SOUTH SYDNEY HIGH

2008
HALF YEARLY
EXAMINATION

Mathematics
Extension 1

General Instructions

- Reading Time- 5 minutes
- Working Time – 1.5 hours
- Write using a black or blue pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Begin each question on a fresh sheet of paper.

Total marks (**48**)

- Attempt Questions 1-4
- All questions are of equal value

Total Marks – 48
Attempt Questions 1-4
All Questions are of equal value

Begin each question on a NEW SHEET of paper, writing your name and question number at the top of the page. Extra paper is available.

QUESTION 1 (12 MARKS) Begin a NEW sheet of writing paper. **Marks**

a) Calculate the acute angle (to the nearest minute) between the lines : **2**

$$2x + y = 4$$
and
$$x - 3y = 6$$

b) Use the table of standard integrals to show that **2**

$$\int_6^{15} \frac{dx}{\sqrt{x^2 + 64}} = \ln(2)$$

c) Solve $\frac{2x-3}{x-1} \leq 4$ **3**

d) Evaluate $\sum_{n=2}^{8} \left(n^2 - n\right)$ **1**

e) Show that $2x - 1$ is a factor of $2x^3 + 5x^2 + x - 2$ **2**

f) Find $\int \sin x \cos x \, dx$ using the substitution $u = \sin x$ **2**

QUESTION 2 (12 MARKS) Begin a NEW sheet of writing paper. **Marks**

a) Find the Cartesian equation of the curve represented by the parametric equations below: **3**

$$x = 2t - 1$$
$$y = t^2 + 2t$$

b) Find the volume of the solid of revolution formed when the section of the curve $y = \cos x$ between $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ is rotated about the x axis. **3**

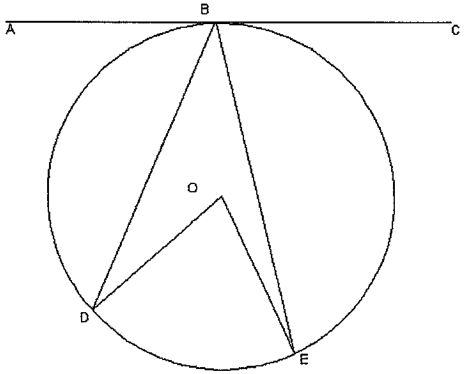
c) Considering the letters which form the word **DESCARTES**.

(i) How many distinct arrangements of the letters are possible? **1**

(ii) How many distinct arrangements are possible if the two letter **S**'s must be placed so that one is at the beginning and one at the end? **1**

(iii) How many distinct arrangements are possible if the two letter **S**'s must be placed together? **1**

d) In the diagram below, O is the centre of the circle, AC is a tangent at B and D and E are points on the circumference. If $\angle$ABD = 80° and $\angle$DBE = 40°, find the size of $\angle$BEO, giving reasons. **3**



QUESTION 3 (12 MARKS) Begin a NEW sheet of writing paper. **Marks**

a) The graph $y = \sin(e^x)$ has a zero close to $x = 1$. Use one application of Newton's method to find a second approximation for this zero, giving your result correct to 3 significant figures. **3**

b) (i) Show that $\sin x - \cos 2x = 2\sin^2 x + \sin x - 1$ **2**

(ii) Hence or otherwise solve $\sin x - \cos 2x = 0$ for $0 \le x \le 2\pi$ **3**

c) For the cubic equation $2x^3 - 3x^2 + 5x - 2 = 0$ with roots, $x = \alpha$, $x = \beta$ and $x = \gamma$, find the value of : **4**

(i) $\alpha^2 + \beta^2 + \gamma^2$

(ii) $\frac{1}{\alpha^2\beta^2} + \frac{1}{\beta^2\gamma^2} + \frac{1}{\alpha^2\gamma^2}$

QUESTION 4 (12 MARKS) Begin a NEW sheet of writing paper. **Marks**

a) The chord joining P $(2ap, ap^2)$ and Q $(2aq, aq^2)$ on the parabola $x^2 = 4ay$ subtends a right angle at the vertex of the parabola.

