Student name/number: ____________________

SOUTH SYDNEY HIGH SCHOOL

2001

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Total marks (84)

- Attempt Questions 1 – 7
- All questions are of equal value

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided on last page
- All necessary working should be shown in every question

Please note that this is a Trial paper only and cannot in any way guarantee the format or the content of the Higher School Certificate Examination.

Question 1 (12 marks) **Start a NEW page.** **Marks**

(a) Find $\int \frac{dx}{\sqrt{9-4x^2}}$ **2**

(b) Differentiate $y = 3e^{\tan 3x}$ **2**

(c) Find all possible values of k if the lines $2x + y + 3 = 0$ and $kx - y + 4 = 0$ intersect at $45°$. **2**

(d) The point $C(10, -7)$ divides the interval AB externally in the ratio 3:5. Find the coordinates of B if A has coordinates $(4, -1)$. **2**

(e) Write down the equation of the horizontal asymptote of $f(x) = \frac{3x}{x-7}$ **1**

(f) Solve $\frac{3}{x+1} \geq 4$ **3**

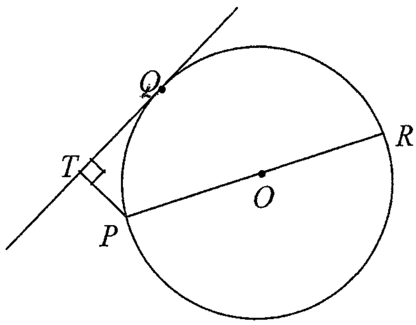
Question 2 (12 marks) **Start a NEW page.** **Marks**

(a) (i) In how many ways can the letters of the word **NONAGON** be arranged? **3**

(ii) Find the probability that the **N**'s are together?

(iii) Find the probability that the vowels are together?

(b) **3**



In the diagram P, Q and R are points on a circle centre O, with PR being a diameter. PT is the perpendicular from P to the tangent at Q.

Copy the diagram into your Writing Booklet

Prove that PQ bisects $\angle RPT$.

(c) Given that one root of $x^3 - 5x^2 - x + k + 6 = 0$ is 3, Find **3**

(i) the value of k

(ii) the sum and the product of the other two roots.

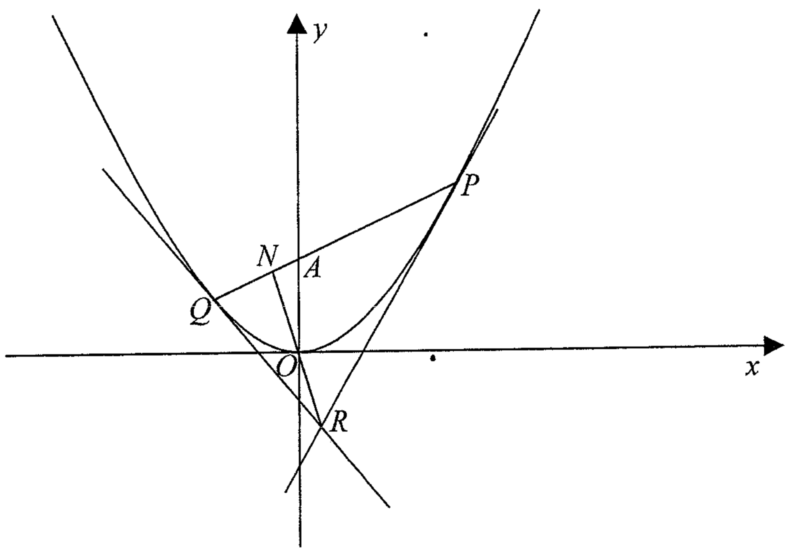
(d) Prove by induction that $5^n + 3$ is divisible by 4 for any integer $n \geq 1$. **3**

Question 3 (12 marks) **Start a NEW page.** **Marks**

(a) Evaluate $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\cos^{-1}\left(-\frac{1}{2}\right)$ **2**

(b) Out of 8 letters of which some are **E**'s and others are different, 6720 different eight-letter words can be formed. How many **E**'s are there? **2**

(c)



