AS^2=9(13-2\sqrt{13}x_1+x_1^2)+4x_1^2-36$

$9AS^2=13x_1^2-18\sqrt{13}x_1+81$

$AS^2=\frac{1}{9}(\sqrt{13}x_1-9)^2$

$AS=\frac{1}{3}(\sqrt{13}x_1-9)$

Similarly for AS'

$(AS')^2=(\sqrt{13}+x_1)^2+(0-y_1)^2$

$(AS')^2=13+2\sqrt{13}x_1+x_1^2+\frac{4x_1^2-36}{9}$

$9(AS')^2=117+18\sqrt{13}x_1+9x_1^2+4x_1^2-36$

$(AS')^2=\frac{1}{9}(13x_1^2+18\sqrt{13}x_1+81)$

$AS'=\frac{1}{3}(\sqrt{13}x_1+9)$

(b) (iii) Continued. Aids To Solns. 8

Now $SB = \sqrt{13} - \dfrac{9}{x_1} = \dfrac{\sqrt{13}\,x_1 - 9}{x_1}$

$S'B = \sqrt{13} + \dfrac{9}{x_1} = \dfrac{\sqrt{13}\,x_1 + 9}{x_1}$

And $\dfrac{SA}{S'A} = \dfrac{\frac{1}{3}(\sqrt{13}\,x_1 - 9)}{\frac{1}{3}(\sqrt{13}\,x_1 + 9)} = \dfrac{\sqrt{13}\,x_1 - 9}{\sqrt{13}\,x_1 + 9}$

Also $\dfrac{SB}{S'B} = \dfrac{(\sqrt{13}\,x_1 - 9) \div x_1}{(\sqrt{13}\,x_1 + 9) \div x_1} = \dfrac{\sqrt{13}\,x_1 - 9}{\sqrt{13}\,x_1 + 9}$.

$$\therefore \frac{SA}{S'A} = \frac{SB}{S'B}$$

Note There are two other methods which can be used for part (iii)

* Aids To Soln's Only 9

Question 4

$f(x) = x - 2\sqrt{x}$

(a) $x \geq 0$

(b) Let $f(x) = 0$

$x - 2\sqrt{x} = 0$

$x = 2\sqrt{x}$

$x^2 = 4x$

$x(x-4) = 0$

$x = 0$ or 4

(c) $f(x) = x - 2x^{1/2}$

$f'(x) = 1 - x^{-\frac{1}{2}}$

$f''(x) = -1 \times -\frac{1}{2} x^{-\frac{3}{2}}$

$= \dfrac{1}{2\sqrt{x^3}}$

Now the domain makes $f''(x) > 0$ for all +ive x which is the condition for concave up.

(d) Stationary pts for $f'(x) = 0$

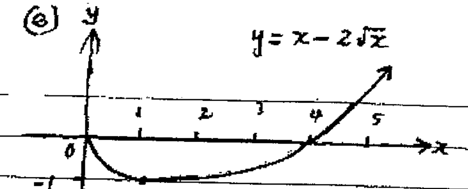
$1 - \dfrac{1}{\sqrt{x}} = 0$

$\sqrt{x} = 1$

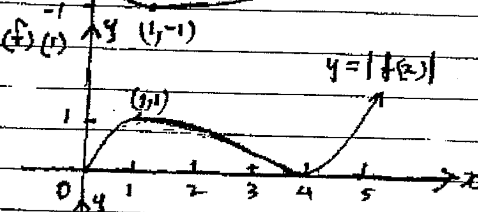
$\therefore x = 1$ and on substitution $y = -1$

Now since the curve is concave up we can say that $(1,-1)$ is a minimum tp.

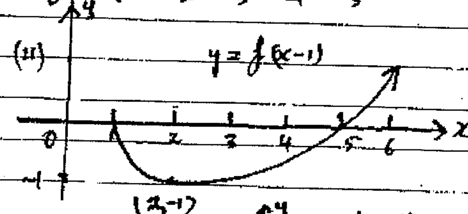
(e)



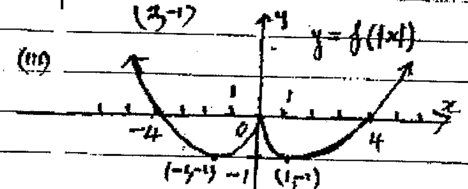
(f) (i)

