Student Name:

Mathematics Teacher:

St Mary's Cathedral College

2010

TERM 3 (HSC) TEST

Wednesday 9 June 2010
(Period 3)

Time allowed: 50 minutes

Year 12 Mathematics
Week 8, Term 3 (HSC)
Mathematics

Content: The Exponential Function
The Logarithmic Function
The Trigonometric Functions

Weighting: 15%

Total Mark: 50

Outcomes to be assessed:

A student:

H3 manipulates algebraic expressions involving logarithmic and exponential functions
H4 expresses practical problems in mathematical terms based on simple given models
H5 applies appropriate techniques from the study of calculus to solve problems
H6 uses the derivative to determine the features of the graph of a function
H8 uses techniques of integration to calculate areas and volumes
H9 communicates using mathematical language, notation, diagrams and graphs.

General Instructions:

- Write your Student Number and the Question Number in the boxes provided in the Writing Booklets.
- Write using blue or black pen. Use pencil for diagrams and graphs.
- Board-approved calculators and templates may be used.
- A table of standard integrals is provided separately.
- All necessary working should be shown for every question.
- Start each question in a new Writing Booklet.
- If you do not attempt a question, you must still hand in the Writing Booklet, with the words 'NOT ATTEMPTED' written clearly on the front cover.
  You may NOT take any Writing Booklets, used or unused, from the examination room.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Question 1 (16 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Copy and complete the table below for $y=\sqrt{2+e^x}$, calculating each value correct to three decimal places. **2**

x	0	1	2
y			

(ii) Use Simpsons's rule with three function values to approximate $\int_0^2 \sqrt{2+e^x}\,dx$. **2**

Give your answer correct to two decimal places.

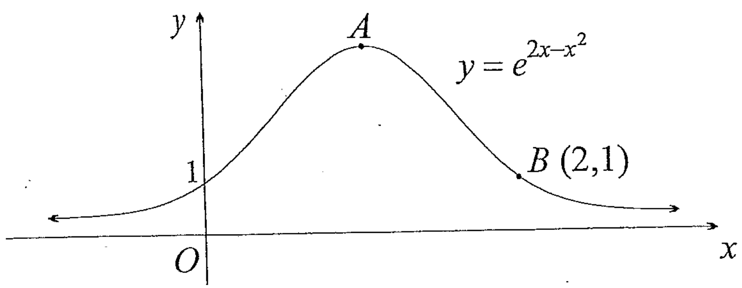
(b) Differentiate the following:

(i) $y=e^{4x}$ **1**

(ii) $y=x^2e^{-x}$ **2**

(c) Evaluate $\int_0^1 e^{2x+1}\,dx$. **2**

(d)



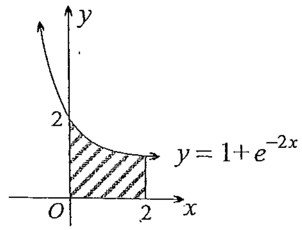
The diagram above shows the curve $y=e^{2x-x^2}$. The point A is a maximum turning point and B is the point $(2,1)$ as shown.

(i) Find the coordinates of the point A. **2**

(ii) Find the equation of the tangent at B. **2**

Marks

(e) **3**



The diagram above shows the curve $y=1+e^{-2x}$.
Find the area of the shaded region.

Question 2 (17 marks) Use a SEPARATE writing booklet. **Marks**

(a) Use the change of base law to calculate $\log_2 94$ correct to one decimal place. **1**

(b) Solve $\log_3(2x-1)=2$. **2**

(c) Differentiate the following:

(i) $\log_e(4x-1)$ **1**

(ii) $\log_e(x+2)^6$ **1**

(iii) $\dfrac{x^2}{\log_e x}$ **2**

Test continues overleaf...

Marks

(c) Find the following:

(i) $\int \frac{4}{x} dx$ **1**

(ii) $\int \frac{1}{1-3x} dx$ **1**

(d) Find the exact value of $\int_0^5 \frac{x}{x^2+5} dx$, giving your answer in simplest exact form. **3**

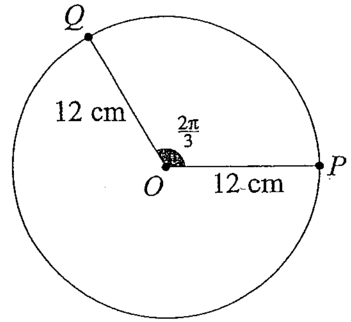
(e) The derivative of a function is given by $\frac{dy}{dx} = e^{-\frac{1}{2}x} - \frac{1}{x+1}$. **3**
The curve $y = f(x)$ passes through $(0,0)$. Find y as a function of x.

