ST IGNATIUS COLLEGE RIVERVIEW

TASK 4

YEAR 12

2004

EXTENSION 2

TRIAL HSC EXAMINATION

Time allowed: 3 hours + 5 minutes reading time.

Instructions to Candidates

- Attempt all questions
- Show all necessary working.
- Marks may be deducted for missing or poorly arranged work.
- Board approved calculators may be used.
- Each question attempted must be returned in a *separate* writing booklet clearly marked Question 1, Question 2 etc, on the cover
- **Each booklet must have your name and the name of your mathematics teacher written on the cover.**

Question 1 (15 marks) Use a SEPARATE writing booklet. Marks

a If $Z_1 = 1+2i$, $Z_2 = 2-i$ and $Z_3 = 1-\sqrt{3}\,i$,
Express in the form $(a+bi)$ where a and b are real.

(i) $Z_1 + Z_2$ 1

(ii) $\dfrac{1}{Z_2}$ 1

(iii) $(Z_3)^3$ 2

b Express $\dfrac{4+3i}{3+i}$ in the form $(a+bi)$ where a and b are real numbers. 2

c (i) Express $Z = \sqrt{3}+i$ in modulus- argument form. 1

(ii) Hence, show that $Z^7 + 64Z = 0$. 3

d (i) Find the square root(s) of $(-8+6i)$. 3

(ii) Hence, solve the equation

$2Z^2 - (3+i)Z + 2 = 0$, expressing Z in the form $(a+bi)$ where a and b are real. 2

Question 2 (15 marks) Use a SEPARATE writing booklet. Marks

a Evaluate

(i) $\int_0^{\frac{\pi}{4}} x\sin 2x\,dx$. 3

(ii) $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$. 2

(iii) $\int_0^{\frac{\pi}{3}} \frac{\tan x}{1+\cos x}\,dx$. (using $t=\tan\frac{x}{2}$). 4

b Show that, if $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$. 6

Then $I_n + I_{n-2} = \frac{1}{n-1}$, where n is an integer and $n \ge 3$

Hence evaluate I_7.

Question 3 (15 marks) Use a SEPARATE writing booklet. Marks

a The point $A\ (a\cos\alpha, b\sin\alpha)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

B is the foot of the perpendicular from A to the x-axis. The normal at A cuts the x-axis at C.

(i) Represent this information with a suitable diagram. 1

(ii) Derive the equation of the normal AC. 3

(iii) Show that the length of CB is $\left|\frac{b^2\cos\alpha}{a}\right|$. 3

b Consider the hyperbola H with equation $4x^2-9y^2=36$. The point $R(x_1,y_1)$ is an arbitrary point on H.

(i) Prove that the equation of the tangent l at R is $4x_1x-9y_1y=36$. 3

(ii) Find the co-ordinates of the point K at which l cuts the x-axis. 1

(iii) Hence, prove that $\frac{SR}{PR}=\frac{SK}{PK}$ where S and P are the foci of H. 4

Question 4 (15 marks) Use a SEPARATE writing booklet. Marks

a The equation $x^3 - 3x + 3 = 0$ has roots which are α, β and γ. Find the equation in x where the roots are α^2, β^2 and γ^2. 4

b The base of a solid is a circle of radius 2 units. A diameter runs through the centre of the base. Any cross section of the solid formed by a plane perpendicular to the given diameter is an equilateral triangle. 5

Show that the volume of the solid is $\frac{32\sqrt{3}}{3}$ units^3.

c The region bounded by the curve $y = \log_e x$, the straight lines $y = 1$ and $x = 3$ is rotated about the y-axis. Find the volume of the resulting solid using the method of cylindrical shells. 6

Question 5 (15 marks) Use a SEPARATE writing booklet. Marks

a Find the four fourth roots of -16 in the form $(a + bi)$. 4

b A function is defined by $f(x) = \frac{\log_e x}{x}$ for $x > 0$.

(i) Find the x intercept. 1

(ii) Find the turning point. 2

(iii) Find the point of inflection. 2

(iv) Sketch the graph of $y = f(x)$. 2

c Consider the function in part (b) sketch

i $y = |f(x)|$. 2

ii $y = \frac{1}{f(x)}$. 2

Question 6 (15 marks) Use a SEPARATE writing booklet. Marks

a Consider the polynomial $Q(x) = ax^4 + bx^3 + cx^2 + dx + e$ where a, b, c, d and e are integers. Suppose α is an integer such that $Q(\alpha) = 0$.

(i) Prove that α is a factor of e. 2

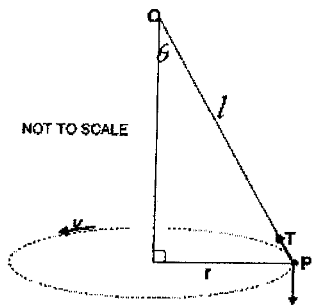
(ii) Prove that the polynomial equation $P(x) = 0$, where $P(x) = 4x^4 - x^3 + 3x^2 + 2x - 3$ does not have an integer root. 2

b It is estimated that the probability that a torpedo will hit its target is $\frac{1}{3}$.

