SOUTH SYDNEY HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

1996

MATHEMATICS

3 UNIT (ADDITIONAL)
AND
3/4 UNIT (COMMON)

Time Allowed - Two hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- Write your student Name / Number on every page of the question paper and your answer sheets.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied.
- Board approved calculators may be used.
- The answers to the seven questions are to be handed in separately clearly marked Question 1, Question 2, etc..
- *The question paper must be handed to the supervisor at the end of the examination.*

Question 1: (Start a new page) **Marks**

a) The letters of the word **AROUND** are written at random on the circumference of a circle. 3

i) How many different permutations are possible?

ii) What is the probability that the ***three*** vowels are together?

b) Solve $\dfrac{1 - x}{1 + x} \leq 1$ 3

c) For the polynomial $P(x) = x^4 - 7x^3 + 12x^2 + 4x - 16$, 4

i) Show that the remainder is zero when $P(x)$ is divided by $x - 2$

ii) Given $x = 2$ is a double root of $P(x)$, find the factors of $P(x)$

d) Find $\int_0^{\pi} (2\sin x - \sin 2x)\, dx$ 2

Question 2: (Start a new page)

a) i) Draw a neat sketch of the curve $y = 3\sin 2x$ for $0 \leq x \leq \pi$ 5

ii) The area between the curve $y = 3\sin 2x$, $x = 0$ and $x = \pi$ and the x-axis is rotated about the x-axis. Find the volume of the solid generated. (Leave your answer in terms of π)

b) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are points on the parabola $x^2 = 4ay$. If PQ is a normal at P, show that $p^2 + pq + 2 = 0$. 3

c) i) Write down the expansion for $\tan(\alpha - \beta)$ 4

ii) The locus of the point $P(x, y)$ is given by

$$\tan^{-1}\left(\frac{y}{x - 2}\right) - \tan^{-1}\left(\frac{y}{x + 2}\right) = \frac{\pi}{4}$$

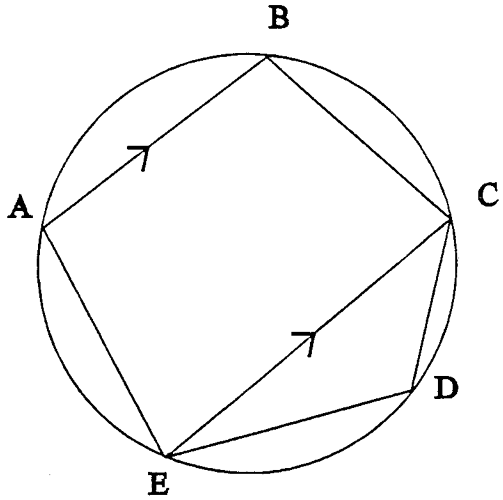
Using Part (i) or otherwise, show that the point P lies on a circle and find its centre and radius.

Question 3: (Start a new page) **Marks**

a) A, B, C, D and E are points on the circumference of a circle such that **AB** ∥ EC 3

Copy the diagram into your workbook.

Prove that $\angle ADE = \angle BDC$



b) A man in a balloon observes a dog due South with an angle of depression of 66°. He also observes a cat on a bearing of 120° from a point on the ground beneath the balloon with an angle of depression of 78°. If the dog and the cat are 100 metres apart, how high is the man in the balloon? 5

c) Susan invests \$6000 for 5 years. The expected return over the 5 years is 8% per annum, compounded monthly. 4

i) What is the expected value of the investment at the end of the 5 years?

ii) In fact, at the end of the second year, the interest rate reduces to 6% per annum and remains at this level for the remaining three years. Calculate the value of the investment at the end of the 5 years.

Question 4: (Start a new page)

a) Find the coefficient of x^3 in the expansion of $\left(2x - \dfrac{1}{x^2}\right)^9$ 3

b) In a factory, it is known that one in every hundred machines produced is defective. If a company buys 10 machines, what is the probability that no more than one is defective? Give your answer as a decimal correct to 3 decimal places. 3

Question 4 (continued) **Marks**

c) The velocity, v ms^{-1}, of a particle at time, t seconds, is given in terms of its position, x m, by the equation 6

$$v = \frac{4}{x} \qquad (x > 0)$$

Initially, $x = 8$.

i) Find the acceleration of the particle when $x = 1$

ii) By expressing v as $\frac{dx}{dt}$, find an expression for x in terms of t.

iii) What is the position of the particle when $t = 2$?

iv) Describe the motion of the particle.

