St Mary's Cathedral College

2004
TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 1

Afternoon Session
Tuesday 10th August 2004.

General Instructions

- Reading time - 5 minutes
- Working time - 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks - 84

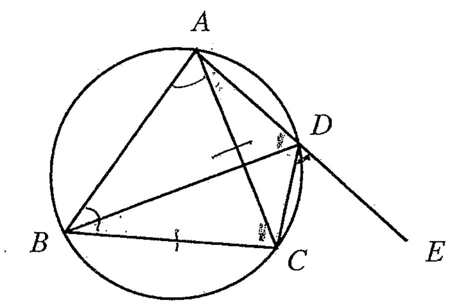
- Attempt Questions 1 – 7
- All questions are of equal value
- Answer each question in a separate booklet.

Question 1 **(12 marks, start a new answer booklet)** **Marks**

(a) Find the acute angle between the lines $2x-3y+7=0$ and $y=\frac{1}{2}x+3$. Answer to the nearest minute. 3

(b) Find the coordinates of the point X, which divides the line AB, where A is (3, 2) and B is (-3, 5), internally in the ratio 3 : 2. 2

(c)



ABC is a triangle in which BC=AC. AD is produced to E.

(i) Copy the diagram into your workbook.

(ii) Give a reason why $\angle CDE = \angle ABC$ 1

(iii) Hence prove that DC bisects $\angle BDE$ 3

(d) The polynomial equation $x^3-2x+1=0$ has a root near $x=0.5$. Using *one* application of Newton's Method, with first approximation $x_1=0.5$, find a more accurate approximation to this root. 3

Question 2 **(12 marks, start a new answer booklet)** **Marks**

(a) Find $\int \frac{x\,dx}{\sqrt{x^2+5}}$, using the substitution $u = x^2+5$. **3**

(b) The equation $2x^3+2x^2+4x+1=0$ has roots α, β and γ.

Find the value of

(i) $\alpha+\beta+\gamma$ **1**

(ii) $\alpha\beta\gamma$ **1**

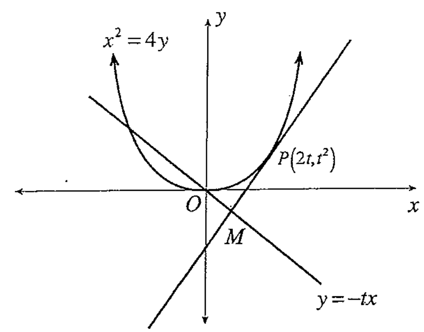
(iii) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$ **2**

(iv) $\alpha^2+\beta^2+\gamma^2$ **2**

(c) Draw a neat sketch of $y = 2\sin^{-1}\frac{x}{3}$. **3**
State the domain and range of this function.

Question 3 **(12 marks, start a new answer booklet)** **Marks**

(a)



$P(2t,t^2)$ is a point on $x^2=4y$. The tangent at the point P, and the line $y=-tx$ intersect at the point M.

(i) Show that the equation of the tangent at P is $tx-y-t^2=0$. **2**

(ii) Find the coordinates of the point M, in terms of t. **2**

(iii) Find the equation of the locus of the point M as t varies. **2**

(b) Consider the function $f(x)=e^x-x$

(i) Prove that the curve $y=f(x)$ is concave up for all values of x. **2**

(ii) Find the coordinates and nature of the stationary point on $y=f(x)$. **2**

(iii) Hence, explain why the results of parts (i) and (ii) prove that $e^x \geq x+1$ for all values of x. **2**

Question 4 **(12 marks, start a new answer booklet)** **Marks**

(a) (i) Make a neat sketch of the parabola $y = f(x)$, where $f(x) = x^2 - 2x - 15 = (x+3)(x-5)$. **1**

(ii) State the largest positive domain for which $f(x)$ has an inverse function. **1**

(iii) Sketch the inverse function for the domain you gave in part (ii). **2**

(iv) Show that the equation of the inverse function is $f^{-1}(x) = 1 + \sqrt{x+16}$. **1**

(b) Prove, by Mathematical Induction, that **4**

$5 + 8 + 11 + \ldots + (2n+3) = \frac{n}{2}(8+2n)$

for all positive integers n.

(c) Find, using a standard integral $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$. **3**

Answer in terms of π .

Question 5 **(12 marks, start a new answer booklet)** **Marks**

(a) Consider the polynomial $P(x) = 2x^3 + 3x^2 - 5x - 6$

(i) Prove, without long division, that $(x+1)$ is a factor of $P(x)$. **2**

(ii) By carrying out a long division, or otherwise, completely factorise $P(x)$. **3**

(b) Find $\int_0^{\frac{\pi}{16}} \sin^2 2x\,dx$ **4**

(c) Solve the inequality $\frac{2x}{x-3} \leq 5$ **3**

Question 6 **(12 marks, start a new answer booklet)** **Marks**

(a) A particle is moving on the number line in Simple Harmonic Motion, according to the equation $\ddot{x} = -9x$.