(i) If 5 torpedoes are fired, what is the probability of 3 successes. 2

(ii) How many torpedoes must be fired so that the probability of at least one success should be greater than 0.9? 2

c



The above diagram shows a light string of length l, fixed at O, and making an angle θ with the vertical as shown in the above diagram. A particle is attached at P. The particle moves with uniform speed v metres / second in a horizontal circle of radius r. The centre of the circle is directly below O.

If the particle is to maintain its motion in a horizontal circle, show by resolving forces vertically and horizontally, that the particle's velocity is given by $v = \sqrt{rg \tan\theta}$. (Note: g is the acceleration due to gravity) 4

d When a polynomial $P(x)$ is divided by $(x-3)$ the remainder is 5 and when it is divided by $(x-4)$ the remainder is 9. Find the remainder when $P(x)$ is divided by $(x-4)(x-3)$. 3

Question 7 (15 marks) Use a SEPARATE writing booklet. Marks

a If $\sin^{-1}x$, $\cos^{-1}x$ and $\sin^{-1}(1-x)$ are acute, show that 6

$\sin(\sin^{-1}x - \cos^{-1}x) = 2x^2 - 1$.

Hence, solve the equation

$\sin^{-1}x - \cos^{-1}x = \sin^{-1}(1-x)$.

b Find the general solution of the equation $3\tan^2 x = 2\sin x$. 5

c Each of the following statements is either true or false. Write 'True' or 'False' for each statement giving a brief reason for your answers. (You are not required to evaluate the integrals).

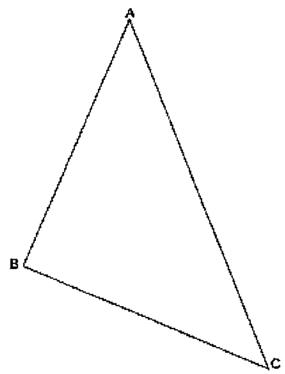
(i) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos x \, dx = 0$. 2

(ii) $\int_{-1}^{1} e^{-x^2} \cos^{-1} x \, dx = 0$. 2

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

a In the Argand diagram, the points A, B and C represent the complex numbers Z_1, Z_2 and Z_3 respectively.

What can you say about triangle ABC if $i(Z_3 - Z_2) = (Z_1 - Z_2)$. **2**



b Solve for x if $|3x+3| + |x-1| \le 4x+3$. **5**

c A particle, projected vertically upward with initial speed u is subjected to forces which create a constant vertical downward acceleration of magnitude g and an acceleration, directed against the motion, of magnitude kv when the speed is v. **8**

(i) Show that the acceleration function is given by $\ddot{x} = -g - kv$.

(ii) Prove that the maximum height reached by the particle after a time T is given by $T = \frac{1}{k}\log_e\left(\frac{g+ku}{g}\right)$.

(iii) Prove that the maximum height reached is $\frac{1}{k}(u - gT)$.

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Question 1 (15 marks)

(a) $z_1 = 1 + 2i$, $z_2 = 2 - i$ and $z_3 = 1 - \sqrt{3}i$

(1) (i) $z_1 + z_2 = 3 + i$

(1) (ii) $\frac{1}{z_2} = \frac{1}{2-i} \cdot \frac{\times(2+i)}{\times(2+i)}$

$= \frac{2}{5} + \frac{1}{5}i$

(2) (iii) $(z_3)^3 = (1 - \sqrt{3}i)^3$

$= 1 - 3\sqrt{3}i + 3(\sqrt{3}i)^2 - (\sqrt{3}i)^3$

$= 1 - 3\sqrt{3}i - 9 + 3\sqrt{3}i$

$= -8$

(2) (b) $\frac{4+3i}{3+i} \cdot \frac{\times(3-i)}{\times(3-i)} = \frac{12 - 4i + 9i + 3}{10}$

$= \frac{3}{2} + \frac{1}{2}i$

(1) (c) (i) $z = \sqrt{3} + i$

$|z| = \sqrt{3+1}$
$= 2$

$\arg z = \tan^{-1}\frac{1}{\sqrt{3}}$
$= \frac{\pi}{6}$

$\therefore z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$\Big\}\ z = 2\,\text{cis}\,\frac{\pi}{6}$

(3) (ii) $z^7 = 2^7\,\text{cis}\,\frac{7\pi}{6}$

$= 128\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$

$= -128\,\text{cis}\,\frac{\pi}{6}$

$64z = 64\left(2\,\text{cis}\,\frac{\pi}{6}\right)$

$= 128\,\text{cis}\,\frac{\pi}{6}$

$z^7 + 64z = 0$ (as required)

Question 1 (Continued)

