SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2014

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 2 hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question if full marks are to be awarded.
- Answer in simplest exact form unless otherwise instructed.

Total Marks – 70 Marks

Section I – **10 Marks**

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – **60 Marks**

- Attempt Questions 11 – 14
- Allow about 1 hour 45 minutes for this section

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R. Boros

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq 1;\quad x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\quad x > 0$

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

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1. If $(7, b)$ divides $(3, -4)$ and $(9, -7)$ internally in the ratio a: 1, find the values of a and b.

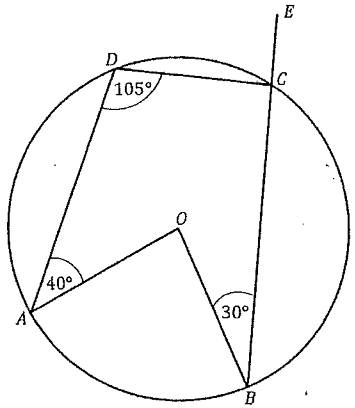
(A) $a = \frac{1}{2},\ b = -\frac{23}{3}$

(B) $a = 2,\ b = -\frac{23}{3}$

(C) $a = \frac{1}{2},\ b = -6$

(D) $a = 2,\ b = -6$

2. In the diagram below, O is the centre of the circle $ABCD$. BCE is a straight line. If $\angle ADC = 105°$, $\angle OBC = 30°$ and $\angle OAD = 40°$, then $\angle DCE =$



(A) 75°

(B) 80°

(C) 85°

(D) 90°

3. $\alpha 3\beta$ is a 3-digit number, where α and β are integers from 1 to 9 inclusive. Find the probability that the 3-digit number is divisible by 5.

(A) $\frac{1}{10}$

(B) $\frac{9}{50}$

(C) $\frac{1}{9}$

(D) $\frac{1}{5}$

4. Let $b > 1$ and $c > 1$. If $a = \log_c \sqrt{b}$, then $a^{-1} =$

(A) $\log_b c^2$

(B) $2 \log_c b$

(C) $\log_c \frac{1}{\sqrt{b}}$

(D) $\log_{\frac{1}{c}} \frac{1}{\sqrt{b}}$

5.

$$\frac{d}{dx}(x \sin^{-1} x) =$$

(A) $\sin^{-1} x - \frac{x}{\sqrt{1-x^2}}$

(B) $\sin^{-1} x + \frac{x}{\sqrt{1-x^2}}$

(C) $\cos^{-1} x + \frac{x}{\sqrt{1-x^2}}$

(D) $\cos^{-1} x - \frac{x}{\sqrt{1-x^2}}$

6. It is given that α and β are roots of the equation $x^2 + 1 = 6x$, then $\alpha - \beta =$

(A) $-4\sqrt{2}$

(B) $4\sqrt{2}$

(C) $\pm 4\sqrt{2}$

(D) 32

7. If ${}^nP_2 = 56$, then

(A) $n = -7$

(B) $n = 8$

(C) $n = 11$

(D) $n = 112$

8. The minimum value of $\frac{1}{\sin^2 x - 2}$ is

(A) $-\frac{1}{2}$

(B) -1

(C) $-\frac{1}{3}$

(D) 0

9.

$$\int \frac{1}{\sqrt{25 - 4x^2}} . dx =$$

(A) $\frac{1}{4}\sin^{-1}\left(\frac{5x}{2}\right) + C$

(B) $\frac{1}{4}\sin^{-1}\left(\frac{2x}{5}\right) + C$

(C) $\frac{1}{2}\sin^{-1}\left(\frac{5x}{2}\right) + C$

(D) $\frac{1}{2}\sin^{-1}\left(\frac{2x}{5}\right) + C$

10. The coefficient of x^{2n} in the binomial expansion of $(1 + x)^{4n}$ is

(A) $\frac{4n!}{2n!2n!}$

(B) $\frac{(4n)!}{2(n!)^2}$

(C) $\frac{(4n)!}{(2n)!}$

(D) None of the above

End of Section A

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

Answer each question in a NEW writing booklet. Extra pages are available

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a NEW Writing Booklet

(a) Determine the acute angle, between the line $x - 3y + 2 = 0$ and the line BC where B is $(-1, -1)$ and C is $(1, 3)$. **2**