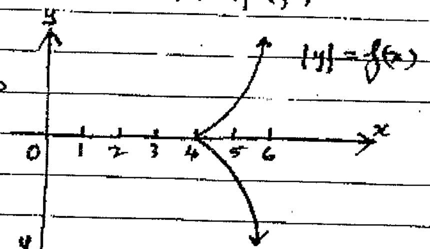


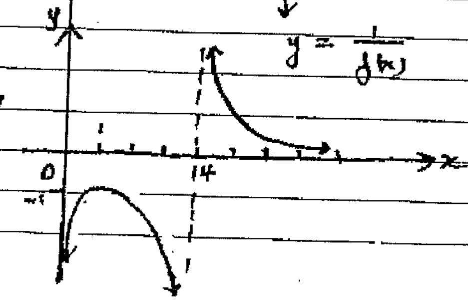
(ii)



(iii)







Question 5

(a) $1\times 1! + 2\times 2! + 3\times 3! + \dots + n\times n! = (n+1)! - 1$, for $n \geq 1$

Let $T_n = n \times n!$

$S_n = (n+1)! - 1$

If $n = 1$

$S_n = (1+1)! - 1 = 2! - 1 = 2 - 1 = 1$ $\therefore$ true for $n = 1$

Assume the result to be true for $n = k$.

$\therefore S_k = (k+1)! - 1$

Consider $n = k+1$

$$\begin{aligned} S_k + T_{k+1} &= (k+1)! - 1 + (k+1)(k+1)! \\ &= (k+1)! + k(k+1)! + (k+1)! - 1 \\ &= 2(k+1)! + k(k+1)! - 1 \\ &= (k+2)(k+1)! - 1 \\ &= (k+2)! - 1 \\ &= ((k+1)+1)! - 1 \end{aligned}$$

which is the same as S_n with $(k+1)$ replacing n.

Hence the result is true for $n = k+1$ if it is true for $n = k$.

Now the result is true for $n = 1$ hence it is true for $n = 2$ and 3 and so on; hence it is true for all integers $n \geq 1$.

(b) $x^3 + 4x^2 - 3x + 1 = 0$.

roots are α, β, γ.

Equation whose roots are $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$

Let $y = \frac{1}{x}$ since $x = \alpha, \beta, \gamma$, $y = \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

Put $x = \frac{1}{y}$ in the above equation

$$\frac{1}{y^3} + \frac{4}{y^2} - \frac{3}{y} + 1 = 0$$

ie $1 + 4y - 3y^2 + y^3 = 0$

The required equation is $x^3 - 3x^2 + 4x + 1 = 0$

(c) $3x^3 - 26x^2 + 52x - 24 = 0$

Let the roots be $\frac{\alpha}{r}, \alpha, \alpha r$ where α is a root and r is the common ratio for the GP.

Sum of roots: $\frac{\alpha}{r} + \alpha + \alpha r = \frac{26}{3}$

$\alpha\left(\frac{1}{r} + 1 + r\right) = \frac{26}{3}$ ---- (1)

Sum of roots 2 at a time $\frac{\alpha}{r} \times \alpha + \alpha \times \alpha r + \frac{\alpha}{r} \times \alpha r = \frac{52}{3}$

$\frac{\alpha^2}{r} + \alpha^2 r + \alpha^2 = \frac{52}{3}$ --- (2)

Product of roots $\frac{\alpha}{r} \times \alpha \times \alpha r = 8$ ---- (3)

$\alpha^3 = 8$

$\alpha = 2$

Sub in (1) $\frac{1}{r} + 1 + r = \frac{13}{3}$

$3 + 3r + 3r^2 = 13r$

$3r^2 - 10r + 3 = 0$

$(3r-1)(r-3) = 0$

$r = 3$ or $r = \frac{1}{3}$

$3r$ -1
r -3

If $r = 3$

roots are

$\frac{2}{3}, 2, 6$

This will reverse for $r = \frac{1}{3}$.

Aids To Solutions

(d) (i) Let $P(x) = (x-k)^2 Q(x)$

$$P'(x) = 2(x-k)^1 \times 1 \times Q(x) + (x-k)^2 Q'(x)$$
$$= 2(x-k)Q(x) + (x-k)^2 Q'(x)$$
$$= (x-k)\left[2Q(x) + (x-k)Q'(x)\right]$$
$$P(k) = (k-k)\left[2Q(x) + (k-k)Q'(x)\right]$$
$$= 0$$