(f) The portion of the curve $x = e^{\frac{1}{y}}$ from $x = 2$ to $x = 5$ is rotated about the x-axis to form a solid. **2**
Show that the volume V of the solid is given by $V = \pi \int_2^5 \frac{1}{(\ln x)^2} dx$.
(*Do not attempt to evaluate this integral.*)

Question 3 (17 marks) Use a SEPARATE writing booklet. **Marks**

(a) Convert 225° to radians. **1**

(b)



The diagram above shows a circle with centre O and radius 12 cm.
The angle POQ is $\frac{2\pi}{3}$ radians.

(i) Find the area of the minor sector POQ. **1**

(ii) Find the perimeter of the minor sector POQ. **2**

(c) (i) Sketch the graph of $y = 4\cos 2x$ for $-\pi \le x \le \pi$, clearly showing the x and y intercepts. **2**

(ii) On the same set of axes, sketch the graph of $y = |x|$. **1**

(iii) Hence write down the number of solutions of the equation $4\cos 2x - |x| = 0$ for $-\pi \le x \le \pi$. **1**

(d) Differentiate the following:

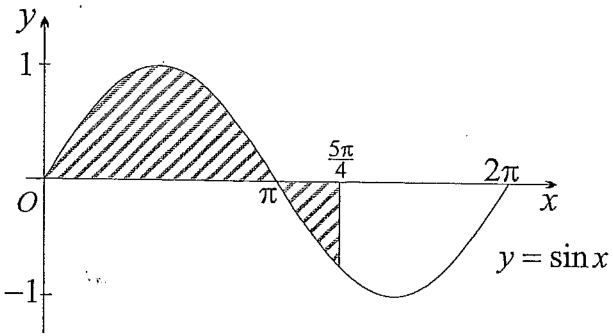
(i) $\cos 5x$ **1**

(ii) $\tan(x^2+1)$ **1**

Test continues overleaf...

(e) Find the exact value of $\int_0^{\frac{\pi}{2}} \sec^2 \frac{x}{3}\, dx$, giving your answer in simplest exact form. **3**

(f) **4**



Find the exact area of the shaded region in the diagram above.

End of test

Question Number
1

Student Name: SOLUTIONS / MARKING SCHEME

Question 1:

(a) (i)

x	0	1	2
y	1.732	2.172	3.064

✓ (one correct)
✓✓ (three correct)

(ii) $\int_0^2 \sqrt{2+e^x}\,dx \doteqdot \frac{2-0}{6}\left(f(0)+4f(1)+f(2)\right)$

$\doteqdot \frac{1}{3}\left(1.732+4\times 2.172+3.064\right)$ ✓ (Follow through errors from table)

$\doteqdot 4.49.$ ✓

(b) (i) $y = e^{4x}$

$\frac{dy}{dx} = e^{4x}\times 4$

$= 4e^{4x}.$ ✓

(ii) $y = x^2 e^{-x}$

$y' = e^{-x}\times 2x + x^2 \times e^{-x}\times(-1)$ ✓ (attempts to apply product rule)

$= 2xe^{-x} - x^2 e^{-x}$ ✓✓ (successfully applies product rule)

