Sydney Secondary College

QUALITY ▪ OPPORTUNITY ▪ DIVERSITY

BLACKWATTLE BAY CAMPUS

2014

TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen. Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11-16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 2-6

10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II Pages 7-13

90 marks

- Attempt Questions 11-16
- Allow about 2 hours and 45 minutes for this section

FAMILY NAME -- FIRST NAME ----------------------------------

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x > 0$

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

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1 Let $z = 1 + i\sqrt{3}$. What is the value of $arg(z)$?

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{3}$

(C) $-\frac{\pi}{6}$

(D) $\frac{\pi}{4}$

2 Given $xy + \sin x + \cos y = 0$ find $\frac{dy}{dx}$

(A) $\frac{y-\cos x}{-x+\sin y}$

(B) $\frac{y-\cos x}{-x-\sin y}$

(C) $\frac{y+\cos x}{x+\sin y}$

(D) $\frac{y+\cos x}{-x+\sin y}$

3 Find the coordinates of the foci of the hyperbola whose equation is $\frac{x^2}{9} - \frac{y^2}{4} = 1$.

(A) $(\pm\sqrt{13}, 0)$

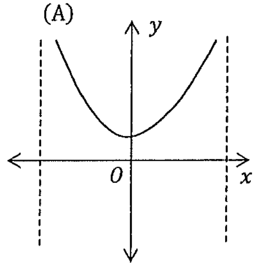
(B) $(\pm\sqrt{5}, 0)$

(C) $(\pm\frac{3\sqrt{5}}{2}, 0)$

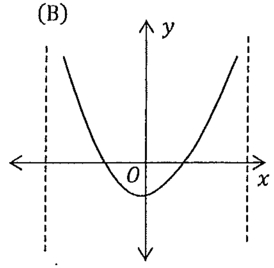
(D) $(\pm\frac{3\sqrt{13}}{2}, 0)$

4 Which of the following best represents the graph of $y = \frac{1}{\sqrt{4-x^2}}$.

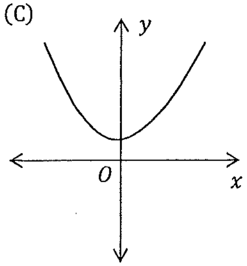
(A)



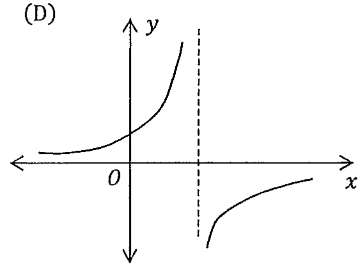
(B)



(C)



(D)



5 The equation $x^3 + px^2 + qx + r = 0$ has roots α, β and γ.

Find the expression for $\alpha^2 + \beta^2 + \gamma^2$.