A parabola is defined by the parametric equations

$$x = 2t, \qquad y = t^2.$$

(i) Find the equations of the tangents at the points $P(2p, p^2)$ and $Q(2q, q^2)$. **2**

(ii) Show that the point of the intersection of the two tangents is at $R(p+q, pq)$. **2**

(iii) Show that the equation of the chord PQ is $(p+q)x - 2y - 2pq = 0$. **2**

(iv) If the points P and Q move on the parabola in such a way that pq remains constant and equal to -2, prove that the chord PQ always passes through the point $A(0,2)$. **1**

(v) Show that RN which passes through O is perpendicular to PQ. **1**

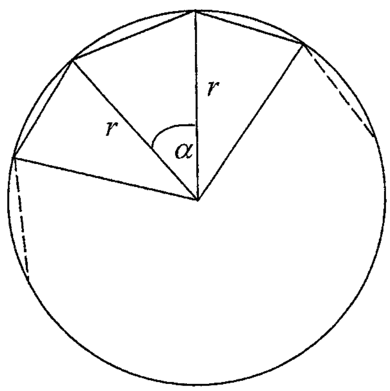
Question 4 (12 marks) **Start a NEW page.** **Marks**

(a) Using the substitution $u = 1 - 2x$, find **3**

$$\int 4x\sqrt{1-2x}\, dx$$

(b) The expression $x^2 + 9x + 4$ has the same remainder whether divided by $x - a$ or $x + b$, where $a \neq -b$. Find the value of $a - b$. **2**

(c)



The diagram above shows a regular n-sided polygon inscribed in a circle of radius r units. Each side of the polygon subtends an angle of α radians at the centre of the circle.

(i) Show that the perimeter of the polygon is $2nr\sin\dfrac{\pi}{n}$ **2**

(ii) Show that the area of the polygon is $\dfrac{1}{2}nr^2\sin\dfrac{2\pi}{n}$. **1**

(iii) Also show that the corresponding area of the circumscribed polygon is **2**

$$nr^2\tan\frac{\pi}{n}.$$

(iv) Deduce the area of the circle by using the inequality, **2**

Area of inscribed polygon < Area of circle < Area of circumscribed polygon

Question 5 (12 marks) **Start a NEW page.** **Marks**

(a) (i) Find the coefficient of x^7 in the expansion of $\left(px^2+\frac{1}{qx}\right)^{11}$. **3**

(ii) If this coefficient is equal to the coefficient of x^{-7} in the expansion of $\left(px-\frac{1}{qx^2}\right)^{11}$, prove that $pq=1$. **3**

(b) (i) By expanding $\cos(2\theta+2\theta)$ or otherwise, show that **3**

$$\cos 4\theta = 1-8\sin^2\theta+8\sin^4\theta.$$

(ii) Hence evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\sin^2\theta-\sin^4\theta)\,d\theta$ **3**

Question 6 (12 marks) **Start a NEW page.** **Marks**

(a) (i) By inserting n arithmetic means between n and $n+1$, show that the arithmetic sequence is **2**

$$n,\ n+\frac{1}{n+1},\ n+\frac{1}{n+2},\ \ldots,\ n+\frac{n+1}{n+1}$$

(ii) Hence show $n+\left(n+\frac{1}{n+1}\right)+\ldots+(n+1)=\frac{(n+2)(2n+1)}{2}$ **1**

(b) A particle is projected vertically upwards from a point 30 metres above the ground. The path of the particle is given by

$$h=6\left(5+9t-3t^2\right)$$

where h is the height in metres above the ground at time t seconds after projection. Find :

(i) the time taken to reach the greatest height. **2**

(ii) the greatest height reached. **1**

(iii) the magnitude and direction of the velocity after $2\frac{1}{2}$ seconds. **2**