$= xe^{-x}(2-x).$

(c) $\int_0^1 e^{2x+1}\,dx = \left[\frac{1}{2}e^{2x+1}\right]_0^1$ ✓

$= \frac{1}{2}\left(e^3 - e^1\right)$

$= \frac{1}{2}e\left(e^2-1\right).$ ✓

(d) (i) $y = e^{2x-x^2}$

$y' = e^{2x-x^2}\times(2-2x)$

$= 2(1-x)e^{2x-x^2},$

which has a zero at $x=1$. ✓

When $x=1$,

$y = e^{2-1}$

$= e,$

so coordinates of A are $(1, e)$. ✓

(ii) At B $y' = 2(1-2)e^{2(2)-(2)^2}$

$= -2e^0$

$= -2,$ ✓

so equation of tangent is

$y-1 = -2(x-2)$

$y-1 = -2x+4$

$y = -2x+5.$ ✓

(e) Area $= \int_0^2 \left(1+e^{-2x}\right)dx$ ✓

$= \left[x - \frac{1}{2}e^{-2x}\right]_0^2$ ✓

$= \left(2-\frac{1}{2}e^{-2(2)}\right) - \left(0-\frac{1}{2}e^{-2(0)}\right)$

$= 2-\frac{1}{2}e^{-4} - \left(-\frac{1}{2}\right)$

$= 2-\frac{1}{2}e^{-4}+\frac{1}{2}$

$= \frac{5}{2}-\frac{1}{2}e^{-4}$

$= \frac{1}{2}\left(5-e^{-4}\right)$ units^2. } ✓ (either form, or equivalent)

Question Number
2

Student Name: SOLUTIONS / MARKING SCHEME

Question 2 :

(a) $\log_2 94 = \frac{\log_e 94}{\log_e 2}$ ✓

$\doteqdot 6\cdot 6.$

(b) $\log_3 (2x-1) = 2$

$3^2 = 2x - 1$ ✓

$9 = 2x - 1$

$10 = 2x$

$x = 5.$ ✓

(c)(i) Let $y = \log_e(4x-1)$.

Then $\frac{dy}{dx} = \frac{1}{4x-1} \times 4$

$= \frac{4}{4x-1}.$ ✓

(ii) Let $y = \log_e(x+2)^6$

$= 6\log_e(x+2).$

Then $y' = 6 \times \frac{1}{x+2}$

$= \frac{6}{x+2}.$ ✓

(iii) Let $y = \frac{x^2}{\log_e x}$.

Then $\frac{dy}{dx} = \frac{\log_e x \times 2x - x^2 \times \frac{1}{x}}{(\log_e x)^2}$ 1 mark: (attempts to apply quotient rule)

$= \frac{2x\log_e x - x}{(\log_e x)^2}$ 2 Marks: (successfully applies quotient rule)

$= \frac{x(2\log_e x - 1)}{(\log_e x)^2}.$

(c)(i) $\int \frac{4}{x}\,dx = 4\int \frac{1}{x}\,dx$

$= 4\ln x + C.$ ✓ (don't penalise if C omitted)

(ii) $\int \frac{1}{1-3x}\,dx = -\frac{1}{3}\ln(1-3x) + C.$ ✓ (applies the standard form $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e(ax+b)+C$)

(d) $\int_0^5 \frac{x}{x^2+5}\,dx = \frac{1}{2}\int_0^5 \frac{2x}{x^2+5}\,dx$ ← Note: Make top the derivative of the bottom.

$= \left[\frac{1}{2}\ln(x^2+5)\right]_0^5$ ✓

$= \frac{1}{2}\left(\ln(5^2+5) - \ln(0^2+5)\right)$

$= \frac{1}{2}(\ln 30 - \ln 5)$ ✓

$= \frac{1}{2}\ln\frac{30}{5}$

$= \frac{1}{2}\ln 6$ or $\ln\sqrt{6}.$ ✓ (last step required for full marks)

(e) $\frac{dy}{dx} = e^{-\frac{1}{2}x} - \frac{1}{x+1}$

$y = -2e^{-\frac{x}{2}} - \ln(x+1) + C.$ ✓ ✓

Substituting $(0,0)$,

$0 = -2e^0 - \ln 1 + C$

$0 = -2 - 0 + C$

$C = 2.$

Hence $y = -2e^{-\frac{x}{2}} - \ln(x+1) + 2.$ ✓

(f) $x = e^{\frac{1}{y}}$

$\frac{1}{y} = \ln x$

$y = \frac{1}{\ln x}.$ ✓

Volume $= \int_2^5 \pi y^2\,dx$ ✓

$= \pi\int_2^5 \frac{1}{(\ln x)^2}\,dx$, as required.