(b) Evaluate **1**

$$\lim_{x \to 0} \frac{3x}{2 \sin 4x}$$

(c) Solve for x, **2**

$$\frac{(x-2)}{(x-1)(x-3)} \geq 0$$

(d) Write down a general solution to the equation $\cos 2x = -\frac{1}{2}$. Leave your answer in terms of π. **2**

(e)

(i) Express $12 \cos x - 5 \sin x$ in the form $A \cos(x + \alpha)$ where A is positive and $0° \leq \alpha \leq 180°$, correct α to the nearest minute. **2**

(ii) Hence find the maximum value of $12 \cos x - 5 \sin x$ and the smallest positive value of x for which this maximum occurs. **2**

(f) Calculate the number of different arrangements which can be made using all the letters of the word BANANA. **1**

Question 11 continues on page 7

(g)

(i) Differentiate $\cot x$ with respect to x. **1**

(ii) Hence differentiate $x \cot x$ with respect to x. **1**

(iii) Hence find **1**

$$\int x \operatorname{cosec}^2 x \, . dx$$

End of Question 11

Question 12 (15 Marks) Start a NEW Writing Booklet

(a) Express $\sin 2\theta$ and $\cos 2\theta$ in terms of $t = \tan\theta$ to show that **2**

$$\frac{1+\sin 2\theta - \cos 2\theta}{1+\sin 2\theta + \cos 2\theta} = \tan\theta$$

(b) In the expansion of $(1+2x)^n(1-x)^2$, the coefficient of x^2 is 9. Find the coefficient of x in the expansion. **3**

(c) If the roots of $x^3 - 6x^2 + 3x + k = 0$ are consecutive terms of an arithmetic series, find k. **2**

(d) Evaluate **2**

$$\int_0^{\frac{3}{4}} x\sqrt{1-x}.dx$$

using the substitution $u = 1 - x$, express your answer in simplest exact form.

(e)
(i) Prove by the Principle of Mathematical Induction that **3**

$$1\times 2^1 + 2\times 2^2 + 3\times 2^3 + \cdots + n\times 2^n = (n-1)\times 2^{n+1} + 2$$

for all positive integers n.

(ii) Using the result of (i), simplify **2**

$$\sum_{r=1}^{n}(r+1)\times 2^r$$

(f) Brian is to celebrate his 16th birthday by having a dinner with 11 other family members. At this dinner, Brian will sit at the head of a non-circular table. In how many ways can everyone be seated? **1**

End of Question 12

Question 13 (15 Marks) Start a NEW Writing Booklet

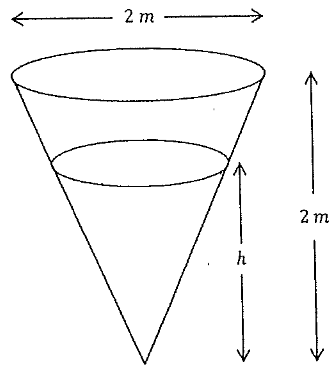
(a) A particle moves up and down so that its vertical displacement, x from a point O, is given by $x = 10 + 8\sin 2t + 6\cos 2t$ where x is in metres and t is in seconds.

(i) Show that the particle moves in Simple Harmonic motion. **1**

(ii) What is the period of the motion? **1**

(iii) What is the amplitude? **1**

(b) A container in the shape of a right cone with both height and diameter $2\ m$ is being filled with water at a rate of $\pi\ m^3/min$.