(3) (d) (i) let $a+bi = \sqrt{-8+6i}$ $(a, b \in \mathbb{R})$

$a^2 - b^2 + 2abi = -8+6i$

$\therefore a^2 - b^2 = -8$ — ①

and $2ab = 6$

$ab = 3$ — ②

from ② $a = \frac{3}{b}$: sub ① : $\left(\frac{3}{b}\right)^2 - b^2 = -8$

$9 - b^4 = -8b^2$

$b^4 - 8b^2 - 9 = 0$

$(b^2-9)(b^2+1) = 0$

$b^2 = -1$ No Real sol^n

$b = \pm 3$

when $b = 3$ $a = 1$

$b = -3$ $a = -1$

The square roots are

$1+3i$ and $-1-3i$

or $\pm(1+3i)$

(2) (ii) $2z^2 - (3+i)z + 2 = 0$

$z = \dfrac{(3+i) \pm \sqrt{(3+i)^2 - 4(2)(2)}}{4}$ $9+6i-1-16$

$= \dfrac{(3+i) \pm \sqrt{-8+6i}}{4}$

$= \dfrac{3+i+1+3i}{4}$ or $\dfrac{3+i-1-3i}{4}$

$z = 1+i$ or $z = \frac{1}{2} - \frac{1}{2}i$

Question 2 (15 marks)

(3) (a) (i) $I = \int_0^{\frac{\pi}{4}} x \sin 2x \, . dx$ let $u = x$ $\frac{du}{dx} = 1$ $\frac{dv}{dx} = \sin 2x$ $V = -\frac{1}{2}\cos 2x$

$= \left[-\frac{1}{2}x\cos 2x\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \frac{1}{2}\cos 2x \, . dx$

$= \left[-\frac{\pi}{8}\cos\frac{\pi}{2} - 0\right] + \frac{1}{4}\left[\sin 2x\right]_0^{\frac{\pi}{4}}$

$= 0 + \frac{1}{4}\left[\sin\frac{\pi}{2} - \sin 0\right]$

$= \frac{1}{4}$

(2) (ii) $I = \int_0^1 \frac{dx}{\sqrt{4-x^2}}$

$= \left[\sin^{-1}\frac{x}{2}\right]_0^1$

$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)$

$= \frac{\pi}{6}$

(4) (iii) $I = \int_0^{\frac{\pi}{3}} \frac{\tan x}{1+\cos x} \, . dx$

let $t = \tan\frac{x}{2}$

$x = 2\tan^{-1} t$

$\frac{dx}{dt} = \frac{2}{1+t^2}$

when $x = 0$ $t = 0$

when $x = \frac{\pi}{3}$ $t = \frac{1}{\sqrt{3}}$

$I = \int_0^{\frac{1}{\sqrt{3}}} \dfrac{\frac{2t}{1-t^2}}{1+\frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2}$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{2t}{1-t^2} \times \frac{\cancel{(1+t^2)}}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{(1+t^2)}} \cdot dt$

Question 2 (Continued)

(a) (iii) $I = \int_0^{\frac{1}{\sqrt{3}}} \frac{2t}{1-t^2} \cdot dt$

$= -\left[\ln(1-t^2)\right]_0^{\frac{1}{\sqrt{3}}}$

$= -\left[\ln\frac{2}{3} - \ln 1\right]$

$= \ln\frac{3}{2}$

(b) $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \cdot dx$

$I_n + I_{n-2} = \int_0^{\frac{\pi}{4}} \tan^n x \cdot dx + \int_0^{\frac{\pi}{4}} \tan^{n-2} x \cdot dx$

(3) $= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \cdot (\tan^2 x + 1)\, dx$

$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \cdot \sec^2 x \, dx$

$= \frac{1}{n-1}\left[\tan^{n-1} x\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{n-1}\left[\left(\tan\frac{\pi}{4}\right)^{n-1} - 0\right]$

$= \frac{1}{n-1}$ (as required)

let $u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$du = \sec^2 x \cdot dx$

$\int u^{n-2} \cdot du$

$= \frac{1}{n-1} \cdot u^{n-1}$

Question 2 (Continued)

(3) (b) Using $I_n + I_{n-2} = \frac{1}{n-1}$, $n \geq 3$

$n=7$ Then $I_7 + I_5 = \frac{1}{6}$

$I_7 = \frac{1}{6} - I_5$

$n=5$ Then $I_5 + I_3 = \frac{1}{4}$

$I_5 = \frac{1}{4} - I_3$

$n=3$ Then $I_3 + I_1 = \frac{1}{2}$

$I_1 = \int_0^{\frac{\pi}{4}} \tan x \cdot dx$ $\quad \tan x = \frac{\sin x}{\cos x}$

$= -\left[\ln \cos x\right]_0^{\frac{\pi}{4}}$ $\quad = \frac{f'(x)}{f(x)}$

$= -\left[\ln\frac{1}{\sqrt{2}} - \ln 1\right]$

$= \ln\sqrt{2}$ or $\frac{1}{2}\ln 2$

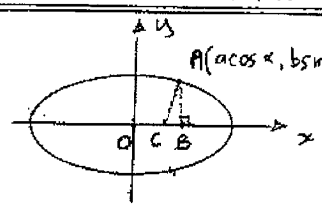
$I_7 = \frac{1}{6} - \frac{1}{4} + \frac{1}{2} - \frac{1}{2}\ln 2$ $\quad \frac{2}{12} - \frac{3}{12} + \frac{6}{12}$