(i) Show that $pq = -4$ **1**

(ii) Show that the locus of the point M, the midpoint of PQ, is also a parabola and give its vertex. **3**

b) (i) By considering the cases where a positive integer k is even $(k = 2x)$ and odd $(k = 2x+1)$, show that $k^2 + k$ is always even. i.e. $k^2 + k = 2m$, where m is also an integer. **2**

(ii) Prove, by Mathematical induction, that for all positive integral values of n, **3**

$n^3 + 5n$ is divisible by 6.

c) (i) Given the series expansion for $e^h = 1 + \frac{h}{1!} + \frac{h^2}{2!} + \frac{h^3}{3!} + ...$ show that **1**

$$\frac{e^h - 1}{h} = 1 + \frac{h}{2!} + \frac{h^2}{3!} + ...$$

(iii) Hence, use the definition $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ to find the derivative of $f(x) = e^x$. **2**

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a}\sin ax, a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax, a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax, a \neq 0$$

$$\int \sec ax \tan ax dx = -\frac{1}{a}\sec ax, a \neq 0$$

$$\int \frac{1}{a + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, a > 0, -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

Note: $\ln x = \log_e x, x > 0$

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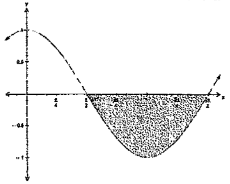
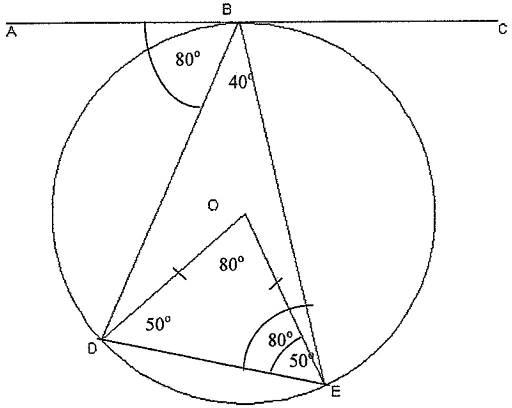
Solutions

Solutions Question 1	Marks/Comments
1. a) For $2x+y=4$ $m_1=-2$ and for $x-3y=6$ $m_2=\frac{1}{3}$ so: $\tan\theta=\left\|\frac{m_1-m_2}{1+m_1m_2}\right\|$ $=\left\|\frac{-2-\frac{1}{3}}{1+-2\times\frac{1}{3}}\right\|$ ✓ $=\left\|\frac{-2\frac{1}{3}}{\frac{1}{3}}\right\|$ $\tan\theta=\|-7\|$ ✓ $\theta=81°52'$	2 marks for correct formula, substitution and evaluation. 1 mark if a single error is made in one step.
1 b) From Standard Integrals $\int_6^{15}\frac{dx}{\sqrt{x^2+64}}=[\ln(x+\sqrt{x^2+64})]_6^{15}$ $=\left(\ln\left(15+\sqrt{15^2+64}\right)\right)-\left(\ln\left(6+\sqrt{6^2+64}\right)\right)$ $=\left(\ln\left(15+\sqrt{289}\right)\right)-\left(\ln\left(6+\sqrt{100}\right)\right)$ $=(\ln(32))-(\ln(16))$ $=\ln(2)$	2 marks for correct substitution, and simplification. 1 mark if a single error is made in the substitution or the simplification.
1 c) $\frac{2x-3}{x-1}\leq 4$ $x\neq 1$ $2x-3=4x-4$ $x=\frac{1}{2}$ try $x=0$ $3\leq 4$✓ try $x=0.75$ $6\leq 4$ ✗ try $x=2$ $1\leq 4$✓ $x\leq\frac{1}{2},x>1$ $\frac{2x-3}{x-1}\leq 4$ $(x-1)^2\frac{2x-3}{x-1}\leq 4(x-1)^2$ $(x-1)(2x-3)\leq 4(x-1)^2$ $2x^2-5x+3\leq 4x^2-8x+4$ $0\leq 2x^2-3x+1$ $(2x-1)(x-1)\geq 0$ $x\leq\frac{1}{2}'x>1$	1 for values 1 for test 1 for statement. No 3rd mark if $x\geq 1$ Similar break up of marks for alternative methods.