(i) Show that **2**

$$\frac{dV}{dt} = \frac{\pi h^2}{4}.\frac{dh}{dt}$$

(ii) Find the rate of change of height h of the water when the container is $\frac{1}{8}th$ full by volume. **2**

Question 13 continues on page 10

(c) The rate of change in the number of members of the Sydney Boys High School Old Boys Mathematical Society, M, is given by

$$\frac{dM}{dt} = k(M - 50)$$

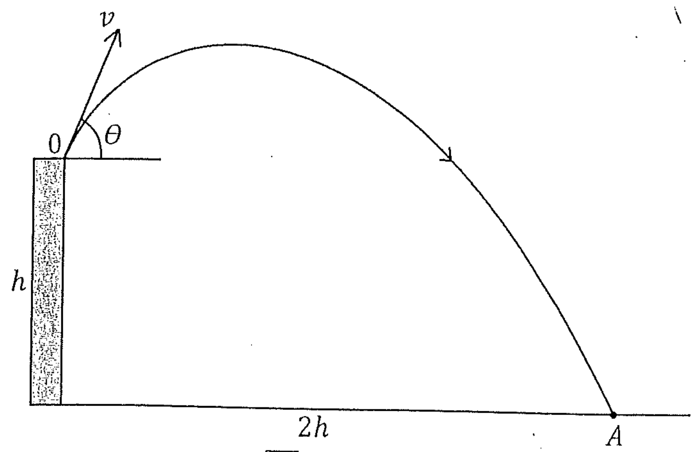
The number of members of this society at the start of 1995 was 70.

(i) Show that $M = 20e^{kt} + 50$. **1**

(ii) In 2000, the number of members was 150. Find the number of members in 2005. **1**

(iii) There is a year that this society will eventually become a "ghost society" with no members. Do you agree? Give reasons. **1**

(d)



A projectile is fired with speed $\sqrt{\frac{4gh}{3}}$ at an angle θ to the horizontal from the top of a cliff of height h and the projectile strikes the ground a horizontal distance $2h$ from the base of the cliff.

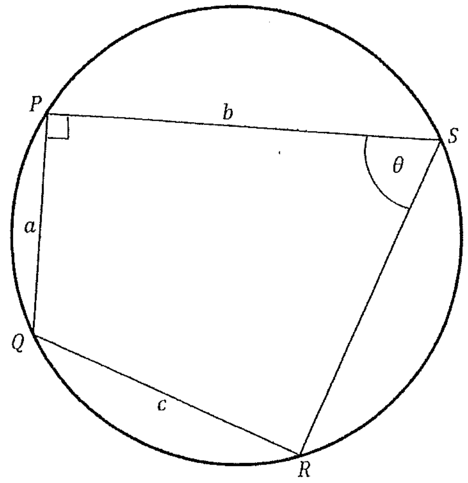
You may assume $y = Vt\sin\theta - \frac{1}{2}gt^2$ and $x = Vt\cos\theta$.

(i) Show that $y = x\tan\theta - \frac{gx^2}{2V^2}(1 + \tan^2\theta)$. **1**

(ii) Find the 2 possible values of $\tan\theta$. **2**

Question 13 continues on page 11

(e) In the diagram below, $PQRS$ is a cyclic quadrilateral, $\angle QPS = 90°$ and $\angle PSR = \theta$, $PQ = a$, $PS = b$ and $QR = c$. **2**



Show that $(a^2 + b^2)\sin^2\theta = a^2 + c^2 + 2ac\cos\theta$.

End of Question 13

Question 14 (15 Marks) Start a NEW Writing Booklet

(a) At an election, 30% of the voters favoured party A. If 5 voters were selected at random, what is the probability (as a decimal) that

(i) exactly 3 favoured party A. **1**

(ii) at most 2 favoured party A. **1**

(b)

(i) Show that **3**

$$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$

(ii) If x satisfies the equation $\tan 3x = \cot 2x$, show that x also satisfies the equation $5\tan^4 x - 10\tan^2 x + 1 = 0$. **2**