(iv) the magnitude and direction of the acceleration. **1**

(c) The present temperature of a star is $8500^0 C$ and it is losing heat continuously in a way that in t million years, its temperature $T^0 C$ may be calculated from the equation

$$T=T_0e^{-0.06t}.$$

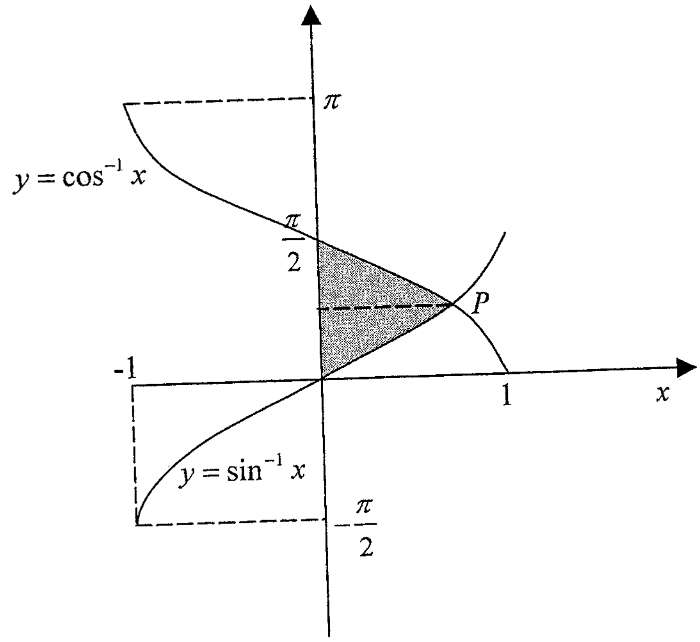
(i) Find the temperature of the star in 4 million years (to the nearest degree). **1**

(ii) After how many years from now will the temperature of the star be **halved.** **2**

Marks

Question 7 (12 marks) **Start a NEW page.**

(a)



The diagram above shows the shaded area between the curves $y = \sin^{-1} x,\ y = \cos^{-1} x$ and the y-axis.

(i) Show that the point of intersection P is $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$. **1**

(ii) Hence, show that the shaded area is equal to $\left(2-\sqrt{2}\right)$ sq. units. **3**

(iii) Show that the volume of solid generated by rotating this area about the y-axis is given by $\frac{\pi}{2}\left[\frac{\pi}{2}-1\right]$ cubic units. **3**

(b) The rate at which a metal block cools in air is assumed to be proportional to the difference between its temperature T and the constant temperature A of the surrounding air. This can be expressed by the differential equation;

$$\frac{dT}{dt} = k(T-A)$$

where t is the time in hours and k is a constant.

(i) Show that $T = A + Be^{kt}$ is a solution to the differential equation, given that B is a constant. **2**

(ii) A metal block which has been heated to 80°C cools to 40°C in two hours. If the air temperature around the metal block is 20°C, find the temperature of the metal block after one further hour has elapsed. Give your answer correct to the nearest degree. **3**

End of paper

Solutions

Question 1 (12 marks)

(a) $\int \frac{dx}{\sqrt{9-4x^2}} = \int \frac{dx}{2\sqrt{\frac{9}{4}-x^2}}$ ✓

$= \frac{1}{2}\int \frac{dx}{\sqrt{\frac{9}{4}-x^2}}$

$= \frac{1}{2}\sin^{-1}\frac{2x}{3}+C$ ✓

(b) $y = 3e^{\tan 3x}$

$\frac{dy}{dx} = 3e^{\tan 3x} \times 3\sec^2 3x$ ✓✓

$= 9e^{\tan 3x}\sec^2 3x$

(c) $\tan\theta = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right|$ $\theta = 45°$, $m_1 = k$, $m_2 = -2$