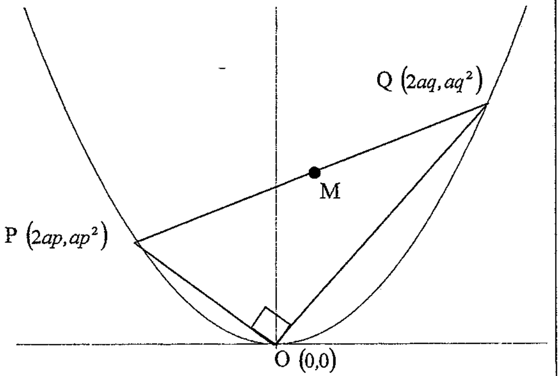
1. d) $\sum_{n=2}^{8}\left(n^2-n\right)=(4-2)+(9-3)+(16-4)+(25-5)$ $+(36-6)+(49-7)+(64-8)$ $=2+6+12+20+30+42+56$ $=168$	1 mark
1. e) $2x-1$ is a factor of $P(x)=2x^3+5x^2+x-2$ if $P\left(\frac{1}{2}\right)=0$ by factor theorem. $P\left(\frac{1}{2}\right)=2\left(\frac{1}{2}\right)^3+5\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)-2$ $=\frac{1}{4}+\frac{5}{4}+\frac{1}{2}-2$ $=0 \quad \therefore 2x\text{-}1$ is a factor	2 marks for substitution and explanation, or other valid method such as dividing. 1 mark if error in substitution, but right reasoning to substitute $x=\frac{1}{2}$
1. f) If $u=\sin x$, then $\frac{du}{dx}=\cos x$ and $du=\cos x dx$ $\int \sin x \; \cos x \; dx=\int u du$ $=\frac{u^2}{2}+c$ $=\frac{\sin^2 x}{2}+c$	2 marks for correct substitution and integration. 1 mark if error made in substitution or integration.

Solutions Question 2	Marks/Comments
2 a) $x=2t-1\Rightarrow t=\frac{x+1}{2}$ $y=t^2+2t$ $y=\left(\frac{x+1}{2}\right)^2+2\left(\frac{x+1}{2}\right)$ OR $4y=x^2+2x+1+4x+4$ $4y=x^2+6x+5$ $\qquad 4y+4=x^2+6x+9$ $y=\frac{x^2+6x+5}{4}$ $\qquad 4(y+1)=(x+3)^2$	1 mark for making t the subject of either equation. 1 for substitution into the other equation 1 for simplifying to either form.
2 b) $y=\cos x$ $V=\pi\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} y^2 dx$ $=\pi\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}(\cos x)^2 dx$ $=\pi\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\frac{(1+\cos 2x)}{2}dx$ $=\frac{\pi}{2}\left[x+\frac{1}{2}\sin 2x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$ $=\frac{\pi}{2}\left[\left(\frac{3\pi}{2}+\frac{1}{2}\sin 2\left(\frac{3\pi}{2}\right)\right)-\left(\frac{\pi}{2}+\frac{1}{2}\sin 2\left(\frac{\pi}{2}\right)\right)\right]$ $=\frac{\pi}{2}\left[\left(\frac{3\pi}{2}+\frac{1}{2}\sin 3\pi\right)-\left(\frac{\pi}{2}+\frac{1}{2}\sin \pi\right)\right]$ $=\frac{\pi}{2}\left[\left(\frac{3\pi}{2}\right)-\left(\frac{\pi}{2}\right)\right]$ $=\frac{\pi}{2}[\pi]$ $=\frac{\pi^2}{2}units^2$	 1 for substituting into Volume formula. 1 for finding integral of $\cos^2 x$ 1 for evaluating resulting definite integral.