(iii) Using the result of (ii), deduce that **4**

$$\tan\frac{\pi}{5} = \sqrt{5 - 2\sqrt{5}}$$

(c) In the expansion of $(1+x)^n$, let S_1 be the terms containing the coefficients

$${}^nC_0,\ {}^nC_2,\ {}^nC_4, \ldots$$

whilst S_2 be the terms containing the coefficients

$${}^nC_1,\ {}^nC_3,\ {}^nC_5, \ldots$$

Prove that,

(i) $4S_1S_2 = (1+x)^{2n} - (1-x)^{2n}$ **2**

(ii) $(S_1)^2 - (S_2)^2 = (1-x^2)^n$ **2**

End of paper

Student Number: SOLUTIONS

Mathematics Extension 1
Trial HSC 2014

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ☒ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

correct

A ☒ B ☒ C ○ D ○

Section I: **Multiple choice answer sheet.**

Completely colour the cell representing your answer. Use black pen.

1. D
2. C
3. C
4. A
5. B
6. C
7. B
8. B
9. D
10. D

Question 11

a $B(-1,-1)$ $C(1,3)$

$m_1 = \frac{3+1}{1+1}$

$= 2$

$x - 3y + 2 = 0$

$3y = x + 2$

$y = \frac{1}{3}x + \frac{2}{3}$ (1)

$\therefore m_2 = \frac{1}{3}$

$\tan\alpha = \frac{2 - \frac{1}{3}}{1 + 2 \cdot \frac{1}{3}}$ (1)

$\alpha = 45^\circ$

b $\lim_{x\to 0} \frac{3x}{2\sin 4x}$

$= \lim_{x\to 0} \frac{4}{4} \times \frac{3x}{2\sin 4x}$

$= \lim_{x\to 0} \frac{3}{8} \cdot \frac{4x}{\sin 4x}$ (1)

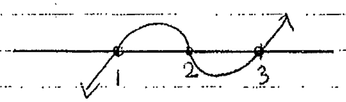
$= 3/8$

c $\frac{x-2}{(x-1)(x-3)} \geq 0$

$x \neq 1, 3$ (1)

consider

$y = (x-2)(x-1)(x-3)$



$\therefore 1 < x \leq 2, \; x > 3$ (1)

d. $\cos 2x = -\frac{1}{2}$

S | A
T | C

$\cos\alpha = \frac{1}{2}$

$\alpha = \pi/3$

$2x = \frac{2\pi}{3} + 2k\pi, \; \frac{4\pi}{3} + 2k\pi$

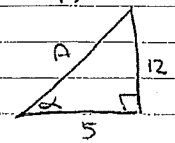
$x = \frac{\pi}{3} + k\pi$ (1), $\frac{2\pi}{3} + k\pi$ (1)

e. i $12\cos x - 5\sin x \equiv A\cos(x+\alpha)$

$\equiv A\cos x\cos\alpha - A\sin x\sin\alpha$

$12 = A\cos\alpha$, $5 = A\sin\alpha$

$\cos\alpha = \frac{12}{A}$, $\sin\alpha = \frac{5}{A}$



$\therefore A = 13$ (1)

$\tan\alpha = 5/12$

$\alpha = 22^\circ 37'$ (1)

ii $12\cos x - 5\sin x \equiv 13\cos(x + 22^\circ 37')$

Max value $= 13$

occurs when $\cos(x + 22^\circ 37') = 1$ (1)

$\cos(x + 22^\circ 37') = 1$

$x + 22^\circ 37' = 360$

$x = 337^\circ 23'$ (1)

$-\frac{1}{2}$ for neg angle

(5.89)

f. Banana $= \frac{6!}{2!\,3!} = 60$ (1)

g. i) $\frac{d}{dx}\cot x = \frac{1}{\tan x}\,dx$ $\quad u = 1 \quad u' = 0$

$v = \tan x \quad v' = \sec^2 x$

$= \frac{\cancel{\tan}\ 0 - \sec^2 x}{\tan^2 x}$

$= \frac{-1}{\cos^2 x} \times \frac{\cos^2 x}{\sin^2 x}$

$= -\mathrm{cosec}^2 x$ (1)

ii. $\frac{d(x\cot x)}{dx} = x\times(-\mathrm{cosec}^2 x) + \cot x$

$= \cot x - x\,\mathrm{cosec}^2 x$ (1)

iii $\int x\,\mathrm{cosec}^2 x\,dx = -\int -x\,\mathrm{cosec}^2 x\,dx$

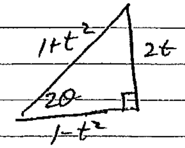
$= -\int(\cot x - x\,\mathrm{cosec}^2 x - \cot x)\,dx$