$\tan 45° = \left|\frac{k+2}{1-2k}\right|$

$1 = \left|\frac{k+2}{1-2k}\right|$ ✓

$\therefore 1-2k = k+2$ or $-(1-2k) = k+2$

$-1 = 3k$ $\quad -1+2k = k+2$

$-\frac{1}{3} = k$ $\quad k = 3$

$\therefore$ possible values of $k = -\frac{1}{3}$ or 3 ✓

(d) $A(4, -1)$, $B(x, y)$, $C(10, -7)$ ratio $= -3 : 5$

$P(X, Y) = \left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n}\right)$

For x; $10 = \frac{5\times 4 + (-3)\times x}{-3+5}$

$20 = 20 - 3x$

$x = 0$ ✓

For y; $-7 = \frac{5\times(-1) + (-3)\times y}{-3+5}$

$-14 = -5 - 3y$

$3y = 9$

$y = 3$

$\therefore B(0, 3)$ ✓

(e) horizontal asymptote $y = \lim_{x\to\infty} f(x)$

$\therefore y = \lim_{x\to\infty} \frac{3x}{x-7}$

$= \lim_{x\to\infty} \frac{3}{1-\frac{7}{x}}$

$= \frac{3}{1-0}$

$\therefore$ horizontal asymptote is $y = 3$ ✓

(f) $\frac{3}{x+1} \geq 4$, Note $x \neq -1$

$3(x+1) \geq 4(x+1)^2$ ✓

$3x+3 \geq 4x^2 + 8x + 4$

$0 \geq 4x^2 + 5x + 1$

$0 \geq (4x+1)(x+1)$ but $x \neq -1$ ✓

$\therefore -1 < x \leq -\frac{1}{4}$ ✓

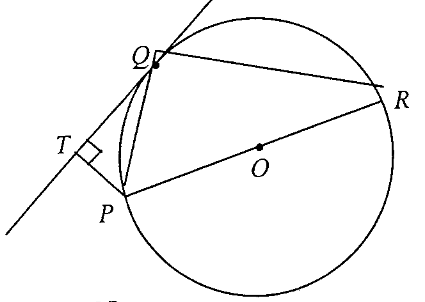
uestion 2 (12 marks)

)(i) $\frac{7!}{3!2!} = 420$ ✓

(ii) $\frac{5!}{2!} \div \frac{7!}{3!2!} = \frac{1}{7}$ ✓

(iii) $\frac{\frac{5!}{3!} \times \frac{3!}{2!}}{\frac{7!}{2!3!}} = \frac{1}{7}$ ✓

b)



Join PQ, QO and QR.

Let $\angle QPR = \alpha$

$\angle PQR = 90^\circ$ (angle in a semi-circle) ✓

$\therefore \angle QRP = 180^\circ - 90^\circ - \alpha$ (angle sum of $\Delta\, PQR$)

$= 90^\circ - \alpha$

$\angle QRP = \angle TQP$ (angle in alternate segment)✓

$\therefore \angle TQP = 90^\circ - \alpha$

$\therefore \angle QPT = 180^\circ - 90^\circ - (90^\circ - \alpha)$ (angle sum of $\Delta\, QPT$)

$\therefore \angle QPT = \alpha = \angle QPR$

Hence PQ bisects $\angle RPT$ ✓

(c) (i) $x = 3$ satisfies the equation

$3^3 - 5(3)^2 - 3 + k + 6 = 0$

$\therefore\ k = 15$ ✓

(ii) eqn becomes $x^3 - 5x^2 - x + 21 = 0$

sum of roots; $\alpha+\beta+\gamma = -\frac{b}{a}$

$\alpha+\beta+3 = 5$

$\alpha+\beta = 2$ ✓

product of roots; $\alpha\beta\gamma = -\frac{d}{a}$

$3\alpha\beta = -21$

$\alpha\beta = -7$ ✓

$\therefore$ the sum of the other two roots is 2

$\therefore$ the product of the other two roots is -7

(d) Prove $5^n + 3$ is divisible by 4

step 1 : Prove true for $n = 1$

$5^1 + 3 = 8$ which is divisible by 4

$\therefore$ true for $n = 1$. ✓

Step 2 : Assume true for $n = k$.

i.e. $5^k + 3 = 4p$ for some integer p

Step 3: Prove true for $n = k + 1$.

$5^{k+1} + 3 = 5^{k+1} + 15 - 12$

$= 5(5^n + 3) - 12$ ✓

$= 5 \times 4p - 12$

$= 4(5p - 3)$

which is divisible by 4 ✓

$\therefore$ true for $n = k + 1$.

Hence if it is true for $n = k$, then it is true for $n = k + 1$. We have proved that it is true for $n = 1$, so it must be true for $n = 2$. If it is true for $n = 2$, then it must be true for $n = 3$, and so on. Hence it is true for all $n \geq 1$.