(i) State the period of motion. **1**

(ii) The particle is initially at the origin with velocity 10m/s. Find the amplitude of the motion, and hence write down the equation for v^2 (where v is velocity), in terms of x. **2**

(iii) Write down, in terms of a trigonometric function, the equation of motion for x in terms of time, t. **1**

(b) Solve, using any appropriate method, the equation $2\sin x - 3\cos x = 3$, on the interval $0 \leq x \leq 2\pi$. **4**

(c) The water in a swimming pool in Siberia is cooling according to Newton's Law of Cooling, $\frac{dT}{dt} = k(T - B)$, where T is the temperature of the water, and B is the constant temperature of the surrounding ground and air.

The temperature of the ground and air surrounding the pool remains a constant $-10°$C.

The pool initially has a temperature of $18°$C.
Ice will start to form in the pool when the temperature reaches $0°$C.

(i) After 24 hours the pool is found to have a temperature of $14°$C. Find the equation for temperature in the form $T = B + Ae^{kt}$, where A and k are also numerical constants. **2**

(ii) After how many hours will ice start to form? Calculate your answer to one decimal place. **2**

Question 7 (12 marks, start a new answer booklet) **Marks**

(a)

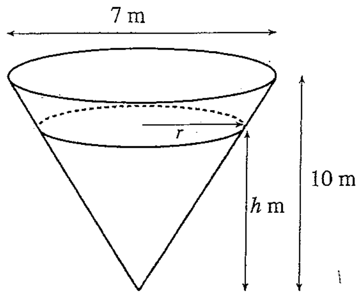


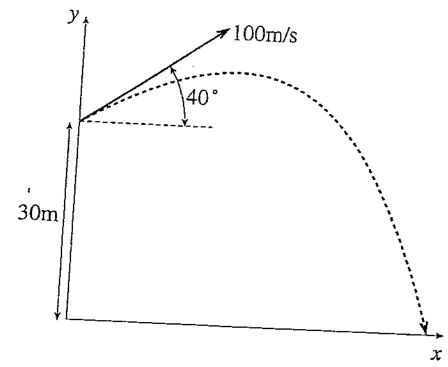
DIAGRAM NOT TO SCALE

(a) A water tower, made in the shape of an inverted cone of height 10 metres and radius 7m, is filled to a depth of h metres. The top-surface of the water inside it is a circle of radius r metres.

(i) Show that $r = \dfrac{7h}{20}$, and hence find an expression for the area of the top-surface of the water in terms of the depth, h. **2**

(ii) The water is let out of the tank so that the depth is reducing by 0.1m/hour. Find the rate at which the area of the top-surface of the water is changing at the moment when the depth is 4 metres. **2**

(b)



A particle is projected from a 30m cliff with a velocity of 100m/s, and an angle of projection of 40° elevation from the horizontal.

(i) If acceleration due to gravity is 10m/s^2, prove that the vertical and horizontal equations of motion are $y = (100\sin 40°)t - 5t^2 + 30$, and $x = (100\cos 40°)t$ respectively.
All initial velocities must be properly justified, and all working shown. **4**

(ii) Find the horizontal range of the particle. **2**

(iii) Find the Cartesian Equation of motion in exact form, in terms of y, x and $\tan 40°$ **2**

END OF EXAMINATION

③

SOLUTIONS	MARKS/COMMENTS
(d) Let $f(x) = x^3 - 2x + 1$ $f'(x) = 3x^2 - 2$ $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$	
$= 0.5 - \frac{f(0.5)}{f'(0.5)}$	$\boxed{1}$ Understanding the basic formula and finding $f(x)$, $f'(x)$.
$= 0.5 - \frac{(0.5)^3 - 2\times0.5+1}{3(0.5)^2 - 2}$	$\boxed{1}$ Completely correct substitution.
$= \cancel{0.58333333}$ 0.6 by calc.	$\boxed{1}$ ~~Any more accurate~~ answer ~~than $x = 0.5$~~ [0.58, 0.583, 0.58333... $0.58\dot{3}$]

④

SOLUTIONS	MARKS/COMMENTS
Question ②	
a) Let $u = x^2+5$ $du = 2x\,dx$ $\therefore \int \frac{x\,dx}{\sqrt{x^2+5}} = \frac{1}{2}\int \frac{2x\,dx}{\sqrt{x^2+5}}$	
$= \frac{1}{2}\int \frac{du}{\sqrt{u}}$	$\boxed{1}$ Correct substitution and integral in terms of u only
$= \frac{1}{2}\int u^{-\frac{1}{2}}\,du$ $= \cancel{\frac{1}{2}}\left[\frac{u^{\frac{1}{2}}}{\cancel{\frac{1}{2}}}\right] + C$	$\boxed{1}$ Correct Primitive
$= u^{\frac{1}{2}} + C$ $= \sqrt{u} + C$ $= \sqrt{x^2+5} + C$	$\boxed{1}$ Correct final answer in terms of x.
(b) i) $\alpha+\beta+\gamma = -\frac{b}{a}$ $= -\frac{2}{2}$ $= -1$	$\boxed{1}$ Answer only
ii) $\alpha\beta\gamma = -\frac{d}{a}$ $= -\frac{1}{2}$	$\boxed{1}$ Answer only
iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ $= \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$	$\boxed{1}$ Expansion in terms of $\alpha+\beta+\gamma$ etc.
Now $\beta\gamma+\alpha\gamma+\alpha\beta = \frac{c}{a} = \frac{4}{2}$ P.T.O.	