$= -\int\left(\cot x - x\,\mathrm{cosec}^2 x - \frac{\cos x}{\sin x}\right)dx$

$= -\left[x\cot x - \ln(\sin x)\right] + C$

$= \ln(\sin x) - x\cot x + C$ (1)

12) a) $t = \tan\theta$



$LHS = \frac{1+\sin 2\theta - \cos 2\theta}{1+\sin 2\theta + \cos 2\theta}$

$= \frac{1 + \frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \times \frac{1+t^2}{1+t^2}$

$= \frac{\cancel{1}+t^2+2t-\cancel{1}+t^2}{1+\cancel{t^2}+2t+1-\cancel{t^2}}$

$= \frac{2t^2+2t}{2t+2}$

$= \frac{\cancel{2}t\cancel{(t+1)}}{\cancel{2(t+1)}}$

$= t$

$= \tan\theta$

$= RHS$

b) $(1+2x)^n(1-x)^2 \equiv \left({}^nC_0 + {}^nC_1(2x) + {}^nC_2(2x)^2 + \ldots\right)\left(1-2x+x^2\right)$

coefficient of x^2 is 9

${}^nC_0 + {}^nC_1(2)(-2) + {}^nC_2(2)^2 = 9$

$1 - 4n + \frac{4n(n-1)}{2} = 9$

$1 - 4n + 2n^2 - 2n = 9$

$2n^2 - 6n - 8 = 0$

$n^2 - 3n - 4 = 0$ $\quad \times\ -4,\ +\ -3$

$(n-4)(n+1) = 0$

$n = 4,\ -1$

coefficient of x is

$${}^4C_0(-2)+{}^4C_1(2) = -2+8$$
$$= 6$$

c) let roots be $\alpha-\beta, \alpha, \alpha+\beta$

$$(\alpha-\beta)+\alpha+(\alpha+\beta) = -\frac{b}{a}$$

$$3\alpha = -\frac{(-6)}{1}$$

$$\alpha = 2$$

since α is a root $P(2)=0$

$$(2)^3-6(2)^2+3(2)+k=0$$
$$-10+k=0$$
$$k=10$$

d) $\int_0^{3/4} x\sqrt{1-x}\,dx$

$u=1-x$

$\frac{du}{dx}=-1$

$dx=-du$

limit change when $x=0, u=1$

$x=3/4, u=\frac{1}{4}$

$$= \int_1^{\frac{1}{4}} (1-u)\sqrt{u}\,.-du$$

$$= \int_{\frac{1}{4}}^{1} \left(u^{\frac{1}{2}}-u^{3/2}\right)du$$

$$= \left[\frac{2}{3}u^{3/2}-\frac{2}{5}u^{5/2}\right]_{\frac{1}{4}}^{1}$$

$$= \frac{2}{3}(1)^{3/2}-\frac{2}{5}(1)^{5/2}-\left(\frac{2}{3}\left(\frac{1}{4}\right)^{3/2}-\frac{2}{5}\left(\frac{1}{4}\right)^{5/2}\right)$$

$$= \frac{47}{240}$$

e) i) Prove $1\times2^1+2\times2^2+3\times2^3+\ldots+n\times2^n=(n-1)\times2^{n+1}+2$