2 c) i) There are $\frac{9!}{2!2!} = 90720$ arrangements with the S's and E's repeated. ii) When the two S's are placed there are $\frac{7!}{2!} = 2520$ arrangements of the remaining 7 letters with E's repeated. iii) Consider the two S's as a single unit, so eight to arrange in $\frac{8!}{2!} = 20160$ ways with E's repeated.	1 mark. 1 mark 1 mark
2 d)  ∠BED = ∠ABD = 80° Angle between tangent and chord is equal to the angle in alternate segment. ∠DOE = 2× ∠DBE = 80° Angle at the centre is twice angle at circumference on same arc. DO = OE Equal radii ∠OED = ∠ODE = 50° Equal angles in isosceles ΔODE ∠BEO = ∠BED – ∠OED = 80° – 50° =30°	1 mark for ∠BED with reasons. 1 mark for ∠DOE with reasons. 1 mark for ∠BEO with reasons.

Solutions Question 3	Marks/Comments
3(a) By Newtons Method $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ Here $x_1 = 1$, $f(x) = \sin(e^x)$ $f'(x) = \frac{d}{dx}\left[\sin(e^x)\right]$ $= e^x \cos(e^x)$ So $x_2 = 1 - \frac{\sin(e^1)}{e^1 \cos(e^1)}$ $= 1 - (-0{\cdot}1657)$ $= 1{\cdot}1657 = 1{\cdot}17$ (3 Significant figures)	1 for derivative 1 for sub in formula 1 for correct evaluation.
3(b) (i) $\sin x - \cos 2x = \sin x - (1 - 2\sin^2)$ $= \sin x - 1 + 2\sin^2 x$ $= 2\sin^2 x + \sin x - 1$	1 mark for expression for cos2x 1 for simplifying
3(b) (ii) $\sin x - \cos 2x = 0$ $2\sin^2 x + \sin x - 1 = 0$ $2\sin^2 x + 2\sin x - \sin x - 1 = 0$ $2\sin x(\sin x + 1) - (\sin x + 1) = 0$ $(2\sin x - 1)(\sin x + 1) = 0$ So $\sin x = \frac{1}{2}$ or $\sin x = -1$ $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$	1 for factorising quadratic equation 1 for values of sin$x$ 1 for the solutions for x

3(c) $2x^3-3x^2+5x-2=0$ $\alpha+\beta+\gamma=\frac{-b}{a}=\frac{3}{2}$ $\alpha\beta+\beta\gamma+\alpha\gamma.=\frac{c}{a}=\frac{5}{2}$ $\alpha\beta\gamma=\frac{-d}{a}=\frac{2}{2}=1$	1 mark for correctly re arranging the required expression in terms of sums and products.
(i) $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\alpha\gamma)$ $=\left(\frac{3}{2}\right)^2-2\left(\frac{5}{2}\right)$ $=-\frac{11}{4}=-2\frac{3}{4}$	1 mark for correctly substituting and simplifying.
(ii) $\frac{1}{\alpha^2\beta^2}+\frac{1}{\beta^2\gamma^2}+\frac{1}{\alpha^2\gamma^2}=\frac{\alpha^2+\beta^2+\gamma^2}{\alpha^2\beta^2\gamma^2}$	1 mark for correctly re arranging the required expression in terms of sums and products.
$=\frac{\alpha^2+\beta^2+\gamma^2}{(\alpha\beta\gamma)^2}$ $=\frac{-11}{4}\Big/(1)^2$ $=\frac{-11}{4}=-2\frac{3}{4}$	1 mark for correctly substituting and simplifying.