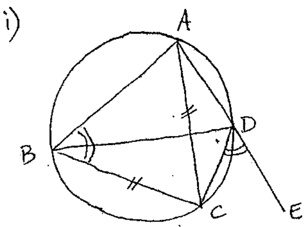
① TRIAL EXT 1 2004

SOLUTIONS	MARKS/COMMENTS
Question ①	
(a) $2x - 3y + 7 = 0$ $-3y = -2x - 7$ $y = \frac{2}{3}x + \frac{7}{3}$ gradient is $\frac{2}{3}$	$\boxed{1}$ Gradients correct.
Gradient of $y = \frac{1}{2}x + 3$ is $\frac{1}{2}$ $\tan\theta = \left\lvert\frac{m_1 - m_2}{1 + m_1 m_2}\right\rvert$	
$= \left\lvert\frac{\frac{2}{3} - \frac{1}{2}}{1 + \frac{2}{3}\times\frac{1}{2}}\right\rvert$	$\boxed{1}$ Correct substitution in formula
$= \left\lvert\frac{\frac{1}{6}}{\frac{4}{3}}\right\rvert$ $= \frac{1}{8}$ $\theta = \tan^{-1}\frac{1}{8}$	
$= 7°8'$ (nearest minute)	$\boxed{1}$ Correct angle to nearest minute (ignore rounding errors)
(b) $X = \left(\frac{3(-3)+2(3)}{3+2}, \frac{3(5)+2(2)}{3+2}\right)$	$\boxed{1}$ Correct substitution in formula
$= \left(-\frac{3}{5}, \frac{19}{5}\right)$	$\boxed{1}$ Correct coordinates or equivalent expression (CNE ISE) (−1 if not in brackets).
(3, 2) (−3, 5) 3 : 2	N.B 2003 HSC, Q.1 Marking guidelines • Correct answer or equivalent expression (2) • One coordinate correct (1)

$x = \frac{3(-3)+2(3)}{5}$ $= -\frac{3}{5}$ $y = \frac{3(5)+2(2)}{5}$ $= \frac{19}{5}$

Comments. "It is important to show working since credit was given for the CNE even if the ..."

②

SOLUTIONS	MARKS/COMMENTS
(c) i) 	
ii) The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.	$\boxed{1}$ Theorem must be correctly and completely stated. Note: A two-part answer using "opposite angles of a cyclic quadrilateral are supplementary" is obviously acceptable but it must be complete.
iii) $\angle CDE = \angle ABC$ (reason given above)	
$= \angle BAC$ (angles opposite equal sides in $\triangle ABC$, $BC = AC$.)	$\boxed{1}$ Use of isosceles $\triangle ABC$
$= \angle BDC$ (angles in the same segment on the same arc are equal).	$\boxed{1}$ Use of angles in the same segment
$\therefore$ DC bisects $\angle BDE$ ($\angle CDE = \angle BDC$)	$\boxed{1}$ logical connection to this final conclusion.

⑦

SOLUTIONS | **MARKS/COMMENTS**

Sub into ② [easier]

$y = -t\left(\frac{t}{2}\right)$

$y = -\frac{t^2}{2}$

$\therefore$ M is $\left(\frac{t}{2}, -\frac{t^2}{2}\right)$ — $\boxed{1}$ Completed coordinates properly justified

iii) Let $(x, y) = \left(\frac{t}{2}, -\frac{t^2}{2}\right)$

$x = \frac{t}{2}$ ——— ③

$y = -\frac{t^2}{2}$ ——— ④

From ③ $t = 2x$ sub into ④ — $\boxed{1}$ Appropriate method substantially completed.

$y = -\frac{(2x)^2}{2}$

$y = -\frac{4x^2}{2}$

$y = -2x^2$ — $\boxed{1}$ Justified Equation.

b) i) $f(x) = e^x - x$

$f'(x) = e^x - 1$

$f''(x) = e^x > 0$ for all x. — $\boxed{1}$ Second derivative

$\therefore y = f(x)$ is concave up for all x. — $\boxed{1}$ Properly explained conclusion.