Prove true for $n=1$

$LHS = 1\times2^1$ $\quad$ $RHS=(1-1)\times2^{1+1}+2$

$= 2$ $\quad$ $= 2$

$LHS=RHS$

$\therefore$ true for $n=1$

Assume true for $n=k$, where $k\in\mathbb{N}$

$$1\times2^1+2\times2^2+3\times2^3+\ldots+k\times2^k=(k-1)\times2^{k+1}+2$$

Prove true for $n=k+1$

ie $1\times2^1+2\times2^2+3\times2^3+\ldots+k\times2^k+(k+1)\times2^{k+1} = k\times2^{k+2}+2$

$$LHS = \underbrace{1\times2^1+2\times2^2+3\times2^3+\ldots+k\times2^k}+(k+1)\times2^{k+1}$$

$$= (k-1)\times2^{k+1}+2+(k+1)\times2^{k+1}$$

$$= 2^{k+1}(k\cancel{-1}+k\cancel{+1})+2$$

$$= 2^{k+1}(2k)+2$$

$$= k\times2^{k+2}+2$$

$$= RHS$$

$\therefore$ true for $n=k+1$

$\therefore$ true by induction for positive integers n.

ii) $\sum_{r=1}^{n}(r+1)2^r$

$$= 2\times2^1+3\times2^2+4\times2^3+\ldots+(n+1)\times2^n$$

$$= \frac{1}{2}\left(2\times2^2+3\times2^3+4\times2^4+\ldots+(n+1)\times2^{n+1}\right)$$

$$= \frac{1}{2}\left(1\times2^1+2\times2^2+3\times2^3+\ldots+(n+1)\times2^{n+1}\right)-1$$

$$= \frac{1}{2}\left(n\times2^{n+2}+2\right)-1$$

$= n \times 2^{n+1} + 1 - 1$

$= n \times 2^{n+1}$

OR

$\sum_{r=1}^{n} (r+1)2^r = \sum_{r=1}^{n} r \times 2^r + \sum_{r=1}^{n} 2^r$

from (i)

geometric series

$S_n = \frac{a(r^n - 1)}{r-1}$

$\sum_{r=1}^{n} r \times 2^r = (n-1)2^{n+1} + 2$

$S_n = \frac{2(2^n - 1)}{2-1}$

$= 2^{n+1} - 2$

$\therefore \sum_{r=1}^{n} (r+1)2^r = (n-1)2^{n+1} \cancel{+2} + 2^{n+1} \cancel{-2}$

$= 2^{n+1}(n \cancel{-1} \cancel{+1})$

$= n \times 2^{n+1}$

f) $1 \times 11! = 39916800$

Question 13

(a) $x = 10 + 8\sin 2t + 6\cos 2t$

(i) $\dot{x} = 16\cos 2t - 12\sin 2t$

$\ddot{x} = -32\sin 2t - 24\cos 2t$

$\ddot{x} = -4(8\sin 2t + 6\cos 2t)$

$\therefore \ddot{x} = -4(x - 10)$

Now let $X = x - 10$

$\therefore \ddot{X} = -4X$ and thus the motion is SHM.

(ii) Clearly $n = 2$.

$\therefore T = \frac{2\pi}{2}$

$= \pi$

(iii) Amplitude

$a = \sqrt{8^2 + 6^2}$

$= 10$

(b) (i) $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}$

$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$

$= \frac{1}{12}\pi h^3$

$\frac{dV}{dh} = \frac{1}{4}\pi h^2$

$\therefore \frac{dV}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt}$

(ii) Maximum Volume

$V_{\max} = \frac{1}{12}\pi(2)^3$

One eighth full means

$V = \frac{8\pi}{12} \div 8$

$= \frac{\pi}{12}$

Thus

$\frac{\pi}{12} = \frac{1}{12}\pi h^3$

$h = 1\text{m}$

We seek $\dfrac{dh}{dt}$ when $h = 1$.

$$\begin{aligned}\frac{dh}{dt} &= \frac{dh}{dV}\frac{dV}{dt}\\ &= \frac{4}{\pi h^2}\pi\\ &= 4 \text{ m/min}\end{aligned}$$

(c) $\dfrac{dM}{dt} = k(M-50)$

When $t = 0$, $M = 70$.