Question 5: (Start a new page)

a) Find $\int x(3x - 1)^3 \, dx$ using the substitution $u = 3x - 1$ 2

b) i) Show that there is a root of the equation $\log_e x - \sin x = 0$ between $x = 2$ and $x = 3$ 4

ii) Using a first approximation of $x = 2.5$, use Newton's Method twice to obtain a better approximation.

c) A stone is thrown from the top of a building, 15 metres high, with an initial velocity of 26 ms^{-1} at an angle of $\tan^{-1}\left(\frac{5}{12}\right)$ to the horizontal. If the acceleration due to gravity is taken as 10 ms^{-2}, find, by deriving the relevant equations of motion, 6

i) the greatest height above the ground that the stone reaches.

ii) the time of flight and the range of the stone.

iii) the velocity and direction of motion after 2 seconds.

Question 6: (Start a new page) **Marks**

a) A particle moves in Simple Harmonic Motion so that its position, x cm, at any time, t, is given by the equation **6**

$$x = 4\cos\left(2t - \frac{\pi}{2}\right)$$

i) Find $\dot{x}$ and $\ddot{x}$

ii) Show that $v^2 = 4(16 - x^2)$

iii) Determine the time taken for the particle to first reach $x = 2$ and find the velocity at this time.

b) i) Differentiate $(x + 1)e^{-x}$ **3**

ii) Hence find the area of the region bounded by $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 2$

c) Consider the statement **3**

$$5^n - 1 \text{ is divisible by } 4.$$

Show that, if the statement is true for $n = k$, then it is true for $n = k + 1$

Question 7: (Start a new page) **Marks**

a) The rate of change of temperature, $T°$, of an object is given by the equation 5

$$\frac{dT}{dt} = k(T - 16) \text{ degrees per minute, where } k \text{ is a constant.}$$

i) Show that the function $T = 16 + Pe^{kt}$, where P is a constant and t is the time in minutes, satisfies this equation.

ii) If initially $T = 0$ and after 10 minutes $T = 12$, find the values of P and k.

iii) Find the temperature of the object after a further 5 minutes.

iv) Sketch the graph of T as function of t and describe its behaviour as t continues to increase.

b) If $f(n) = 2(\log_e 2)^n - n \times f(n - 1)$ and $f(0) = 2$, show that 2

$$f(4) = 2(\log_e 2)^4 - 8(\log_e 2)^3 + 24(\log_e 2)^2 - 48\log_e 2 + 48$$

c) i) Show that 5

$$(1)\ 1 + \frac{2}{1 + \sqrt{5}} = \frac{1 + \sqrt{5}}{2} \qquad (2)\ 1 + \frac{2}{1 - \sqrt{5}} = \frac{1 - \sqrt{5}}{2}$$

ii) Hence, or otherwise, show that

$$\left(\frac{1 + \sqrt{5}}{2}\right)^k - \left(\frac{1 - \sqrt{5}}{2}\right)^k + \left(\frac{1 + \sqrt{5}}{2}\right)^{k - 1} - \left(\frac{1 - \sqrt{5}}{2}\right)^{k - 1}$$
$$= \left(\frac{1 + \sqrt{5}}{2}\right)^{k + 1} - \left(\frac{1 - \sqrt{5}}{2}\right)^{k + 1}$$

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

1996

NSW Independent Trial HSC: Solutions to Mathematics 3 Unit Paper

Q1(a)(i) $5! = 120$

(ii) $P(3\text{ vowels}) = \frac{3!3!}{5!} = \frac{3}{10}$

(b) $\frac{1-x}{1+x} \le 1$

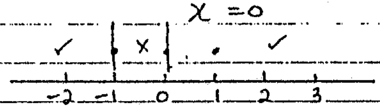
Critical values at: $x=-1$

and $\frac{1-x}{1+x} = 1$

$1-x = 1+x$

$0 = 2x$

$x = 0$



Test: $x=1 \Rightarrow \frac{1-1}{1+1} \le 1$ ✓

$x=-\frac{1}{2} \Rightarrow \frac{1-(-\frac{1}{2})}{1+(-\frac{1}{2})} = 3 \not\le 1$ ×