$= \frac{5}{12} - \frac{1}{2}\ln 2$

$\doteqdot 0.07$

Question 3

(1) (a) (i)



(3) (ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by implicit diff^n we have

$$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{2x}{a^2} \cdot \frac{b^2}{2y}$$

$$m_T = \frac{-b^2 x}{a^2 y}$$

$$m_N = \frac{a^2 y}{b^2 x}$$

Equation of Normal at A (acos α, bsin α) : $y - b\sin\alpha = \frac{a^2 b \sin\alpha}{b^2 a \cos\alpha}\left(x - a\cos\alpha\right)$

$$(b\cos\alpha)y - b^2\sin\alpha\cos\alpha = a\sin\alpha\,(x - a\cos\alpha)$$

$$= (a\sin\alpha)x - a^2\sin\alpha\cos\alpha$$

$\boxed{\div \text{ by } \sin\alpha\cos\alpha}$ $\frac{by}{\sin\alpha} - b^2 = \frac{ax}{\cos\alpha} - a^2$

$$a^2 - b^2 = \frac{ax}{\cos\alpha} - \frac{by}{\sin\alpha}$$

(3) (iii) put $y = 0$: $a^2 - b^2 = \frac{ax}{\cos\alpha}$

$$x = \frac{\cos\alpha\,(a^2 - b^2)}{a}$$

Question 3 (Continued)

(a) (iii) $CB = |OB - OC|$

$$= \left|a\cos\alpha - \frac{a^2 - b^2}{a}\cdot\cos\alpha\right|$$

$$= \cos\alpha\left[\frac{a^2 - a^2 + b^2}{a}\right]$$

$$= \left|\frac{b^2\cos\alpha}{a}\right| \quad \text{(as required)}$$

(b) $4x^2 - 9y^2 = 36$ R (x_1, y_1) on Hyperbola

(3) (i) By implicit differentiation

$$8x - 18y\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{8x}{18y}$$

$$= \frac{4x}{9y}$$

Equation of tangent at R (x_1, y_1) : $y - y_1 = \frac{4x_1}{9y_1}\left(x - x_1\right)$

$$9y_1 y - 9y_1^2 = 4x_1 x - 4x_1^2$$

Since R(x_1, y_1) lies on H Then $4x_1^2 - 9y_1^2 = 36$

$$36 = 4x_1 x - 9y_1 y \quad \text{(as required)}$$

(ii)

Question 3 (Continued)

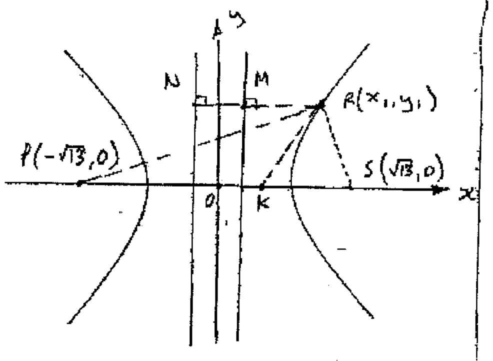
(1) (b) (ii) l cuts x-axis when $y = 0$

$$36 = 4x_1 x - 0$$

$$x = \frac{9}{x_1}$$

$$\therefore K\left(\frac{9}{x_1}, 0\right)$$

(4) (iii) Prove $\frac{SR}{PR} = \frac{SK}{PK}$



$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

$a = 3$

$b = 2$

$$b^2 = a^2(e^2 - 1)$$

$$e^2 = 1 + \frac{b^2}{a^2}$$

$$= 1 + \frac{4}{9}$$

$$e = \frac{\sqrt{13}}{3}$$

$$S(\pm ae, 0)$$

$$S(\pm\sqrt{13}, 0)$$

$$x = \pm\frac{a}{e}$$

$$x = \pm\frac{9}{\sqrt{13}}$$

$$SK = OS - OK = \sqrt{13} - \frac{9}{x_1}$$

$$PK = OP + OK = \sqrt{13} + \frac{9}{x_1}$$

$$\frac{SK}{PK} = \frac{\sqrt{13} - \frac{9}{x_1}}{\sqrt{13} + \frac{9}{x_1}}$$

$$= \frac{x_1\sqrt{13} - 9}{x_1\sqrt{13} + 9}$$

Question 3 (Continued)

(b) (iii) By def^n of Hyperbola, we have

$$\frac{SR}{MR} = e \Rightarrow SR = eMR$$

and

$$\frac{PR}{NR} = e \Rightarrow PR = eNR$$

$$\frac{SR}{PR} = \frac{MR}{NR}$$

$$MR = x_1 - \frac{9}{\sqrt{13}} \quad \text{and} \quad NR = x_1 + \frac{9}{\sqrt{13}}$$

$$\frac{MR}{NR} = \frac{x_1 - \frac{9}{\sqrt{13}}}{x_1 + \frac{9}{\sqrt{13}}}$$

$$\therefore \frac{SR}{PR} = \frac{x_1\sqrt{13} - 9}{x_1\sqrt{13} + 9}$$

$$= \frac{SK}{PK}$$

(as required)

Question 4 (15 marks)

(4) (a) $x^3 - 3x + 3 = 0$ roots α, β and γ.