i.e. k is a zero of $P'(x)$

(ii) Let $P(x) = x^{2n} - nx^{n+1} + nx^{n-1} - 1$

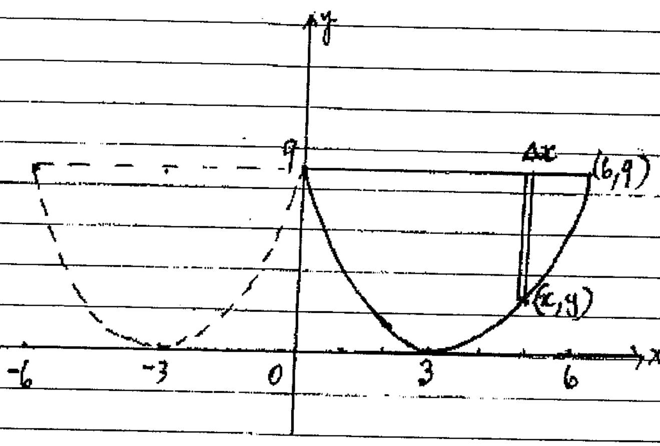
$$P(1) = 1 - n + n - 1 = 0$$
$$P'(x) = 2nx^{2n-1} - n(n+1)x^{n} + n(n-1)x^{n-2}$$
$$P'(1) = 2n - n^2 - n + n^2 - n$$
$$= 0$$

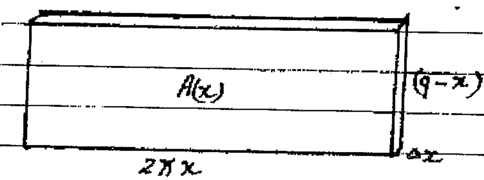
so $x = 1$ is a root of $P(x) = 0$ and $P'(x) = 0$
hence a double root of $P(x) = 0$.

Aids To Solutions

Question 6

(a) $y = (x-3)^2$ and $y = 9$





$$A(x) = 2\pi x(9-x)$$
$$\Delta V \doteq 2\pi x(9-x)\Delta x$$

Now $9 - y = 9 - (x-3)^2$
$$= 9 - x^2 + 6x - 9$$
$$= 6x - x^2$$

$$V = \lim_{\Delta x \to 0} \sum_{x=0}^{6} 2\pi x(6x - x^2)\Delta x$$
$$V = 2\pi \int_0^6 (6x^2 - x^3)\,dx$$
$$= 2\pi\left[2x^3 - \frac{x^4}{4}\right]_0^6$$
$$V = 2\pi(432 - 324 - 0)$$
$$V = 216\pi \text{ units}^3$$

Aids To Solutions

(b) $y = 5 - x^2$ ---- ①

$y = \frac{1}{4}x^2$ --- ②

$5 - x^2 = \frac{1}{4}x^2$

$5 = x^2 + \frac{1}{4}x^2$

$\frac{5}{4}x^2 = 5$

$x^2 = 4$

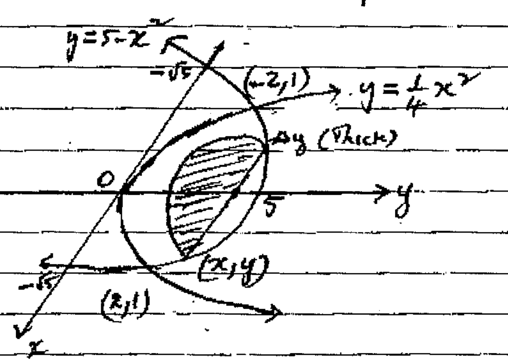
$x = \pm 2$

hence from ① $y = 1$

$V = \frac{\pi}{2}\left[\left(25 - \frac{25}{2}\right) - \left(5 - \frac{1}{2}\right) + 2 - 0\right]$

$V = \frac{\pi}{2}(20 - 12 + 2)$

$V = 5\pi$ units^3



Area of semicircular disc

$y = 5 - x^2$ $\qquad = \frac{1}{4}x^2$

$A = \pi(x)^2 \times \frac{1}{2}$ $\qquad A = \pi(x)^2 \times \frac{1}{2}$

Volume of Oasis : $\Delta V \doteq \frac{\pi x^2}{2}\Delta y$ $\qquad \Delta V \doteq \frac{\pi x^2}{2}\Delta y$

$$V = \lim_{\Delta y \to 0} \sum_{y=1}^{5} \pi\frac{5-y}{2}\Delta y + \lim_{\Delta y \to 0} \sum_{y=0}^{1} \pi \cdot \frac{4y}{2}\Delta y$$

$$V = \frac{\pi}{2}\int_1^5 (5-y)\,dy + \frac{\pi}{2}\int_0^1 4y\,dy.$$

$$V = \frac{\pi}{2}\left\{\left[5y - \frac{y^2}{2}\right]_1^5 + \left[2y^2\right]_0^1\right\}$$

Aids To Solutions

(c) (i) arc PQ $\doteq$ chord PQ

because Q is very close to P

Δx is very small

Using Pythagoras.