(A) $q^2 - 2p$

(B) $p^2 - 2r$

(C) $p^2 - 2q$

(D) $p^2 + 2q$

6 $\int \frac{cos^3 x + sin^3 x}{\cos x + \sin x}\, dx =$

(A) $x - 2sin^2 x + c$

(B) $x - \frac{1}{2} sin^2 x + c$

(C) $x + 2sin^2 x + c$

(D) $x + \frac{1}{2} sin^2 x + c$

7 Given that $\int cos^n x\ dx = \frac{1}{n}\ [cos^{n-1} x\ \sin x + (n-1) \int cos^{n-2} x\ dx]$

Find $\int cos^4 x\ dx$

(A) $\frac{1}{4} cos^3 x\ sin\ x + \frac{3}{8} (\cos x \sin x + x) + c$

(B) $\frac{1}{4} cos^3 x\ sin\ x + \frac{3}{8} (\cos x \sin x + x) + c$

(C) $\frac{1}{4} cos^3 x\ sin\ x + \frac{3}{8} (\cos x \sin x + x) + c$

(D) $\frac{1}{4} cos^3 x\ sin\ x + \frac{3}{8} (\cos x \sin x + x) + c$

8 The locus of $|z - 2i| = |z - 2 + i|$ is :

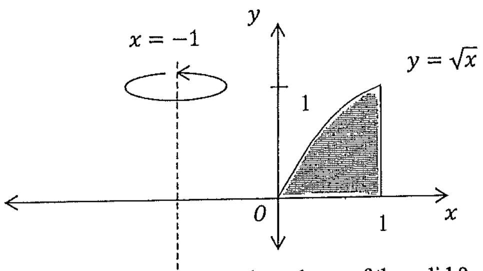
(A) A circle

(B) An ellipse

(C) A straight line

(D) A parabola

9 The region bounded by the curve $y = \sqrt{x}$, the x – axis and the line $x = 1$ is rotated about the line $x = -1$ to form a solid.



Which integral represents the volume of the solid ?

(A) $2\pi \int_0^1 (x-1)\sqrt{x}\ dx$

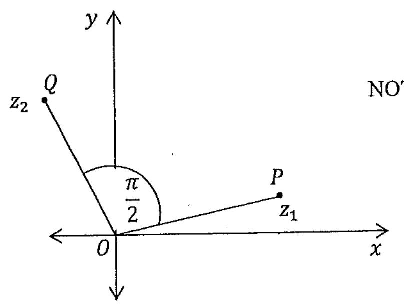
(B) $2\pi \int_0^1 (x+1)\sqrt{x}\ dx$

(C) $2\pi \int_0^1 (x+1)^2\, x\ dx$

(D) $2\pi \int_0^1 (x+2)\, x\ dx$

10 In the Argand diagram, $OP = OQ$ and $\angle POQ = \dfrac{\pi}{2}$.

The points P and Q correspond to the complex numbers z_1 and z_2 respectively.



NOT TO SCALE

The value of $z_1^{\,2} + z_2^{\,2}$ is :

(A) z_2

(B) -1

(C) 1

(D) 0

Section II

90 Marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) If $z = 1 + 3i$ and $w = 2 - i$, express in the form $a + bi$ where a and b are real.

(i) $\bar{z} - w$ **1**

(ii) zw **1**

(b) (i) On the Argand diagram sketch the region $|z - 3 + 3i| \le 2$ **2**

(ii) Find the maximum value of $|z|$ **1**

(c) By completing the square, evaluate $\displaystyle\int_{-1}^{1} \frac{1}{x^2 - 2x + 5}\, dx$ **3**

(d) (i) Write $z = -1 + i\sqrt{3}$ in modulus-argument form. **2**

(ii) Hence, express $z^8 - 16z^4$ in the form $x + yi$, where x and y are real. **2**

(e) On separate number planes, sketch the following, showing main features.

(i) $y = -\sqrt{x+1}$ **1**

(ii) $y = \dfrac{1}{\sqrt{x+1}}$ **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Find real numbers A and B such that **2**

$$\frac{7x+1}{(x+1)(x-2)} \equiv \frac{A}{x+1} + \frac{B}{x-2}$$

(ii) Hence find $\int \frac{7x+1}{(x+1)(x-2)}\, dx$ **1**

(b) Consider the hyperbola $L : 4x^2 - 16y^2 = 64$ with foci at S and S'.

(i) Find the coordinates of the foci. **1**

(ii) What are the equations of the directrices ? **1**

(iii) Sketch L showing its main features. **2**

(iv) Find the gradient of the tangent to the curve at $P(4\sec\theta,\ 2\tan\theta)$. **2**

(v) Show that the equation of the tangent at P is $x = (2\sin\theta)y + 4\cos\theta$. **2**

(c) If $I_n = \int_1^e (1 - \log_e x)^n\, dx$ for $n = 0, 1, 2, \ldots$

(i) Show that $I_n = -1 + nI_{n-1}$ for $n = 1, 2, 3, \ldots$ **2**

(Hint : use integration by parts)