⑧

SOLUTIONS | **MARKS/COMMENTS**

ii) Stationary points when $f'(x) = 0$

$e^x - 1 = 0$

$e^x = 1$

$x = 0$ — $\boxed{1}$ location of x-value of stationary point.

when $x = 0$ $f(x) = f(0)$

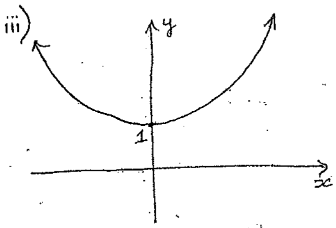
$= e^0 - 0$

$= 1 - 0$

$= 1$

$\therefore$ Absolute minimum turning stationary point at $(0, 1)$ — $\boxed{1}$ Coordinates and nature ("Absolute" need not be stated).

iii)



Since the curve is always concave up $(0, 1)$ is an absolute minimum turning point.

Hence $f(x) \geq 1$ for all x. — $\boxed{1}$ Properly explaining why $f(x) \geq 1$ for all x.

$\therefore$ $e^x - x \geq 1$

$e^x \geq x + 1$ for all x. — $\boxed{1}$ Extended to required conclusion.

⑤

SOLUTIONS | **MARKS/COMMENTS**

$= 2$

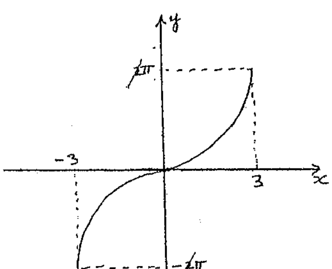
$\therefore \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{2}{\left(-\frac{1}{2}\right)}$

$= -4$ — $\boxed{1}$ Answer

iv) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$ — $\boxed{1}$ Expansion in terms of $\alpha + \beta + \gamma$ etc.

$= (-1)^2 - 2(2)$

$= -3$ — $\boxed{1}$ Answer



$\boxed{1}$ Basic Curve Shape Correct

$\boxed{1}$ Correct Domain and Range Indicated on graph as critical endpoints.

(These are connected so one may be given as follow through)

Domain: $-3 \leq x \leq 3$

Range: $-\frac{1}{2}\pi \leq y \leq \frac{1}{2}\pi$ — $\boxed{1}$ Correct statement of both domain and range

$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$2\left(-\frac{\pi}{2}\right) \leq 2y \leq 2\left(\frac{\pi}{2}\right)$

$-\pi \leq y \leq \pi.$

⑥

SOLUTIONS | **MARKS/COMMENTS**

Question ③

(a) i) $x^2 = 4y \Rightarrow y = \frac{x^2}{4}$

$\therefore \frac{dy}{dx} = \frac{2x}{4}$

$= \frac{x}{2}$

When $x = 2t$

$\frac{dy}{dx} = \frac{2t}{2}$

$= t$ — $\boxed{1}$ Correctly Justified gradient of tangent.

Equation

$(y - t^2) = t(x - 2t)$

$y - t^2 = tx - 2t^2$

$0 = tx - y - t^2$

$tx - y - t^2 = 0$ — $\boxed{1}$ Correctly Justified equation.

ii) $tx - y - t^2 = 0$ ——— ①

$y = -tx$ ——— ②

Sub ② into ①

$tx - (-tx) - t^2 = 0$

$2tx - t^2 = 0$

$t(2x - t) = 0$

$t \neq 0$ so $2x - t = 0$ — $\boxed{1}$ Any simultaneous solution substantially completed.

$2x = t$

$x = \frac{t}{2}$

Suppose that the statement is true for a positive integer k.
That is, suppose $5+8+11+\dots+(2k+3)=\frac{k}{2}(8+2k)$
We prove the statement for $n=k+1$.
That is, we prove $5+8+11+\dots+(2k+3)+(2k+5)=\frac{k+1}{2}(10+2k)$.

SOLUTIONS

Now $LHS = 5+8+11+\dots+(2k+3)+(2k+5)$

$= \frac{k}{2}(8+2k) + (2k+5)$

by Inductive Assumption

$= \frac{8k}{2} + \frac{2k^2}{2} + 2k+5$

$= k^2+6k+5$

$= (k+1)(k+5)$

$= RHS$

We have proven the result true for $n=1$, and also for $n=k+1$ on the assumption it is true for $n=k$
$\therefore$ It is true for $n=1+1=2$ and $n=1+2=3$ and so on for all positive integers n.

(c) $\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_0^1$

$= \sin^{-1}\frac{1}{2} - \sin^{-1}\frac{0}{2}$

$= \frac{\pi}{6} - 0$

$= \frac{\pi}{6}$

MARKS/COMMENTS

$L.H.S = \underbrace{5+8+11+\dots+(2k+3)}+(2k+5)$

$= \frac{k}{2}(8+2k)+(2k+5)$ by the induction hypothesis

$= 4k+k^2+2k+5$

$= k^2+6k+5$

$= (k+5)(k+1)$

$= \frac{1}{2}(10+2k)(k+1)$

$= \frac{k+1}{2}(10+2k) = R.H.S.$

Hence, if the statement is true for the integer k, then it is also true for the next integer $k+1$.