α^2, β^2 and γ^2 will satisfy $x^{\frac{3}{2}} - 3x^{\frac{1}{2}} + 3 = 0$

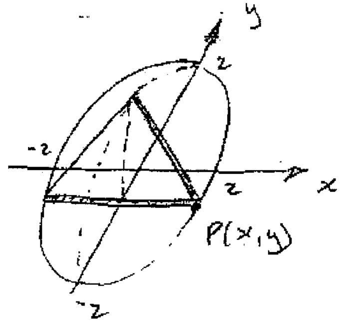
i.e. $x^{\frac{1}{2}}(x - 3) = -3$

Squaring: $x(x-3)^2 = 9$

$x(x^2 - 6x + 9) = 9$

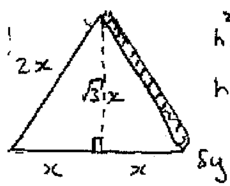
$x^3 - 6x^2 + 9x - 9 = 0$

(5) (b)



Base $x^2 + y^2 = 4$

Typical cross section



$h^2 = 4x^2 - x^2$

$h = \sqrt{3}x$

$\delta A = \frac{1}{2}bh$

$= \frac{1}{2} \cdot 2x \cdot \sqrt{3}x$

$= \sqrt{3}x^2$

Typical Volume

$\delta V \doteqdot \sqrt{3}x^2 . \delta y$

$V = \int_{-2}^{2} \sqrt{3}x^2 . dy$

Using $x^2 = 4 - y^2$

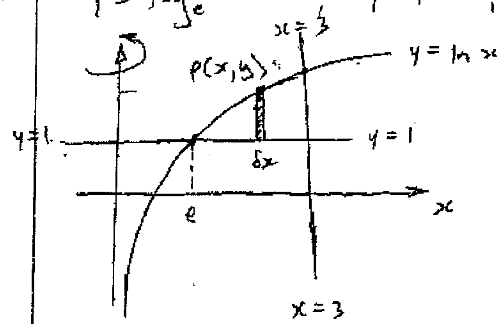
$= 2\sqrt{3}\int_0^2 (4 - y^2) . dy$

$= 2\sqrt{3}\left[4y - \frac{y^3}{3}\right]_0^2$

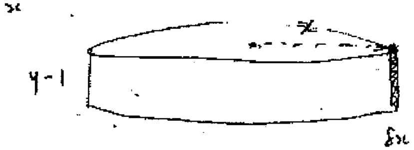
$= 2\sqrt{3}\left[8 - \frac{8}{3}\right] = \frac{32\sqrt{3}}{3}$ cubic units

Question 4 (Continued)

(6) (c) $y = \log_e x$ $y = 1$, $x = 3$.



Typical Shell



$\delta V = 2\pi x(y-1) . \delta x$

$= 2\pi x(\ln x - 1) . \delta x$

$V = 2\pi \int_e^3 (x\ln x - x) . dx$ L.I.A.T.E.

$\boxed{\begin{array}{ll} \text{Integ. By Parts} & \\ u = \ln x & \frac{dV}{dx} = x \\ \frac{du}{dx} = \frac{1}{x} & V = \frac{1}{2}x^2 \end{array}}$

$\frac{V}{2\pi} = \int_e^3 x\ln x . dx - \left[\frac{x^2}{2}\right]_e^3$

$= \left[\frac{1}{2}x^2 \ln x\right]_e^3 - \int_e^3 \frac{1}{2}x^2 . \frac{1}{x} . dx - \left[\frac{x^2}{2}\right]_e^3$

$= \frac{9}{2}\ln 3 - \frac{1}{2}e^2 - \left[\frac{x^2}{4}\right]_e^3 - \left[\frac{x^2}{2}\right]_e^3$

$= \frac{9}{2}\ln 3 - \frac{1}{2}e^2 - \left[\frac{3x^2}{4}\right]_e^3$

$= \frac{9}{2}\ln 3 - \frac{1}{2}e^2 - \frac{27}{4} + \frac{3e^2}{4}$

$V = \pi\left[9\ln 3 - e^2 - \frac{27}{2} + \frac{3e^2}{2}\right]$

$= \pi\left[9\ln 3 - \frac{27}{2} + \frac{e^2}{2}\right]$ units^3.

Question 5 (15 marks)

(4) (a) let $z = r \operatorname{cis} \theta$

where $z^4 = -16$

$r^4 \operatorname{cis} 4\theta = 16 \operatorname{cis} \pi$

$\therefore r = 2$ and $4\theta = \pi + 2n\pi$, $n \in J$

$\theta = \dfrac{\pi + 2n\pi}{4}$

$z_1 = 2 \operatorname{cis} \frac{\pi}{4} = 2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right) = \sqrt{2} + \sqrt{2}\, i$

$z_2 = 2 \operatorname{cis} \frac{3\pi}{4} = \qquad = -\sqrt{2} + \sqrt{2}\, i$

$z_3 = 2 \operatorname{cis} \frac{5\pi}{4} = \qquad = -\sqrt{2} - \sqrt{2}\, i$

$z_4 = 2 \operatorname{cis} \frac{7\pi}{4} = \qquad = \sqrt{2} - \sqrt{2}\, i$

(b) $f(x) = \dfrac{\ln x}{x}$, for $x > 0$.