(i) Consider

$$M = 20e^{kt} + 50$$

$$\begin{aligned}\frac{dM}{dt} &= 20ke^{kt}\\ &= k\left(20e^{kt}\right)\\ &= k(M-50)\end{aligned}$$

$\therefore$ Satisfies D.E.

(ii) When $t = 5$, $M = 150$.

$$\begin{aligned}150 &= 20e^{5k} + 50\\ 100 &= 20e^{5k}\\ 5 &= e^{5k}\\ k &= \frac{\ln 5}{5}\\ &\approx 0.3219\end{aligned}$$

Thus when $t = 10$

$$\begin{aligned}M &= 20e^{10k} + 50\\ &= 550\end{aligned}$$

(iii) No, as $k > 0$ membership always increases.

(d) Given $y = Vt\sin\theta - \frac{1}{2}gt^2$ and $x = Vt\cos\theta$

(i) $t = \dfrac{x}{V\cos\theta}$, substitute to obtain

$$y = \frac{x\sin\theta}{\cos\theta} - \frac{1}{2}g\left(\frac{x^2}{V^2\cos^2\theta}\right)$$

$y = x\tan\theta - \frac{gx^2}{2V^2}\left(1+\tan^2\theta\right)$ as required.

(ii) The point A is $(2h,-h)$. Substitute:

$$-h = 2h\tan\theta - \frac{g(2h)^2}{2V^2}\left(1+\tan^2\theta\right)$$

But $V^2 = \dfrac{4gh}{3}$

$$-h = 2h\tan\theta - \frac{3h}{2}\left(1+\tan^2\theta\right)$$

Thus $3\tan^2\theta - 4\tan\theta + 1 = 0$

So $\tan\theta = 1$ or $\dfrac{1}{3}$

(e) Required to prove $\left(a^2+b^2\right)\sin^2\theta = a^2 + c^2 + 2ac\cos\theta$

Join PR, QS. QS is the diameter, so
$\angle QPR = 90^0$ (angle in a semicircle).

In ΔPQS $\quad QS^2 = a^2 + b^2$ (Pythagoras's Thm)

$$\sin\angle PQS = \frac{b}{QS}$$

$$\therefore QS = \frac{b}{\sin\angle PQS}$$

In quad $PQRS$ $\quad \angle PQR = 180^0 - \theta$ (opposite angles of cyclic quadrilateral)
$\angle PRS = \angle PQS$ (angles in same segment)

In ΔPQR

$$\begin{aligned}PR^2 &= a^2 + c^2 - 2ac\cos\left(180^0 - \theta\right)\\ &= a^2 + c^2 + 2ac\cos\theta\end{aligned}$$

In ΔPRS

$$\begin{aligned}\frac{PR}{\sin\theta} &= \frac{b}{\sin\angle PRS}\\ &= \frac{b}{\sin\angle PQS}\\ &= QS \qquad \text{from above}\end{aligned}$$

$\therefore PR = QS\sin\theta$

$PR^2 = QS^2\sin^2\theta$

$= a^2 + c^2 + 2ac\cos\theta$

Thus $(a^2+b^2)\sin^2\theta = a^2+c^2+2ac\cos\theta$ QED

QUESTION 14 (X1)

(a) (i) $P(x=3) = \binom{5}{3}(0.3)^3(0.7)^2$

$\boxed{\doteqdot 0.1323.}$

(ii) $P(x=0)+P(x=1)+P(x=2)$

$= \binom{5}{0}(0.3)^0(0.7)^5 + \binom{5}{1}(0.3)^1.(0.7)^4 + \binom{5}{2}(0.3)^2(0.7)^3$