Question Number: 3

Student Name: SOLUTIONS / MARKING SCHEME

Question 3:

(a) $225^\circ = \frac{5\pi}{4}$ radians. ✓

(b)(i) $A = \frac{1}{2} r^2 \theta$

$= \frac{1}{2} \times 12^2 \times \frac{2\pi}{3}$

$= 48\pi \text{ cm}^2$. ✓

(ii) $P = 12 + 12 + 12 \times \frac{2\pi}{3}$ ✓ ✓

$= 24 + 8\pi$ cm.

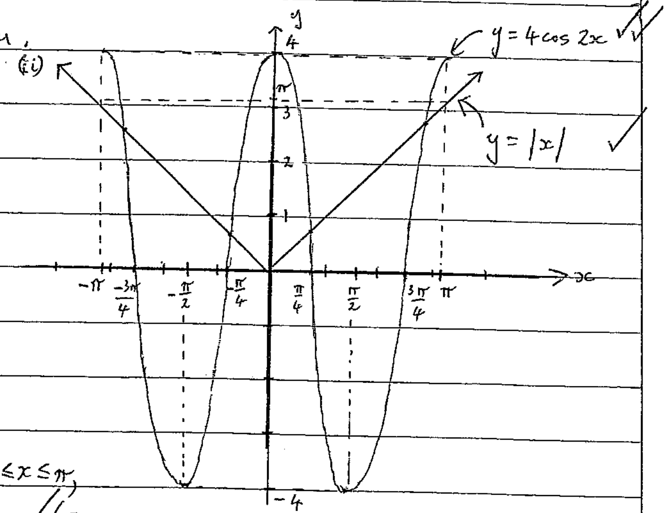
(c)(i) $y = 4\cos 2x$

Amplitude $= 4$

Period $= \frac{2\pi}{b}$

$= \frac{2\pi}{2}$

$= \pi$.



(iii) $4\cos 2x - |x| = 0, \ -\pi \le x \le \pi,$

has 4 solutions. ✓ (follow through errors from graphs)

(d)(i) $\frac{d}{dx} \cos 5x = -\sin 5x \times 5$

$= -5 \sin 5x$. ✓

(ii) $\frac{d}{dx} \tan(x^2+1) = \sec^2(x^2+1) \times 2x$

$= 2x \sec^2(x^2+1)$. ✓

(e) $\int_0^{\frac{\pi}{2}} \sec^2 \frac{x}{3} \, dx = \left[3 \tan \frac{x}{3}\right]_0^{\frac{\pi}{2}}$ ✓✓

$= 3\left(\tan \frac{\pi}{6} - \tan 0\right)$

$= 3\left(\frac{1}{\sqrt{3}} - 0\right)$

$= \frac{3}{\sqrt{3}}$ ✓

$= \sqrt{3}$

(f) 1. For the region above the x-axis,

$\int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi}$ ✓

$= -(\cos \pi - \cos 0)$

$= -(-1-1)$

$= -(-2)$

$= 2$, ✓

and so area above $= 2$ units^2.

2. For the region below the x-axis,

$\int_{\pi}^{\frac{5\pi}{4}} \sin x \, dx = [-\cos x]_{\pi}^{\frac{5\pi}{4}}$

$= -\left(\cos \frac{5\pi}{4} - \cos \pi\right)$

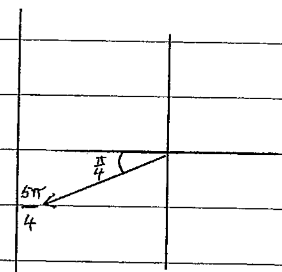
$= -\left(-\cos \frac{\pi}{4} - \cos \pi\right)$

$= -\left(-\frac{1}{\sqrt{2}} - (-1)\right)$

$= \frac{1}{\sqrt{2}} - 1$ ✓

$= -\left(1 - \frac{1}{\sqrt{2}}\right)$

$= -\left(1 - \frac{\sqrt{2}}{2}\right)$,



$\cos \frac{5\pi}{4} = -\cos \frac{\pi}{4}$

and so area below $= 1 - \frac{\sqrt{2}}{2}$ units^2.

So, total area $= 2 + \left(1 - \frac{\sqrt{2}}{2}\right)$

$= \left(3 - \frac{\sqrt{2}}{2}\right)$ units^2. ✓