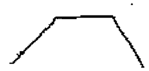
(1) (i) let $f(x) = 0$: $\dfrac{\ln x}{x} = 0$

$x = 1$

(2) (ii) $f'(x) = \dfrac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2}$

$= \dfrac{1 - \ln x}{x^2}$

x	2	e	3
$f'(x)$	+	0	−



put $f'(x) = 0$: $\dfrac{1 - \ln x}{x^2} = 0$

$\ln x = 1$

$x = e$

and $f(e) = \dfrac{1}{e}$

Max T.P at $\left(e, \frac{1}{e}\right)$.

Question 5 (continued)

(b) (iii) $f'(x) = \dfrac{1 - \ln x}{x^2}$

(2) $f''(x) = \dfrac{x^2 \cdot -\frac{1}{x} - (1 - \ln x) \cdot 2x}{x^4}$

$= \dfrac{-x - 2x + 2x \ln x}{x^4}$

$= \dfrac{2 \ln x - 3}{x^3}$

When $f''(x) = 0$: $2 \ln x - 3 = 0$

$\ln x = \frac{3}{2}$

$x = e^{\frac{3}{2}}$

when $x = e^{\frac{3}{2}}$: $y = \dfrac{\frac{3}{2}}{e^{\frac{3}{2}}}$

$= \dfrac{3}{2e^{\frac{3}{2}}}$

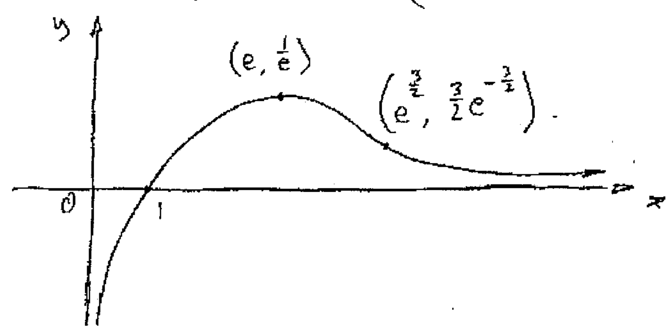
Concavity Test

x	e	$e^{\frac{3}{2}}$	e^2
$f''(x)$	−	0	+

$\therefore$ I.P. at $\left(e^{\frac{3}{2}}, \dfrac{3}{2e^{\frac{3}{2}}}\right)$

(iv) as $x \to \infty$ $\dfrac{\ln x}{x} \to 0$ (Since x dominates $\ln x$)

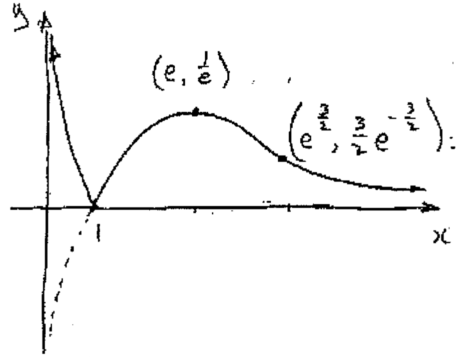
(2)



Question 5 (Continued)

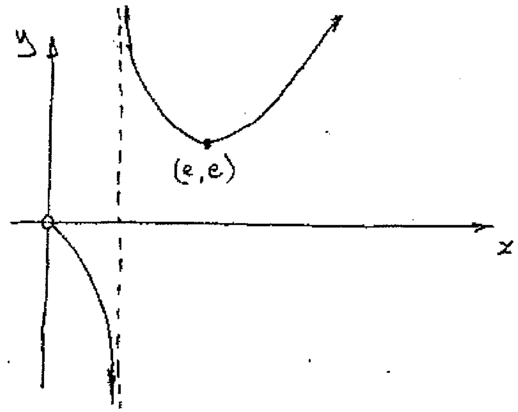
(c) (i) $y = |f(x)|$

(2)



(ii) $y = \dfrac{1}{f(x)}$

(2)



Question 6 (15 marks)

(a) $Q(x) = ax^4 + bx^3 + cx^2 + dx + e$

(2) (i) given α is an integer : $Q(\alpha) = 0$

$Q(\alpha) = a\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + e$

letting $Q(\alpha) = 0$: $a\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha = -e$

$\alpha(a\alpha^3 + b\alpha^2 + c\alpha + d) = -e$.

Since a, b, c, d are integers then $k\alpha = -e$ where k is also an integer

hence α is a factor of e.

(2) (ii) $P(x) = 4x^4 - x^3 + 3x^2 + 2x - 3$

The only possible integer roots are ± 1 and ± 3

$P(1) = 4 - 1 + 3 + 2 - 3 \neq 0$

$P(-1) = 4 + 1 + 3 - 2 - 3 \neq 0$

$P(3) = 324 - 27 + 27 + 6 - 3 \neq 0$

$P(-3) = 324 + 27 + 27 - 6 - 3 \neq 0$

$\therefore$ $P(x) = 0$ does not have an integer root.