$\boxed{\doteqdot 0.83692}$

(b) (i) Let $t = \tan x$ $\therefore$

$\tan 3x = \tan(2x+x)$

$= \dfrac{\tan 2x + \tan x}{1 - \tan 2x.\tan x.}$

$= \dfrac{\frac{2t}{1-t^2} + t}{1 - \frac{2t}{1-t^2}.t.}$

$= \dfrac{2t + t(1-t^2)}{1-t^2-2t^2}$

$= \dfrac{3t - t^3}{1-3t^2}$

$= \dfrac{3\tan x - \tan^3 x}{1 - 3\tan x.}$

(ii) Let $\tan 3x = \cot 2x.$

ie. $\tan 3x = \tan(\frac{\pi}{2} - 2x)$

$3x = \frac{\pi}{2} - 2x$

$5x = \frac{\pi}{2}$

$x = \frac{\pi}{10}.$

OR. $\tan 3x = \cot 2x.$

$\Rightarrow \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} = \dfrac{1}{\tan 2x.}$

(ii) (CONTD) i.e. $\dfrac{3t-t^3}{1-3t^2} = \dfrac{1-t^2}{2t}$

$$2t(3t-t^3) = (1-3t^2)(1-t^2)$$

$$6t^2-2t^4 = 1-t^2-3t^2+3t^4$$

$$5t^4-10t^2+1 = 0 \qquad (A)$$

(iii) From $x=\frac{\pi}{10}$ is also a solution to (A)

$$\text{i.e. } t^2 = \frac{10\pm\sqrt{100-20}}{10} = \frac{10\pm4\sqrt5}{10} = \frac{5\pm2\sqrt5}{5}$$

$$\text{i.e. } t = \pm\sqrt{\frac{5\pm2\sqrt5}{5}}$$

$$\text{i.e. } \tan\frac{\pi}{10} = \pm\sqrt{\frac{5\pm2\sqrt5}{5}}$$

clearly $\tan\frac{\pi}{10} > 0$

$$\therefore \tan\frac{\pi}{10} = \sqrt{\frac{5\pm2\sqrt5}{5}}$$

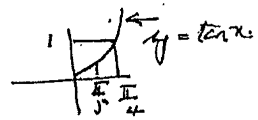
now
$$\tan\frac{\pi}{5} = \frac{2\tan\frac{\pi}{10}}{1-\tan^2\frac{\pi}{10}} = \frac{2\sqrt{\frac{5\pm2\sqrt5}{5}}}{1-\frac{5\pm2\sqrt5}{5}}$$

(iii) CONTD.

$$= \frac{2\sqrt{\frac{5\pm2\sqrt5}{5}}}{\mp\frac{2\sqrt5}{5}} = \mp\frac{\sqrt{\frac{5\pm2\sqrt5}{5}}}{\frac{\sqrt5}{5}} = \mp\sqrt{5\pm2\sqrt5} = \sqrt{5\pm2\sqrt5}$$

(clearly $\tan\frac{\pi}{5} > 0$)

Also



$\tan\frac{\pi}{5} < \tan\frac{\pi}{4} = 1$

$\therefore \tan\frac{\pi}{5} \neq \sqrt{5+2\sqrt5}$ which is >1.

$$\therefore \boxed{\tan\frac{\pi}{5} = \sqrt{5-2\sqrt5}}$$

(c) Now $S_1 = \binom{n}{0} + \binom{n}{2}x^2 + \binom{n}{4}x^4 + \dots$

& $S_2 = \binom{n}{1}x + \binom{n}{3}x^3 + \binom{n}{5}x^5 + \dots$

$\therefore S_1+S_2 = (1+x)^n$

& $S_1-S_2 = (1-x)^n$

(i)
$$\begin{aligned} \text{RHS} &= (1+x)^{2n}-(1-x)^{2n} \\ &= (S_1+S_2)^2-(S_1-S_2)^2 \\ &= S_1^2+2S_1S_2+S_2^2-(S_1^2-2S_1S_2+S_2^2) \\ &= 4S_1S_2 \\ &= \text{LHS} \end{aligned}$$

(ii). $LHS = S_1^2 - S_2^2$

$= (S_1 + S_2)(S_1 - S_2)$

$= (1+x)^n (1-x)^n$

$= (1-x^2)^n$

$= RHS.$