$x=-2 \Rightarrow \frac{1-(-2)}{1+(-2)} = -3 \le 1$ ✓

$\therefore x < -1$ and $x \ge 0$

(c)(i) $P(2) = 2^4 - 7\times2^3 + 12\times2^2 + 4\times2 - 16$

$= 0$

$\therefore$ Remainder is 0

(ii) $P(x) = (x-2)(x-2)Q(x)$

$P(-1) = 0 \quad \therefore (x+1)$ is a factor

$\therefore P(x) = (x-2)^2(x+1)(x-4)$

(d) $\int_0^{\pi} 2\sin x - \sin 2x \, dx$

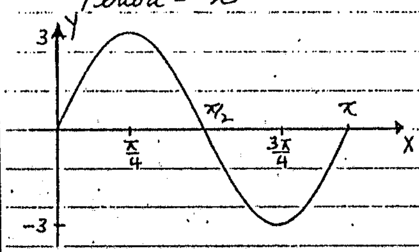
$= \left[-2\cos x + \frac{1}{2}\cos 2x\right]_0^{\pi}$

$= \left(-2\cos\pi + \frac{1}{2}\cos 2\pi\right) - \left(-2\cos 0 + \frac{1}{2}\cos 0\right)$

$= 4$

Q2(a)(i) Amplitude $= 3$

Period $= \pi$



(ii) $V = \pi\int_a^b y^2\,dx$

$= \pi\int_0^{\pi} 9\sin^2 2x\,dx$

$= 9\pi\int_0^{\pi}\frac{1}{2}(1-\cos 4x)\,dx$

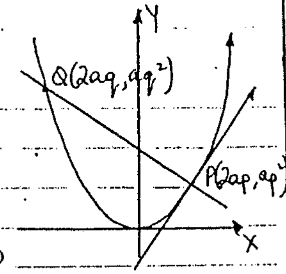
$= \frac{9\pi}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi}$

$= \frac{9\pi}{2}\left[\left(\pi - \frac{1}{4}\sin 4\pi\right) - \left(0 - \frac{1}{4}\sin 0\right)\right]$

$= \frac{9\pi^2}{2}$

3 Unit Solutions Q2 (continued)

Q2(b)



Gradient of PQ

$= \frac{ap^2 - aq^2}{2ap - 2aq}$

$= \frac{p+q}{2}$

$m_{tangent} = p$

$\Rightarrow m_{normal} = -\frac{1}{p}$

So $-\frac{1}{p} = \frac{p+q}{2}$

$-2 = p^2 + pq$

$p^2 + pq + 2 = 0$

(c)(i) $\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

(ii) $\tan\left[\tan^{-1}\left(\frac{y}{x-2}\right) - \tan^{-1}\left(\frac{y}{x+2}\right)\right] = \tan\frac{\pi}{4}$

$\frac{\frac{y}{x-2} - \frac{y}{x+2}}{1 + \frac{y}{x-2}\times\frac{y}{x+2}} = 1$ (from (i))

$y(x+2) - y(x-2) = (x-2)(x+2) + y^2$

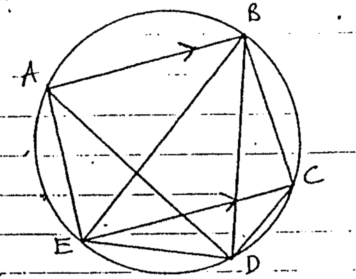
$yx + 2y - yx + 2y = x^2 - 4 + y^2$

$x^2 + y^2 - 4y = 4$

$x^2 + (y-2)^2 = 8$

which is a circle, centre $(0,2)$ and radius $\sqrt{8}$.

Q3(a)



Construct AD, BD and BE

Then $\angle BEC = \angle BDC$ (angles on same arc)

$\angle BEC = \angle ABE$ (alternate angles on parallel lines)

$\angle ABE = \angle ADE$ (angles on same arc)

$\therefore \angle BDC = \angle ADE$

3 UNIT SOLUTIONS Q3 (continued).

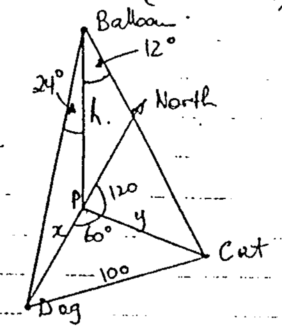
3(b)