(b) (2) (i)

$P(H) = \frac{1}{3}$

$P(M) = \frac{2}{3}$

<u>Five Torpedoes fired</u>

$P(3\text{ Hits}) = {}^5C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2$

$= \dfrac{40}{243}$.

(2) (ii) $P\left(\begin{smallmatrix}\text{at least}\\ 1\text{ hit}\end{smallmatrix}\right) = 1 - P\left(\begin{smallmatrix}\text{no}\\ \text{hits}\end{smallmatrix}\right)$

$= 1 - \left(\frac{2}{3}\right)^n$

$\therefore 1 - \left(\frac{2}{3}\right)^n > 0.9$

$\left(\frac{2}{3}\right)^n < 0.1$

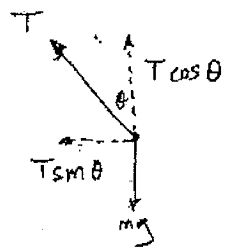
$\left(\frac{2}{3}\right)^5 = 0.132$

$\left(\frac{2}{3}\right)^6 = 0.088$

Requires 6 torpedoes

Question 6 (Continued)

(c) (4) Forces on P



Vertically (zero net force)

$T\cos\theta = mg$ — ①

Radially

$T\sin\theta = \frac{mv^2}{r}$ — ②

② ÷ ① $\quad \tan\theta = \dfrac{\frac{mv^2}{r}}{mg}$

$= \dfrac{v^2}{gr}$

$v^2 = gr\tan\theta$

$v = \sqrt{gr\tan\theta}$ (as required)

(d) (3) $P(x) = (x-3)(x-4) + R(x)$

Degree of $R(x) < 2$

let $R(x) = ax + b$

$P(3) = 5 \quad : \quad 5 = 3a + b$ — ①

$P(4) = 9 \quad : \quad 9 = 4a + b$ — ②

② − ① $\quad : \quad 4 = a$

sub ② $\quad 9 = 16 + b$

$b = -7$

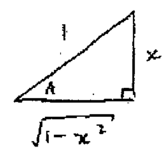
$\therefore R(x) = 4x - 7$

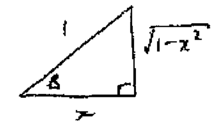
Question 7 (15 marks)

(a) $\sin^{-1}x$, $\cos^{-1}x$ and $\sin^{-1}(1-x)$ are acute

(3) let $A = \sin^{-1}x$ $\qquad$ let $B = \cos^{-1}x$

$\sin A = x$ $\qquad$ $\cos B = x$





$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$= x^2 - \sqrt{1-x^2}\cdot\sqrt{1-x^2}$

$= x^2 - (1-x^2)$

$= 2x^2 - 1$ (as required)

(3) Solve $\sin^{-1}x - \cos^{-1}x = \sin^{-1}(1-x)$

$\sin(A-B) = \sin\left(\sin^{-1}(1-x)\right)$

$2x^2 - 1 = 1 - x$

$2x^2 + x - 2 = 0$

$x = \dfrac{-1 \pm \sqrt{1 + 4(2)(2)}}{4}$

$= \dfrac{-1 \pm \sqrt{17}}{4}$

Since $x = \sin A$ then $-1 \le x \le 1$ $\quad$ $x = \dfrac{-1+\sqrt{17}}{4}$

(≈ 0.78)

Question 7 (continued)

(b) (5)

$$3\tan^2 x = 2\sin x$$

$$3\frac{\sin^2 x}{\cos^2 x} = 2\sin x$$

$$3\sin^2 x = 2\sin x\left(1-\sin^2 x\right)$$

$$3\sin^2 x = 2\sin x - 2\sin^3 x$$

$$2\sin^3 x + 3\sin^2 x - 2\sin x = 0$$

$$\sin x\left(2\sin^2 x + 3\sin x - 2\right) = 0$$

$2\sin x \quad -1$

$\sin x \quad +2$

$$\sin x\left(2\sin x - 1\right)\left(\sin x + 2\right) = 0$$

either $\sin x = 0$ or $\sin x = \frac{1}{2}$ $\quad \sin x \neq -2$

$x = n\pi$ $\quad x = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots$

$$x = n\pi + (-1)^n\cdot\frac{\pi}{6}$$

(c) (i) (2)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3\cos x\,.dx = 0 \qquad \text{True}$$

x^3 - is an odd function
$\cos x$ - is an even function
] odd × even = odd

(ii) (2)

$$\int_{-1}^{1} e^{-x^2}\,.\cos^{-1}x\,.dx = 0 \qquad \text{False}$$

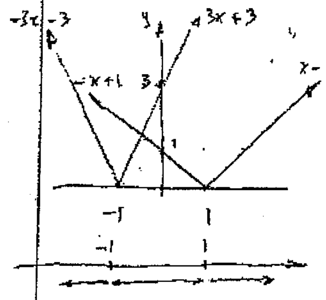
$\cos^{-1}x > 0$ for $-1 < x < 1$
$e^{-x^2} > 0$ for all x
]