Question 3 (12 marks)

(a) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\cos^{-1}\left(-\frac{1}{2}\right)=\frac{\pi}{4}+\left(\pi-\frac{\pi}{3}\right)$ ✓

$$=\frac{\pi}{4}+\frac{2\pi}{3}=\frac{11\pi}{12}. \checkmark$$

(b) Let the number of e's be n.

$$\therefore \frac{8!}{n!}=6720 \checkmark$$

$$\therefore n!=\frac{8!}{6720}=6$$

$$\therefore n=3 \checkmark$$

(c) (i) $\frac{dy}{dx}=\frac{dy}{dp}\times\frac{dp}{dx}$

$$=2p\times\frac{1}{2}=p \checkmark$$

$\therefore$ the equation of the tangent at P is

$$y-p^2=p(x-2p)$$

$$y-p^2=px-2p^2$$

$$\therefore px-y-p^2=0 \checkmark$$

$\therefore$ similarly, the equation of the tangent at Q is

$$\therefore qx-y-q^2=0$$

(ii) Solving simultaneously the equations of tangents at P and Q

$$px-y-p^2=0 \;\ldots\ldots\; (1)$$

$$qx-y-q^2=0 \;\ldots\ldots\; (2)$$

(1) – (2) gives

$$(p-q)x-(p^2-q^2)=0$$

$$x=\frac{(p+q)\cancel{(p-q)}}{\cancel{(p-q)}}=p+q \checkmark$$

Substituting into (1) gives

$$y=p(p+q)-p^2=pq \checkmark$$

(iii) Gradient of chord PQ is

$$m_{PQ}=\frac{p^2-q^2}{2p-2q}=\frac{(p+q)\cancel{(p-q)}}{2\cancel{(p-q)}}$$

$$=\frac{p+q}{2} \checkmark$$

$\therefore$ the equation of the chord PQ is

$$y-p^2=\frac{p+q}{2}(x-2p)$$

$$2(y-p^2)=(p+q)(x-2p)$$

$$2y-2p^2=(p+q)x-2p^2-2pq \checkmark$$

$$\therefore (p+q)x-2y-2pq=0 \text{ as required.}$$

(iv) If $pq=-2$ then

$$(p+q)x-2y+4=0$$

At $x=0,\ 2y=4$

$$\therefore y=2 \checkmark$$

$\therefore$ the coordinates of A is $(0,2)$ as required.

(v) The gradient of RN is

$$m_{RN}=\frac{pq-0}{(p+q)-0}=\frac{-2}{p+q}$$

Since $m_{PQ}\times m_{RN}=\frac{\cancel{p+q}}{2}\times\frac{-2}{\cancel{p+q}}=-1$ ✓

$\therefore PQ$ is perpendicular to RN.

Question 4 (12 marks)

(a) $u = 1 - 2x \qquad du = -2dx$

$2x = 1 - u \qquad dx = -\dfrac{du}{2}$ ✓

$$\int 4x\sqrt{1-2x}\,dx = \int 2(1-u)\sqrt{u}\,\frac{du}{-2}$$

$$= -\int (u^{\frac{1}{2}} - u^{\frac{3}{2}})du \checkmark$$

$$= -\left(\tfrac{2}{3}u^{\frac{3}{2}} - \tfrac{2}{5}u^{\frac{5}{2}}\right) + C$$

$$= \tfrac{2}{5}(1-2x)^{\frac{5}{2}} - \tfrac{2}{3}(1-2x)^{\frac{3}{2}} + C \checkmark$$

(b) Let $P(x) = x^2 + 9x + 4$

$P(a) = a^2 + 9a + 4$ .... (i)

$P(-b) = (-b)^2 - 9b + 4$ .... (ii)

Since $P(a) = P(-b)$

(i) – (ii) $\qquad a^2 - b^2 + 9(a+b) = 0$ ✓

$(a+b)(a-b) + 9(a+b) = 0$

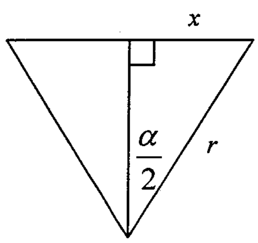
$(a+b)[a-b+9] = 0$

Since $a + b \neq 0 \Rightarrow a \neq -b$

$\therefore\ a - b + 9 = 0$

$\therefore\ a - b = -9$ ✓

(c) (i) Since $\alpha = \dfrac{2\pi}{n}$



$\sin\left(\dfrac{\alpha}{2}\right) = \dfrac{x}{r} \qquad \therefore\ x = r\sin\left(\dfrac{\pi}{n}\right)$ ✓