1 Inductive step, clearly using the inductive assumption.

But we know that the statement is true for $n=1$ so it must be true for $n=2, n=3$, and so on for all positive integral values of n.

1 Summary step.

1 Primitive

1 Clear substitution (no compromise steps)

1 Answer in radians.

SOLUTIONS

Question 5

(a) i) $P(x) = 2x^3+3x^2-5x-6$

$P(-1) = 2\times(-1)^3 \ 3\times(-1)^2 - 5\times(-1) - 6$

$= -2+3+5-6$

$= 0$

$\therefore (x+1)$ is a factor of $P(x)$

ii)
$$\begin{array}{r} 2x^2+x-6 \\ x+1\,\overline{)\,2x^3+3x^2-5x-6} \\ \underline{2x^3+2x^2} \\ x^2-5x \\ \underline{x^2+x} \\ -6x-6 \\ \underline{-6x-6} \\ 0 \end{array}$$

$\therefore P(x) = (x+1)(2x^2+x-6)$

$= (x+1)(2x-3)(x+2)$

b) $\int_0^{\frac{\pi}{16}} \sin^2 2x \, dx$

$= \int_0^{\frac{\pi}{16}} \frac{1}{2}(1-\cos 4x)\,dx$

$= \frac{1}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{16}}$

$= \frac{1}{2}\left[\left(\frac{\pi}{16} - \frac{1}{4}\sin 4\left(\frac{\pi}{16}\right)\right) - \left(0 - \frac{1}{4}\sin 4(0)\right)\right]$

MARKS/COMMENTS

1 Understanding the need to substitute $x=-1$

1 Showing clearly that $P(-1)=0$ and stating or indicating this is sufficient.

1 long division process understood

1 Correct long division resulting in this step

1 Final factorised form.

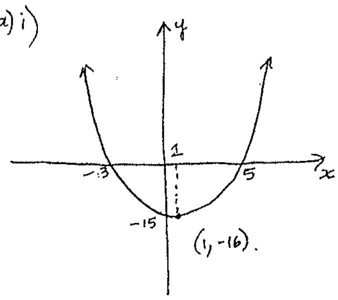
1 Correct Trig result

1 Correct Primitive

1 Substitution

SOLUTIONS

(a) i)

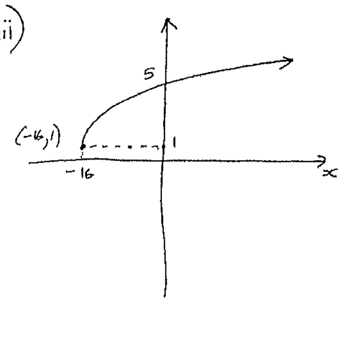


When $x=1$, $y = 1^2-2\times1-15$
$= -16$

$\therefore$ Vertex is $(1,-16)$

ii) $x \geq 1$

iii)



MARKS/COMMENTS

1 Sketch must have at least two of the following three
- x-ints
- y-ints
- vertex

in addition to
$\Rightarrow$ concave up parabola

1 Also accept $x>1$

Basic shape

1 Critical endpoint indicated (still award mark if "5" is not shown).

SOLUTIONS

iv) $y = x^2-2x-15$

Inverse $x = y^2-2y-15$

$x+15 = y^2-2y$

$x+16 = y^2-2y+1$

$x+16 = (y-1)^2$

$y-1 = \pm\sqrt{x+16}$

$y = 1 \pm \sqrt{x+16}$

$y = 1+\sqrt{x+16} \quad (y \geq 1)$

$\therefore f^{-1}(x) = 1+\sqrt{x+16}$

b) When $n=1$

$LHS = 2\times1+3$
$=5$

$RHS = \frac{1}{2}(7+3\times1)$

$= \frac{1}{2}\times 10$

$= 5$

$= LHS$

So the statement ~~result~~ is true for $n=1$

Assuming true for $n=1$

ie. $5+8+11+\dots+(2k+3) = \frac{k}{2}(8+2k)$

We must prove true for $n=k+1$

ie. $5+8+11+\dots+(2(k+1)+3) = \frac{k+1}{2}(8+2(k+1))$

ie. $5+8+11+\dots+(2k+5) = \frac{k+1}{2}(10+2k)$

ie. $5+8+11+\dots+(2k+5) = (k+1)(k+5)$

MARKS/COMMENTS

1 Essentials of $x-y$ swap and completing the square shown (mark generously.)

1 Proof for $n=1$

1 Inductive assumption and understanding of what must be proven.

(15)