$\tan 24 = \frac{x}{h}$

$\therefore x = h \tan 24$

$\tan 12 = \frac{y}{h}$

$\therefore y = h \tan 12$



$100^2 = x^2 + y^2 - 2xy \cos 60$

$= h^2 \tan^2 24 + h^2 \tan^2 12 - 2 . h \tan 24 \times h \tan 12 \times \frac{1}{2}$

$= h^2 (\tan^2 24 + \tan^2 12 - \tan 24 \tan 12)$

$\therefore h^2 = 100^2 / (\tan^2 24 + \tan^2 12 - \tan 24 \tan 12)$

$h = 259.26$ m $\sim$ 260 metres

(c)(i) $P = 6000$, $n = 60$, $r = 8/12$

$A = P(1 + \frac{r}{100})^n$

$= 6000 \left(1 + \frac{8/12}{100}\right)^{60}$

$= \$8939.07$

(ii) Calculate the accumulated value after 2 years. This becomes the principal for the remaining 3 years.

$P = 6000$; $n = 24$; $r = 8/12$

$\Rightarrow A = 6000 \left(1 + \frac{8/12}{100}\right)^{24} = \$7037.33$

then $P = 7037.33$, $n = 36$; $r = 6/12$

$\Rightarrow A = 7037.33 \left(1 + \frac{6/12}{100}\right)^{36} = \$8421.43$

Q4(a) $(a+b)^n = \sum_0^n {}^nC_r \, a^{n-r} b^r$

$\left(2x - \frac{1}{x^2}\right)^9 = \sum_0^9 {}^9C_r (2x)^{9-r} \left(-\frac{1}{x^2}\right)^r$

$\therefore$ Each term $= {}^9C_r \, 2^{9-r} x^{9-r} \frac{(-1)^r}{x^{2r}}$

$= {}^9C_r \, 2^{9-r} (-1)^r x^{9-3r}$

We require the coefficient of x^3:

$\therefore 9 - 3r = 3$

$\Rightarrow r = 2$

giving ${}^9C_2 \, 2^7 (-1)^2 x^3 = {}^+4608 x^3$

$\therefore$ coefficient is ${}^+4608$

(b) Let p = probability of defective $= \frac{1}{100}$

q = probability of non defective $= \frac{99}{100}$

$n = 10$

Let X = no of defective machines.

Then $P(X=r) = {}^nC_r \, p^r q^{n-r}$

$= {}^{10}C_r \left(\frac{1}{100}\right)^r \left(\frac{99}{100}\right)^{10-r}$

P(no more than 1)

$= P(X=0) + P(X=1)$

$= {}^{10}C_0 \left(\frac{1}{100}\right)^0 \left(\frac{99}{100}\right)^{10} + {}^{10}C_1 \left(\frac{1}{100}\right)^1 \left(\frac{99}{100}\right)^9$

$= 0.996$

3 UNIT SOLUTIONS Q4 (continued)

Q4(c)(i) $v = \frac{4}{x}$

$a = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$

$= -\frac{4}{x^2} \times \frac{4}{x} = \frac{-16}{x^3}$

$\therefore$ when $x = 1$, $a = \frac{-16}{1} = -16 \text{ ms}^{-2}$

(ii) $\frac{dx}{dt} = \frac{4}{x}$

$\therefore \frac{dt}{dx} = \frac{x}{4}$

$t = \frac{x^2}{8} + C$

At $t = 0$, $x = 8 \Rightarrow C = -8$

$\therefore t = \frac{x^2}{8} - 8$

$8t + 64 = x^2$

$\Rightarrow x = \sqrt{8t + 64}$ (since $x > 0$)

(iii) At $t = 2$, $x = \sqrt{80}$

(iv) The particle moves away from $x = 8$ with velocity decreasing and acceleration, acting in a negative direction, also decreasing.