Each side of the polygon $= 2r\sin\left(\dfrac{\pi}{n}\right)$

Perimeter of the polygon is

$n \times 2r\sin\left(\dfrac{\pi}{n}\right) = 2nr\sin\left(\dfrac{\pi}{n}\right)$ ✓

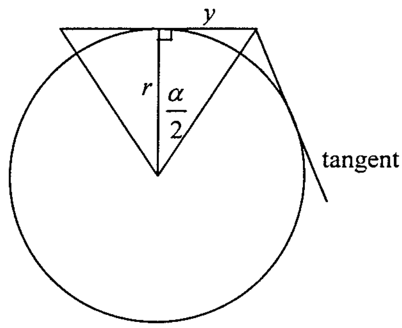
(ii) Area of each triangle is $\dfrac{1}{2}r^2\sin\alpha$

$= \dfrac{1}{2}r^2\sin\left(\dfrac{2\pi}{n}\right)$

Area of the polygon is $n \times$ Area of each triangle

$= \dfrac{1}{2}nr^2\sin\left(\dfrac{2\pi}{n}\right)$ sq. units. ✓

(iii) For the circumscribed polygon



$\tan\dfrac{\alpha}{2} = \dfrac{y}{r}$

$y = r\tan\dfrac{\pi}{n}$ ✓

Area of the each triangle is

$\dfrac{1}{2}bh = \dfrac{1}{2}\times 2r\tan\dfrac{\pi}{n}\times r \ = r^2\tan\dfrac{\pi}{n}$ ✓

Area of the circumscribed polygon is

$n \times r^2\tan\dfrac{\pi}{n}$ as required.

(iv) Using the inequality

$$A_{\text{insribed polygon}} < A_{\text{circle}} < A_{\text{circumscribed polygon}}$$

$$\frac{1}{2}nr^2\sin\frac{2\pi}{n} < A_{\text{circle}} < nr^2\tan\frac{\pi}{n}$$

taking the limit as n approaches infinity

$$\lim_{n\to\infty}\frac{1}{2}nr^2\sin\frac{2\pi}{n} = A_{\text{circle}} = \lim_{n\to\infty} nr^2\tan\frac{\pi}{n}$$

$$\lim_{n\to\infty}\frac{1}{2}r^2\times\frac{\sin\frac{2\pi}{n}}{\frac{1}{n}} = A_{\text{circle}} = \lim_{n\to\infty} r^2\frac{\tan\frac{\pi}{n}}{\frac{1}{n}} \checkmark$$

But note that $n \to \infty$; then $\dfrac{1}{n} \to 0$

Let $h=\dfrac{1}{n}$, $\therefore \lim\limits_{h\to 0}\dfrac{1}{2}r^2\dfrac{\sin 2\pi h}{h}$

$$=\lim_{h\to 0}\frac{1}{2}r^2\times\frac{\sin 2\pi h}{2\pi h}\times 2\pi \checkmark$$

$$\text{(Note: }\lim_{h\to 0}\frac{\sin 2\pi h}{2\pi h}=1)$$

$$=\frac{1}{2}r^2\times 1\times 2\pi=\pi r^2$$

Hence, the area of the circle is πr^2.

Or alternately use,

$$\lim_{h\to 0}r^2\frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}\times\pi=\pi r^2 .\checkmark\checkmark$$

Question 5 (12 marks)

(a)(i) $T_{r+1} = {}^{11}C_r\left(px^2\right)^{11-r}\left(\dfrac{1}{qx}\right)^r = Kx^r$

where K is a constant.