SOLUTIONS | **MARKS/COMMENTS**

$\therefore \tan\frac{x}{2} = \frac{3}{2}$

Basic angle $\frac{x}{2}$ is 0.982793723

$\therefore \frac{x}{2} = 0.982793723 + 2k\pi$, or $(\pi + 0.982793723) + 2k\pi$

$= 0.982793723 + 2k\pi$, or $4.1243863 + 2k\pi$

$x = 1.965587447 + 4k\pi$ or $8.248772754 + 4k\pi$ $(k \in \mathbb{Z})$

1 Substantial progress in finding correct angles

On $0 \le x \le 2\pi$

$x = 1.965587447$ only

Check $x = \pi$

$LHS = 2\sin\pi - 3\cos\pi$
$= 2\times 0 - 3\times(-1)$
$= 3$
$= RHS$

$\therefore x = 1.965587447,\ \pi.$

1 Fully correct answer.

Method 2

Let $2\sin x - 3\cos x = A\sin(x-\alpha)$

$2\sin x - 3\cos x = A\sin x\cos\alpha - A\cos x\sin\alpha$

$2\sin x - 3\cos x = A\cos\alpha.\sin x - A\sin\alpha\cos x$

1 Correct Auxiliary form or method started.

We need $A\cos\alpha = 2$ — (1)

$A\sin\alpha = 3$ — (2)

(16)

SOLUTIONS | **MARKS/COMMENTS**

(2) ÷ (1) gives

$\frac{A\sin\alpha}{A\cos\alpha} = \frac{3}{2}$

$\tan\alpha = \frac{3}{2}$

$\alpha = 0.982793723$ will do (1st quad.)

1 Either A or α correctly found

Adding and squaring

$(A\sin\alpha)^2 + (A\cos\alpha)^2 = 3^2 + 2^2$

$A^2(\sin^2\alpha + \cos^2\alpha) = 13$

1 Other constant found.

$A^2 = 13$

$A = \sqrt{13}$ will do (taking $A>0$)

$\therefore \sqrt{13}\sin(x - 0.982793723) = 3$

$\sin(x - 0.982793723) = \frac{3}{\sqrt{13}}$

Basic angle for $x - 0.982793723$

$x - 0.982793723 = 0.982793723 + 2k\pi$, $(\pi - 0.982793723) + 2k\pi$

$x = 1.965587447 + 4k\pi$, $\pi + 2k\pi$ $(k \in \mathbb{Z})$

On $0 \le x \le 2\pi$

$x = 1.965587447,\ \pi$

1 Final Answer.

(13)

SOLUTIONS | **MARKS/COMMENTS**

$= \frac{1}{2}\left[\left(\frac{\pi}{16} - \frac{1}{4}\sin\frac{\pi}{4}\right) - \left(0 - \frac{1}{4}\sin 0\right)\right]$

$= \frac{1}{2}\left[\left(\frac{\pi}{16} - \frac{1}{4}\times\frac{1}{\sqrt{2}}\right) - (0-0)\right]$

$= \frac{\pi}{32} - \frac{1}{8\sqrt{2}}$

1 Any correct answer that results from the correct substitution.

c) $\frac{2x}{x-3} \le 5$

$\frac{2x}{x-3}(x-3)^2 \le 5(x-3)^2$

1 Start of any legitimate method.

$2x(x-3) \le 5(x-3)^2$

$0 \le 5(x-3)^2 - 2x(x-3)$

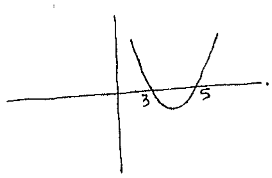
$(x-3)[5(x-3) - 2x] \ge 0$

$(x-3)(5x-15-2x) \ge 0$

$(x-3)(3x-15) \ge 0$

$3(x-3)(x-5) \ge 0$

1 Method leads to partially correct answer (eg. only one detail incorrect).



$x \le 3$ or $x \ge 5$

But $x \ne 3$

$\therefore x < 3$ or $x \ge 5$

1 Correct Answer in all respects.

(14)

SOLUTIONS | **MARKS/COMMENTS**

Question 6

a) i) $\ddot{x} = -9x$ is in form $\ddot{x} = -n^2x$ where $n = 3$.

Period $= \frac{2\pi}{n}$
$= \frac{2\pi}{3}$

1 Answer only.

$\because \ddot{x}$ is a function of x

OR $\frac{d}{dx}(\frac{1}{2}v^2) = -9x$
so $d(\frac{1}{2}v^2) = -9x\,dx$
so $\int d(\frac{1}{2}v^2) = \int -9x\,dx$
so $\frac{1}{2}v^2 = -\frac{9x^2}{2} + c$
when $x=0$, $v=10$
$\Rightarrow c = 50$

$\Rightarrow$ Hence $\frac{1}{2}v^2 = -\frac{9x^2}{2} + 50$
$v^2 = -9x^2 + 100$
so $v^2 = 9(\frac{100}{9} - x^2)$
$v^2 = n^2(a^2 - x^2)$
$\therefore a = \frac{10}{3}$.