Q5(a) If $u = 3x - 1$

$du = 3\,dx \Rightarrow dx = \frac{1}{3} du$

Also $x = \frac{1}{3}(u+1)$

$\therefore I = \int x (3x-1)^3 dx$

$= \int \frac{1}{3}(u+1) . u^3 . \frac{1}{3} du$

$= \frac{1}{9} \int u^4 + u^3 \, du$

$= \frac{1}{9}\left(\frac{u^5}{5} + \frac{u^4}{4}\right) + C$

$= \frac{1}{45}(3x-1)^5 + \frac{1}{36}(3x-1)^4 + C$

(b)(i) $f(x) = \ln x - \sin x$

$f(2) = \ln 2 - \sin 2 = -0.216 < 0$

$f(3) = \ln 3 - \sin 3 = +0.957 > 0$

+ $f(x)$ is continuous between $x = 2$, $x = 3$

$\therefore$ A root exists between $x = 2$ + $x = 3$

(ii) $f'(x) = 1/x - \cos x$

so $x_1 = x - \frac{f(x)}{f'(x)}$

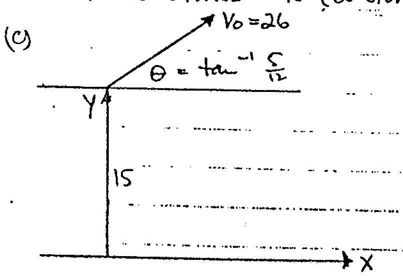
$= x - \frac{\ln x - \sin x}{1/x - \cos x}$

If $x = 2.5$, $x_1 = 2.235$

and $x_2 = 2.219$

3 UNIT SOLUTIONS Q5 (continued)

(c)



As $\theta = \tan^{-1}\frac{5}{12}$, $\sin\theta = \frac{5}{13}$,
$\cos\theta = \frac{12}{13}$

$x'' = 0$	$y'' = -g$
$x' = V\cos\theta$	$y' = -gt + V\sin\theta$
$= 26 \times \frac{12}{13}$	$= -10t + 26 \times \frac{5}{13}$
$= 24$	$= -10t + 10$
$x = 24t$	$y = -5t^2 + 10t + 15$

(i) Max height when $y' = 0$
$-10t + 10 = 0$
$t = 1$

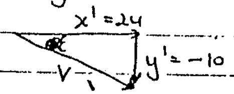
whence $y = -5 \times 1^2 + 10 \times 1 + 15 = 20\text{ m}$

(ii) Time of flight when $y = 0$:

$-5t^2 + 10t + 15 = 0$
$-5(t^2 - 2t - 3) = 0$
$-5(t-3)(t+1) = 0$
$\Rightarrow t = 3$ seconds

Range $= 24 \times 3$
$= 72\text{ m}$.

(iii) At $t = 2$:
$x' = 24$; $y' = -10$



Velocity, $v = 26$
+ $\tan\alpha = \frac{10}{24} = \frac{5}{12}$

$\therefore \alpha = \tan^{-1}\left(\frac{5}{12}\right)$ down.

Alternatively, since the motion is parabolic and $t = 1$ is the turning point, $v = 26$ + $\alpha = \tan^{-1}\left(\frac{5}{12}\right)$

3 UNIT SOLUTIONS Q6

6(a)(i) $x = 4\cos\left(2t - \frac{\pi}{2}\right)$

$v = \dot{x} = -8\sin\left(2t - \frac{\pi}{2}\right)$
$a = \ddot{x} = -16\cos\left(2t - \frac{\pi}{2}\right)$

(ii) $v^2 = 64\sin^2\left(2t - \frac{\pi}{2}\right)$
$= 64\left(1 - \cos^2\left(2t - \frac{\pi}{2}\right)\right)$
$= 4\left(16 - 16\cos^2\left(2t - \frac{\pi}{2}\right)\right)$
$= 4(16 - x^2)$

(iii) At $x = 2$, $4\cos\left(2t - \frac{\pi}{2}\right) = 2$
$\cos\left(2t - \frac{\pi}{2}\right) = \frac{1}{2}$
$2t - \frac{\pi}{2} = -\frac{\pi}{3}, \frac{\pi}{3}$
$2t = \frac{\pi}{6}, \frac{5\pi}{6}$
$\therefore t = \frac{\pi}{12}$

also, $v = -8\sin\left(2t - \frac{\pi}{2}\right)$
$= -8\sin\left(\frac{\pi}{6} - \frac{\pi}{2}\right)$
$= +4\sqrt{3}$

(b)(i) $\frac{d}{dx}\left[(x+1)e^{-x}\right] =$

$= -(x+1)e^{-x} + e^{-x}.1$
$= -xe^{-x}$

$\therefore \int xe^{-x}\,dx = -(x+1)e^{-x} + C$

+ $\int_0^2 xe^{-x}\,dx = \left[-(x+1)e^{-x}\right]_0^2$

$= -3e^{-2} + 1$
$= 1 - 3e^{-2}$
$\doteqdot 0.594$

(c) Assume $5^k - 1 = 4M$,
where M is an integer.