$= {}^{11}C_r p^{11-r} \times \dfrac{1}{q^r}\left(x^{22-2r}.x^{-r}\right) = Kx^7$ ✓

Equating the powers of x gives

$22-2r-r=7$

$\therefore\ 3r=15 \Rightarrow r=5$ ✓

$\therefore\ T_6 = {}^{11}C_5\left(px^2\right)^6\left(\dfrac{1}{qx}\right)^5$

$= {}^{11}C_5 p^6 q^{-5} x^7$

The coefficient is ${}^{11}C_5 p^6 q^{-5} = 462p^6q^{-5}$ ✓

(ii) For the expansion $\left(px-\dfrac{1}{qx^2}\right)^{11}$

$T_{s+1} = {}^{11}C_s\left(px\right)^{11-s}\left(-\dfrac{1}{qx^2}\right)^s = Kx^{-7}$ ✓

where K is a constant.

Similarly, comparing powers of x gives

$11-s-2s=-7$

$\therefore\ 3s=18 \Rightarrow s=6$ ✓

Since the coefficients are equal, then

${}^{11}C_5 p^6 q^{-5} = {}^{11}C_6 p^5 q^{-6}$

$\therefore\ \dfrac{p^6q^{-5}}{p^5q^{-6}} = \dfrac{{}^{11}C_6}{{}^{11}C_5}$

$\therefore\ \dfrac{p}{q^{-1}} = 1\ \left(\because\ {}^nC_r = {}^nC_{r-1}\right)$ ✓

$\therefore\ pq=1$ as required.

(b) (i) $\cos(2\theta+2\theta) = \cos^2 2\theta - \sin^2 2\theta$

$= 1-\sin^2 2\theta - \sin^2 2\theta$

$= 1-2\sin^2 2\theta$ ✓

$= 1-2(2\sin\theta\cos\theta)^2$ ✓

$= 1-2\left(4\sin^2\theta\left(1-\sin^2\theta\right)\right)$

$= 1-2\left(4\sin^2\theta-4\sin^4\theta\right)$ ✓

$= 1-8\sin^2\theta+8\sin^4\theta$ as required.

(ii) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(1-8\sin^2\theta+8\sin^4\theta\right)\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos 4\theta\, d\theta$

$\therefore 8\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\sin^2\theta-\sin^4\theta\right)d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(1-\cos 4\theta\right)d\theta$ ✓

$\therefore\ \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\sin^2\theta-\sin^4\theta\right)d\theta$

$= \dfrac{1}{8}\left[\theta-\dfrac{\sin 4\theta}{4}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ ✓

$= \dfrac{1}{8}\left[\left(\dfrac{\pi}{2}-\dfrac{\sin 2\pi}{4}\right)-\left(\dfrac{\pi}{4}-\dfrac{\sin\pi}{4}\right)\right]$

$= \dfrac{1}{8}\left[\left(\dfrac{\pi}{2}-0\right)-\left(\dfrac{\pi}{4}-0\right)\right]$

$= \dfrac{1}{8}\left[\dfrac{\pi}{4}\right] = \dfrac{\pi}{32}$. ✓

Question 6 (12 marks)

(a) (i) Let $a = n$ and $l = T_N = n+1$

where $N = n+2$

$$T_N = n + \left((n+2)-1\right)d = n+1$$

$$(n+1)d = 1$$

$$\therefore\ d = \frac{1}{n+1}\ ✓$$

Hence, the arithmetic sequence is

$$n,\ n+\frac{1}{n+1},\ n+\frac{2}{n+1},\ \ldots.,\ n+\frac{n+1}{n+1}\ ✓$$

(ii) $$n+\left(n+\frac{1}{n+1}\right)+\left(n+\frac{2}{n+1}\right)+\ldots+(n+1)$$