Solutions Question 4	Marks/Comments
4(a) 	
i) Gradient of OP = m₁ $=\frac{ap^2}{2ap}=\frac{p}{2}$ Gradient of OQ = m₂ $=\frac{aq^2}{2aq}=\frac{q}{2}$ Now $m_1m_2=-1$ since <POQ is a right angle. $\frac{p}{2}\frac{q}{2}=-1$ $pq=-4$	1 mark
ii) M has coordinates $\left(\frac{2ap+2aq}{2},\frac{ap^2+aq^2}{2}\right)=\left(a(p+q),\frac{a(p^2+q^2)}{2}\right)$	1 co-ords of M
$x=a(p+q)\Rightarrow p+q=\frac{x}{a}$ $y=\frac{a(p^2+q^2)}{2}$ $\frac{2y}{a}=p^2+q^2$ $\frac{2y}{a}+2pq=p^2+2pq+q^2$	1 for comp square
$\frac{2y}{a}+2(-4)=(p+q)^2$ $\frac{2y}{a}-8=\left(\frac{x}{a}\right)^2$ $2ay-8a^2=x^2$ $x^2=2a(y-4a)$ This is a parabola with vertex $(0,4a)$	1 for equation Other methods possible.

4(b) i) If k is even, i.e $k = 2x$, then $k^2 + k = (2x)^2 + 2x$ $= 4x^2 + 2x$ $= 2(2x^2 + x)$ $= 2m$	If k is odd, i.e $k = 2x+1$, then $k^2 + k = (2x+1)^2 + 2x+1$ $= 4x^2 + 4x + 1 + 2x + 1$ $= 4x^2 + 6x + 2$ $= 2(2x^2 + 3x + 1)$ $= 2m$	1 mark for each case.
$\therefore k^2 + k$ is divisible by 2, no matter whether k is odd or even. ii) When $n = 1$, $n^3 + 5n = 1^3 + 5(1) = 6$ $\therefore n^3 + 5n$ is divisible by 6 when $n = 1$ Assume that $n^3 + 5n$ is divisible by 6 for $n = k$ i.e $k^3 + 5k = 6p$ where p is an integer. Now when $n = k+1$ $(k+1)^3 + 5(k+1) = k^3 + 3k^2 + 3k + 1 + 5k + 5$ $= k^3 + 5k + 3k^2 + 3k + 6$ $= 6p + 3k^2 + 3k + 6$ $= 6p + 6 + 3(k^2 + k)$ * from i) above $= 6p + 6 + 3(2m)$* $= 6p + 6 + 6m$ $= 6(p + m + 1)$ $(k+1)^3 + 5(k+1)$ is divisible by 6 $\therefore$ if true for $n = k$, then also true for $n = k+1$, but since true for $n = 1$, by induction is true for all integral values, $n \geq 1$		3 marks for full proof including all steps. Delete one mark if case n=1 or conclusion not correct or omitted. Delete one mark for simple error in case for k+1

4 (c) (i) $e^h = 1 + \frac{h}{1!} + \frac{h^2}{2!} + \frac{h^3}{3!} + \ldots$	
$e^h - 1 = \frac{h}{1!} + \frac{h^2}{2!} + \frac{h^3}{3!} + \ldots$	1 mark
$\frac{e^h - 1}{h} = \frac{1}{1!} + \frac{h}{2!} + \frac{h^2}{3!} + \ldots$	
$\frac{e^h - 1}{h} = 1 + \frac{h}{2!} + \frac{h^2}{3!} + \ldots$	
(ii) $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$	
$= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h}$	1 mark for substitution and taking e^x out of limit.
$= \lim_{h\to 0} \frac{e^x(e^h - 1)}{h}$	
$= e^x \lim_{h\to 0} \frac{(e^h - 1)}{h}$	
$= e^x \lim_{h\to 0} \left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \ldots\right)$	1 for evaluating limit.
$= e^x\left(1 + \frac{0}{2!} + \frac{0^2}{3!} + \ldots\right)$	
$= e^x(1)$	
$= e^x$	