OR Subst. $v = 0$
so $0 = -9x^2 + 100$
$x^2 = \frac{100}{9}$
$x = \pm\frac{10}{3}$
$\therefore$ a (max disp.) $= \frac{10}{3}$ #

ii) $v^2 = n^2(a^2 - x^2)$

When $v = 10$ and $x = 0$

$10^2 = 3^2(a^2 - 0^2)$

$100 = 9a^2$

$a^2 = \frac{100}{9}$

$a = \frac{10}{3}$ (amplitude)

1 Form and substitution resulting in amplitude of $\frac{10}{3}$

OR $x = a\sin(nt + \alpha)$
when $t=0$, $x=0$
Hence $\sin\alpha = 0$
so $\alpha = 0$.
$\Rightarrow x = \frac{10}{3}\sin 3t$ #

$\therefore v^2 = 9\left(\frac{100}{9} - x^2\right)$

1 For either of these.

or $v^2 = 100 - 9x^2$

(iii) $x = a\sin nt$ (if the particle starts at the middle of motion with positive velocity) $\Rightarrow x = \frac{10}{3}\sin 3t$.

b) Method 1 (t-method)

$2\left(\frac{2t}{1+t^2}\right) - 3\left(\frac{1-t^2}{1+t^2}\right) = 3$ where $t = \tan\frac{x}{2}$

1 Correct t-substitution

$\frac{4t - 3 + 3t^2}{1+t^2} = 3$

$4t - 3 + 3t^2 = 3(1+t^2)$

$4t - 3 + \cancel{3t^2} = 3 + \cancel{3t^2}$

$4t = 6$

$t = \frac{3}{2}$ (P.T.O.

1 Value of t

(19)

SOLUTIONS	MARKS/COMMENTS

$\therefore A = \pi r^2$

$= \pi\left(\frac{7h}{20}\right)^2$

$= \pi \, \frac{49h^2}{400}$

$A = \frac{49\pi h^2}{400}$

$\boxed{1}$ Correct expression for Area in terms of h.

ii) $\frac{dA}{dt} = \frac{dA}{dh} \times \frac{dh}{dt}$

$= \frac{49\pi h}{\cancel{50}\ 200} \times (-0\cdot 1)$

$\boxed{1}$ Use of chain rule and at least one derivative correctly substituted.

When $h = 4$

$\frac{dA}{dt} = \frac{49\pi \times 4}{200\ \cancel{50}} \times (-0\cdot 1)$

CNE ISE

$= -\frac{196\pi}{\cancel{50}\ 200}$

$= -\frac{196\pi}{\cancel{500}\ 2000}$

$= -\frac{49\pi}{\cancel{125}\ 500}$ m²/hour

$= -\cancel{1\cdot 23150432}$ m²/h.

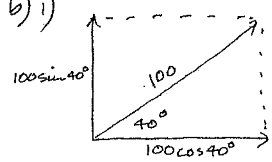
$\boxed{1}$ Answer, justified, in exact or decimal form.

$\doteqdot -0\cdot 3079$ m²/h #

(20)

SOLUTIONS	MARKS/COMMENTS

b) i)



Initial Velocities

$\boxed{1}$ Initial velocities justified.

Vertically

$\ddot{y} = -10$

$\dot{y} = -10t + C_1$

when $t=0$ $\dot{y} = 100\sin 40°$

$100\sin 40° = -10\times 0 + C_1$

$C_1 = 100\sin 40°$

$\therefore \dot{y} = -10t + 100\sin 40°$

$\boxed{1}$ Equation for $\dot{y}$ justified.

$y = -5t^2 + (100\sin 40°)t + c_2$

When $t=0$ $y = 30$

$30 = -5\times 0^2 + (100\sin 40°)(0) + c_2$

$C_2 = 30$

$\therefore y = -5t^2 + (100\sin 40°)t + 30$

$y = (100\sin 40°)t - 5t^2 + 30$

$\boxed{1}$ Equation for y justified.

(17)

SOLUTIONS	MARKS/COMMENTS

c) i) $B = -10$

$\therefore T = B + Ae^{kt}$

When $t=0$, $T = 18$

$\therefore 18 = -10 + Ae^{k\times 0}$

$28 = A\times 1$

$A = 28$

$\therefore T = -10 + 28e^{kt}$

$\boxed{1}$ B and A correctly justified.

When $t = 24$, $T = 14$

$14 = -10 + 28e^{k\times 24}$

$24 = 28e^{24k}$

$e^{24k} = \frac{6}{7}$

$24k = \ln\frac{6}{7}$

$k = \frac{1}{24}\ln\frac{6}{7}$

$(k = -0\cdot 006422944993)$

$T = -10 + 28e^{\frac{t}{24}\ln\frac{6}{7}}$

or

$T = -10 + 28e^{-0\cdot 006422944993t}$

$\boxed{1}$ Justified form

(18)

SOLUTIONS	MARKS/COMMENTS

ii) When $T=0$

$0 = -10 + 28e^{\frac{t}{24}\ln\frac{6}{7}}$

$10 = 28e^{\frac{t}{24}\ln\frac{6}{7}}$

$\boxed{1}$ Substitution of T=0 and substantial progress.