(Using $S_N = \frac{N}{2}(a+l)$) ✓

$$=\frac{n+2}{2}\left(n+(n+1)\right)$$

$$=\frac{(n+2)(2n+1)}{2}\ ✓$$

(b) $h = 6\left(5+9t-3t^2\right)$

Greatest height reached when $\frac{dh}{dt} = 0$ ✓

(i) $\frac{dh}{dt} = 54-36t$

$0 = 9-6t$

$\therefore\ t = 1\frac{1}{2}$ sec. ✓

(i) Greatest height when $t = 1\frac{1}{2}$ s

$h = 70\frac{1}{2}$ m. ✓

(ii) $\frac{dh}{dt} = v$

$\therefore\ v = 54-36t$ ✓

When $t = 2\frac{1}{2}$ sec, $v = -36$ m/s. ✓

(iii) $a = \frac{dv}{dt} = -36$ m/s^2 ✓

(c) i) $T = T_0 e^{-0.06t}$

$T = 8500e^{-0.06\times 4}$

$T = 6686^0 C$ ✓

(ii) $T = T_0 e^{-0.06t}$

$4250 = 8500e^{-0.06t}$

$\frac{1}{2} = e^{-0.06t}$ ✓

$\ln\left(\frac{1}{2}\right) = -0.06t$

$t = 11.55$ million years. ✓

Question 7 (12 marks)

(a) (i) Solving $\sin y = x$ and $\cos y = x$

$\sin y = \cos y$

$\tan y = 1$

$\therefore\ y = \tan^{-1} 1 = \frac{\pi}{4}$

$\therefore\ x = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

The point of intersection is $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$. ✓

(ii) The shaded area is equal to

$$\int_0^{\frac{\pi}{4}} \sin y\, dy + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos y\, dy \ ✓$$

$$= \left[-\cos y\right]_0^{\frac{\pi}{4}} + \left[\sin y\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left[\left(-\cos\frac{\pi}{4}\right) - \left(-\cos 0\right)\right] + \left[\sin\frac{\pi}{2} - \sin\frac{\pi}{4}\right] ✓$$

$$= -\frac{1}{\sqrt{2}} + 1 + 1 - \frac{1}{\sqrt{2}}$$

$$= 2 - \frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} ✓$$

$$= \left(2 - \sqrt{2}\right) \text{ sq. units. as required.}$$

(iii) The volume of solid of revolution about the y-axis is given by

$$V_y = \pi \int_0^{\frac{\pi}{4}} \sin^2 y\, dy + \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 y\, dy \ ✓$$

$$= \pi \int_0^{\frac{\pi}{4}} \frac{1}{2}(1 - \cos 2y)\, dy + \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}(\cos 2y + 1)$$

$$= \frac{\pi}{2}\left[\left(y - \frac{\sin 2y}{2}\right)\right]_0^{\frac{\pi}{4}} + \frac{\pi}{2}\left[\frac{\sin 2y}{2} + y\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} ✓$$

$$= \frac{\pi}{2}\left[\left(\frac{\pi}{4} - \frac{\sin\frac{\pi}{2}}{2}\right) - 0\right] + \frac{\pi}{2}\left[\left(\frac{\sin\pi}{2} + \frac{\pi}{2}\right) - \left(\frac{\sin\frac{\pi}{2}}{2} + \frac{\pi}{4}\right)\right]$$

$$= \frac{\pi}{2}\left[\frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{2} - \frac{1}{2} - \frac{\pi}{4}\right] ✓$$

$$= \frac{\pi}{2}\left[\frac{\pi}{2} - 1\right] \text{ cubic units.}$$

(b)(i) given $T = A + Be^{kt}$

differentiating gives $\frac{dT}{dt} = kBe^{kt}$ ✓

but $\frac{dT}{dt} = k(T - A)$

$= k(A + Be^{kt} - A)$ ✓

$= kBe^{kt}$

$\therefore\ T = A + Be^{kt}$ is a solution

(ii) when $t = 0$, $A = 20°\text{C}$, $T = 80\ °\text{C}$

$\therefore 80 = 20 + Be^0$

$B = 60$ ✓

$\therefore\ T = 20 + 60e^{kt}$

when $t = 2$, $T = 40\ °\text{C}$

$\therefore 40 = 20 + 60e^{2k}$

$20 = 60\,e^{2k}$

$\frac{1}{3} = e^{2k}$

$\therefore k = \frac{1}{2}\ln\frac{1}{3}$ ✓

when $t = 3$,

$T = 20 + 60e^{(3 \times \frac{1}{2}\ln\frac{1}{3})}$

$T = 31{\cdot}547\ldots.$

$\therefore T = 32\ °\text{C}$ ✓