$\Delta z \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}$

(ii) $$\text{Arc } AB = \lim_{\Delta x \to 0} \sum_{x=a}^{b} \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \lim_{\Delta x \to 0} \sum_{x=a}^{b} \sqrt{(\Delta x)^2\left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right)}$$

$$= \lim_{\Delta x \to} \sum_{x=a}^{b} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

(iii) $y = x^{\frac{3}{2}}$

$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$

Now $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{9}{4}x}$

$$l = \int_0^4 \left(1 + \frac{9x}{4}\right)^{1/2} dx$$

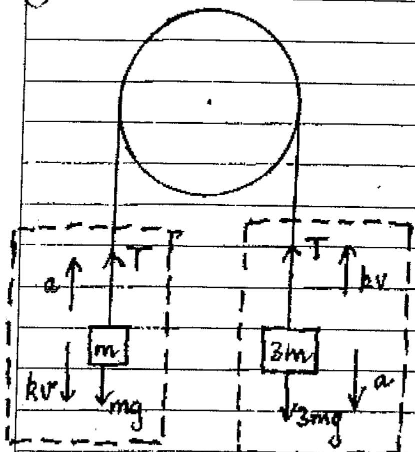
$$= \left[\frac{\left(1 + \frac{9x}{4}\right)^{\frac{3}{2}}}{\frac{3}{2} \times \frac{9}{4}}\right]_0^4$$

$$= \frac{8}{27}\left[10^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$$

$$= \frac{8}{27}\left(10\sqrt{10} - 1\right)$$ units

(a) Question 7

(i)



"a smooth fixed pulley" with "a light inextensible string" means the tensions are equal. Now $m < 3m$ so the particle with mass m will accelerate up while the particle with mass $3m$ will accelerate down.

(ii) using $F = ma$ we have

$\uparrow ma = T - mg - kv$

ie $T = mg + ma + kv$ ---- (1)

also $\downarrow 3ma = (3m)g - T - kv$

$T = 3mg - 3ma - kv$ ---- (2)

Equate (1) and (2)

$mg + ma + kv = 3mg - 3ma - kv$

$4ma = 2mg - 2kv$

$a = \dfrac{mg - kv}{2m}$

But $a = \dfrac{dv}{dt}$ $\therefore$ $\dfrac{dv}{dt} = \dfrac{mg - kv}{2m}$ ---- (3)

(iii) Terminal Velocity occurs when $a = 0$

so $\dfrac{mg - kv}{2m} = 0$

$V = \dfrac{mg}{k}$ ---- (4)

(iv) from (3)

$\dfrac{dt}{dv} = \dfrac{2m}{mg - kv}$

$t = -\dfrac{2m}{k} \ln|mg - kv| + c$

Now when $t = 0, v = 0$

$0 = -\dfrac{2m}{k} \ln|mg| + c$

$c = \dfrac{2m}{k} \ln|mg|$

$\therefore t = \dfrac{2m}{k} \ln\left|\dfrac{mg}{mg - kv}\right|$ ---- (5)

(v) When $v = \frac{1}{2} V_T$

$= \dfrac{mg}{2k}$ from (4)

Sub in (5)

$t = \dfrac{2m}{k} \ln\left|\dfrac{mg}{mg - k \times \frac{mg}{2k}}\right|$

$t = \dfrac{2m}{k} \ln\left|\dfrac{mg}{mg - \frac{mg}{2}}\right|$

$t = \dfrac{2m}{k} \ln 2$

$t = \dfrac{m}{k} \ln 2^2$

$t = \dfrac{m}{k} \ln 4$

now from (4)

$t = \dfrac{m}{\frac{mg}{V_T}} \ln 4$

$t = \dfrac{V_T}{g} \ln 4$ ---- (6)

Aids To Solutions

(b) For $P(z) = 0$

sum of roots : $z_1 + z_2 + z_3 = 1$

Sum of roots two at a time : $z_1 z_2 + z_1 z_3 + z_2 z_3 = 9$

Product of roots : $z_1 z_2 z_3 = 9$

Equation is $z^3 - \left(\frac{-b}{a}\right) z^2 + \frac{c}{a} z - \left(\frac{c}{a}\right) = 0$.