$\frac{5}{14} = e^{\frac{t}{24}\ln\frac{6}{7}}$

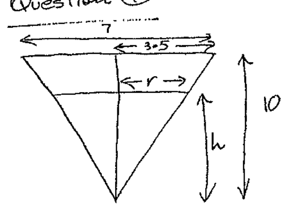
$\frac{t}{24}\ln\frac{6}{7} = \ln\frac{5}{14}$

$t = \frac{24\ln\frac{5}{14}}{\ln\frac{6}{7}}$

$= 160\cdot 3$ hours.

$\boxed{1}$ Final Answer.

Question 7



$\frac{10}{h} = \frac{3\cdot 5}{r}$ (similar triangles)

$10r = 3\cdot 5h$

$r = \frac{3\cdot 5h}{10} = \frac{\frac{7}{2}h}{10} = \frac{7h}{2}\times\frac{1}{10}$

$r = \frac{7h}{\cancel{10}\ 20}$

$\boxed{1}$ Basic similar triangles argument.

21

SOLUTIONS

Horizontally

$\ddot{x} = 0$

$\dot{x} = C_3$

When $t=0$ $\dot{x} = 100\cos 40^\circ$

$\therefore 100\cos 40^\circ = C_3$

$\therefore \dot{x} = 100\cos 40^\circ$

$x = (100\cos 40^\circ)t + C_3$

When $t=0$ $x=0$

$\therefore 0 = (100\cos 40^\circ)(0) + C_3$

$\Rightarrow C_3 = 0$

$\therefore x = (100\cos 40^\circ)t$

CFPA

ii) When $y=0$ (taking origin at foot of cliff).

$(100\sin 40^\circ)t - 5t^2 + 30 = 0$

$$t = \frac{(-100\sin 40^\circ) \pm \sqrt{(100\sin 40^\circ)^2 - 4\times(-5)\times 30}}{2\times(-5)}$$

$= 13.30665441 \quad (t>0)$

At this t-value

$x = (100\cos 40^\circ)(13.3066544)$

$= 1019.348867$ m.

MARKS/COMMENTS

* OR

Taking origin at point of projection,

$\Rightarrow y = -30.$

So $-30 = 100\sin 40^\circ t - 5t^2 + 30$

So $5t^2 - 100\sin 40^\circ t - 60 = 0.$

Hence $t = \dfrac{100\sin 40^\circ \pm \sqrt{(100\sin 40^\circ)^2 - 4(5)(-60)}}{2(5)}$

$= \dfrac{100\sin 40^\circ \pm \sqrt{100^2\sin^2 40^\circ + 1200}}{10}$

$= \dfrac{100\sin 40^\circ \pm 10\sqrt{100\sin^2 40^\circ + 12}}{10}$

$= 10\sin 40^\circ \pm \sqrt{100\sin^2 40^\circ + 12}$

$= 10\sin 40^\circ \pm 7.30\ldots$

So $t = 13.729$ or $t = -0.87$

$\Rightarrow x = (100\cos 40^\circ)(13.729)$

$= 1051.76..$ m.

$\boxed{1}$ Equation for x justified

$\Rightarrow 5t^2 - (100\sin 40^\circ)t - 30 = 0.$

$\therefore t = \dfrac{100\sin 40^\circ \pm \sqrt{(-100\sin 40^\circ)^2 - 4(5)(-30)}}{2(5)}$

$= \dfrac{100\sin 40^\circ \pm \sqrt{100^2\sin^2 40^\circ + 600}}{10}$

$= \dfrac{100\sin 40^\circ \pm 10\sqrt{100\sin^2 40^\circ + 6}}{10}$

$= 10\sin 40^\circ \pm \sqrt{100\sin^2 40^\circ + 6}$

$= 10\sin 40^\circ \pm 6.878$

$= 13.30..\quad (t>0)$

$\boxed{1}$ Value of t.

$\boxed{1}$ Range.

*

22

SOLUTIONS

iii) From $x = (100\cos 40^\circ)t$

$$t = \frac{x}{100\cos 40^\circ}$$

sub into equation for y

$$\therefore y = (100\sin 40^\circ)\left(\frac{x}{100\cos 40^\circ}\right) - 5\left(\frac{x}{100\cos 40^\circ}\right)^2 + 30$$

$$y = (\tan 40^\circ)x - 5\left(\frac{x^2}{100^2\cos^2 40^\circ}\right) + 30$$

$$y = (\tan 40^\circ)x - \frac{5x^2}{100^2\cos^2 40^\circ} + 30$$

$$y = (\tan 40^\circ)x - \frac{x^2}{2000}\sec^2 40^\circ + 30$$

$$y = (\tan 40^\circ)x - \frac{x^2}{2000}(1 + \tan^2 40^\circ) + 30$$

MARKS/COMMENTS

$\boxed{1}$ Value of t substituted into equation for y.

$\boxed{1}$ For equation.