Then $5^{k+1} - 1$
$= 5^k.5^1 - 1$
$= (5^k - 1).5 + 4$
$= 4M \times 5 + 4$ (from above)
$= 4(5M + 1)$

but, since M is an integer, $5M + 1$ is an integer and so $5^{k+1} - 1$ is a multiple of 4

As the statement is true for $n = k+1$ if it is true for $n = k$.

3 UNIT SOLUTIONS Q7

Q7(a)(i) $T = 16 + Pe^{kt}$

$\frac{dT}{dt} = 0 + P.ke^{kt}$

$= k.Pe^{kt}$

$= k(T-16)$

(ii) At $t=0$, $T=0 \Rightarrow 0 = 16 + Pe^{0}$

$\therefore P = -16$

So $T = 16 - 16e^{kt}$

At $t=10$, $T=12 \Rightarrow 12 = 16 - 16e^{10k}$

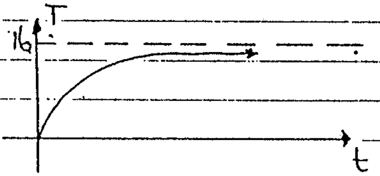
$e^{10k} = 4/16$

$\therefore k \doteq \frac{1}{10}\ln\frac{1}{4} = -0.139$

(iii) At $t=15$, $T = 16 - 16e^{-0.139\times 15}$

$= 14$

(iv)



As $t \to \infty$, $T \to 16$

(b) $f(4) = 2(\ln 2)^4 - 4f(3)$

$f(3) = 2(\ln 2)^3 - 3f(2)$

$f(2) = 2(\ln 2)^2 - 2f(1)$

$f(1) = 2\ln 2 - f(0)$ $\quad f(0) = 2$

$\therefore f(1) = 2\ln 2 - 2$

$f(2) = 2(\ln 2)^2 - 4\ln 2 + 4$

$f(3) = 2(\ln 2)^3 - 6(\ln 2)^2 + 12\ln 2 - 12$

$f(4) = 2(\ln 2)^4 - 8(\ln 2)^3 + 24(\ln 2)^2 - 48\ln 2 + 48$

(c)(i) 1. $1 + \frac{2}{1+\sqrt5} = 1 + \frac{2.(1-\sqrt5)}{1-5}$

$= \frac{-4+2-2\sqrt5}{-4}$

$= \frac{-2-2\sqrt5}{-4} = \frac{1+\sqrt5}{2}$

2. $1 + \frac{2}{1-\sqrt5} = 1 + \frac{2(1+\sqrt5)}{1-5}$

$= \frac{-4+2+2\sqrt5}{-4}$

$= \frac{-2+2\sqrt5}{-4} = \frac{1-\sqrt5}{2}$

(ii) $\text{LHS} = \left(\frac{1+\sqrt5}{2}\right)^k + \left(\frac{1+\sqrt5}{2}\right)^{k-1} - \left[\left(\frac{1-\sqrt5}{2}\right)^k + \left(\frac{1-\sqrt5}{2}\right)^{k-1}\right]$

$= \left(\frac{1+\sqrt5}{2}\right)^k\left[1 + \left(\frac{1+\sqrt5}{2}\right)^{-1}\right] - \left(\frac{1-\sqrt5}{2}\right)^k\left[1 + \left(\frac{1-\sqrt5}{2}\right)^{-1}\right]$

$= \left(\frac{1+\sqrt5}{2}\right)^k\left[1 + \frac{2}{1+\sqrt5}\right] - \left(\frac{1-\sqrt5}{2}\right)^k\left[1 + \frac{2}{1-\sqrt5}\right]$

$= \left(\frac{1+\sqrt5}{2}\right)^k . \left(\frac{1+\sqrt5}{2}\right) - \left(\frac{1-\sqrt5}{2}\right)^k . \left(\frac{1-\sqrt5}{2}\right)$ (from above)

$= \left(\frac{1+\sqrt5}{2}\right)^{k+1} - \left(\frac{1-\sqrt5}{2}\right)^{k+1}$