$$z^3 - z^2 + 9z - 9 = 0$$

$$z^2(z-1) + 9(z-1) = 0$$

$$(z^2+9)(z-1) = 0.$$

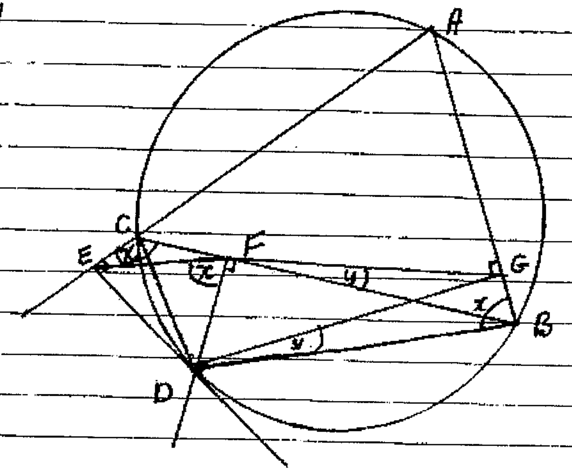
$$(z^2 - 9i^2)(z-1) = 0$$

$$(z-3i)(z+3i)(z-1) = 0$$

$$z = 3i \text{ or } -3i \text{ or } 1.$$

(a) Question 8 Aids To Solutions

(i)



(ii) In quadrilateral DECF The opposite angles CED and CFD are both 90° hence supplementary. This means That The remaining pair ECF and FDE are also supplementary hence DECF is a cyclic quadrilateral

In quadrilateral DFGB $D\hat{F}G = B\hat{G}D = 90°$ and so by converse of "angles in the same segment are equal", The points D, F, G and B are concyclic.

(iii) Let $E\hat{F}D = x$ and $G\hat{F}B = y$.

$E\hat{F}D = E\hat{C}D = x$ (angles in the same segment are equal — concyclic points E, C, F and D)

$E\hat{C}D = A\hat{B}D = x$ (external ∠ of cyclic quadrilateral CABD equals The interior opposite angle)

$G\hat{F}B = B\hat{D}G = y$ (angles in The same segment are equal — concyclic points D, F, G and B)

$x + y = 90°$ (external angle of Δ DGB equals sum of interior opposite angles)

Now $E\hat{F}D\,(x) + D\hat{F}B\,(90°) + G\hat{F}B\,(y) = 180°$ (because $x+y=90°$)

ie EFG is a straight line and so E, F and G are collinear

Aids To Solutions.

(b) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ --- (1)

Differentiate w.r.t x

$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}y' = 0$

$\frac{1}{x^{\frac{1}{3}}} + \frac{1}{y^{\frac{1}{3}}}y' = 0$

$y' = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$

at the point (l, m)

$y' = -\left(\frac{m}{l}\right)^{\frac{1}{3}}$

$\therefore$ Gradient of tangent is

$-\left(\frac{m}{l}\right)^{\frac{1}{3}}$

Equation of tangent

$y - y_1 = m(x - x_1)$

$y - m = -\left(\frac{m}{l}\right)^{\frac{1}{3}}(x - l)$

x-intercept let $y = 0$

$-m = -\left(\frac{m}{l}\right)^{1/3}(x - l)$

$m \times \left(\frac{l}{m}\right)^{\frac{1}{3}} = x - l$

$x_I = l + l^{1/3} m^{2/3}$

$x_I = l^{1/3}(l^{2/3} + m^{2/3})$

~~y-intercept let $x = 0$~~

$y - m = -\left(\frac{m}{l}\right)^{\frac{1}{3}}(-l)$

$y = m + l \times \left(\frac{m}{l}\right)^{1/3}$

$y_I = m + l^{2/3} m^{1/3}$

$y_I = m^{1/3}(m^{2/3} + l^{2/3})$

Now using Pythagoras to find length of tangent, say T

$T^2 = \left[l^{1/3}(l^{2/3} + m^{2/3})\right]^2 + \left[m^{1/3}(m^{2/3} + l^{2/3})\right]^2$

$T^2 = \left(l^{2/3} + m^{2/3}\right)^2\left(l^{2/3} + m^{2/3}\right)$

$T^2 = \left(l^{2/3} + m^{2/3}\right)^3$

Now from (1) $l^{2/3} + m^{2/3} = a^{2/3}$

because (l, m) lies on astroid

$T^2 = (a^{2/3})^3$

$T^2 = a^2$

$T = a$

$\therefore$ The length of the tangent is constant.