$\therefore$ product can never equal zero

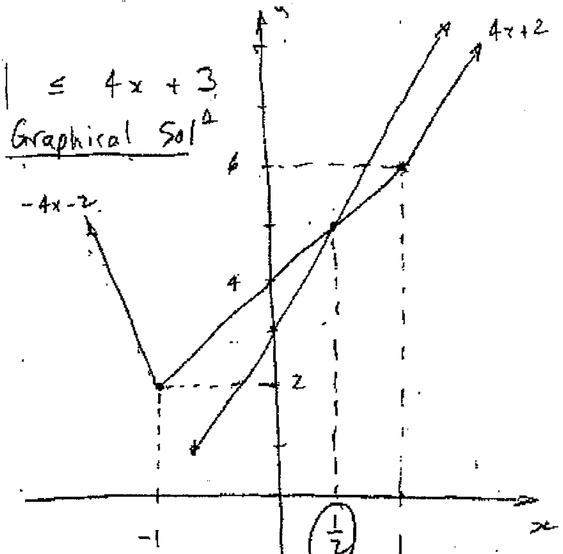
Question 8 (15 marks)

(b) (5)

$$|3x+3| + |x-1| \leq 4x+3$$



Graphical Sol^n



Graphical Sol^n

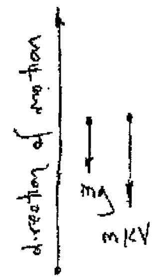
$x \geq \frac{1}{2}$

Algebraic Sol^n

for $x \leq -1$	for $-1 \leq x \leq 1$	for $x \geq 1$
$-4x-2 \leq 4x+3$	$2x+4 \leq 4x+3$	$4x+2 \leq 4x+3$
$-8x \leq 5$	$1 \leq 2x$	$2 \leq 3$
$x \geq -\frac{5}{8}$	$x \geq \frac{1}{2}$	True for all $x \geq 1$
No Sol^n	$\therefore \frac{1}{2} \leq x \leq 1$	

$\boxed{\text{Final Sol}^n: \ x \geq \tfrac{1}{2}}$

(c)



$t = 0$
$V = u$
$x = 0$

(1) (i) Forces on P

$$m\ddot{x} = -mg - mkV$$

$$\ddot{x} = -g - kV$$

(3) (ii)

$$\ddot{x} = -(g+kV)$$

$$\frac{dV}{dt} = -(g+kV)$$

$$\frac{dt}{dV} = \frac{-1}{g+kV}$$

Question 8 (Continued)

(b) $t = \int \frac{-1}{g+kV} \cdot dV$

$= -\frac{1}{k}\ln(g+kV) + C$

when $t=0$, $V=u$ $\quad \therefore \; C = \frac{1}{k}\ln(g+ku)$

$t = \frac{1}{k}\ln\left(\frac{g+ku}{g+kV}\right)$

max height when $V=0$ $\quad T = \frac{1}{k}\ln\left(\frac{g+ku}{g}\right)$ (as required)

(iii) Using $\ddot{x} = -(g+kV)$

(4) $V\cdot\frac{dV}{dx} = -(g+kV)$

$\frac{dV}{dx} = -\left(\frac{g+kV}{V}\right)$

$\frac{dx}{dV} = -\frac{V}{g+kV}$

$= -\frac{1}{k}\left(\frac{kV+g-g}{g+kV}\right)$

$= -\frac{1}{k}\left(1 - \frac{g}{g+kV}\right)$

$x = -\frac{1}{k}\int\left(1-\frac{g}{g+kV}\right)\cdot dV$

$= -\frac{1}{k}\left[V - \frac{g}{k}\ln(g+kV)\right] + C$

when $x=0$, $V=u$ $\quad C = \frac{1}{k}\left[u - \frac{g}{k}\ln(g+ku)\right]$

Question 8 (Continued)

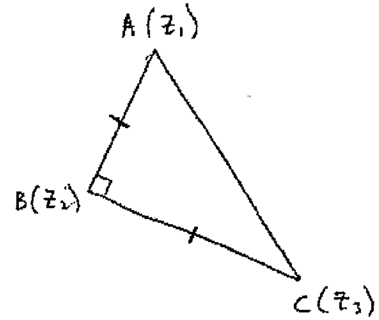
(b) $x = \frac{1}{k}\left[(u-V) - \frac{g}{k}\ln\left(\frac{g+ku}{g+kV}\right)\right]$

Max height when $V=0$ $\quad x = \frac{1}{k}\left[u - \frac{g}{k}\ln\left(\frac{g+ku}{g}\right)\right]$

$= \frac{1}{k}\left[u - gT\right]$ (as required)

(c)

(2)



Note $i(z_3 - z_2)$ rotates the vector by 90° (anti-clockwise).

ABC is a right angled triangle with $AB = BC$

$|z_3 - z_2| = BC$ and $|z_1 - z_2| = AB$