(ii) Hence find the value of I_3. **2**

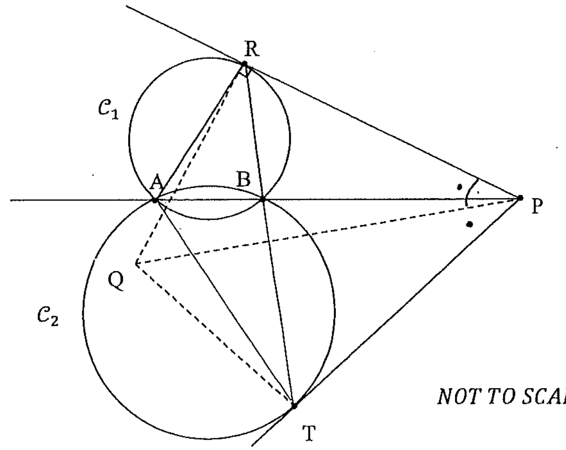
Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A particle of mass m is projected vertically upwards with initial velocity V_0 in a medium where resistance, R, is proportional to the square of the velocity ($R = mkv^2$).

(i) Show that $2kx = \log_e\left(\frac{g+kV_0{}^2}{g+kv^2}\right)$ **3**

(ii) Hence, find the maximum height reached by the particle. **2**

(b) Two circles C_1 and C_2 intersect at the points A and B. Let P be a point on AB produced and let PR and PT be tangents to C_1 and C_2, respectively. Also, $\angle PRQ = 90°$ and $\angle RPQ = \angle TPQ$.



NOT TO SCALE

(i) Prove that $\Delta ARP /// \Delta RBP$. **2**

(ii) Hence, prove that $PR^2 = AP \times BP$. **2**

(iii) Given also that $\Delta ATP /// \Delta TBP$ show that $PT = PR$. **1**

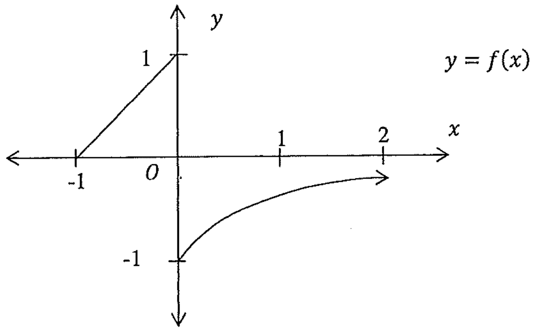
(iv) Prove that QT passes through the centre of C_2. **2**

(c) The region bounded by $0 \le x \le \sqrt{3}$, $0 \le y \le x(3 - x^2)$ is rotated about the y-axis to form a solid. **3**

Use the method of cylindrical shells to find the volume of the solid in terms of π.

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) The function $y = f(x)$ is shown below.



Draw separate one-quarter page sketches of :

(i) $y = |f(x)|$ **2**

(ii) $y = (f(x))^2$ **2**

(iii) $y = \frac{1}{f(x)}$ **2**

(iv) $y = \sqrt{f(x)}$ **2**

(b) Consider the curve $y = \frac{1-x^3}{x}$

(i) Find the coordinates of any x or y intercepts. **1**

(ii) What is the equation of the vertical asymptote ? **1**

(iii) Using long division or otherwise, find the equation of the quadratic asymptote. **1**

(iv) Find the exact coordinates of any stationary points and determine their nature. **2**

(v) Sketch the curve showing its main features. **2**

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) The roots of $3x^3 - 9x^2 + 6x + 2 = 0$, are α, β, and γ:

(i) Find the cubic polynomial equation whose roots are α^2, β^2 , and γ^2. **2**

(ii) Hence evaluate $\alpha^3\beta\gamma + \alpha\beta^3\gamma + \alpha\beta\gamma^3$. **2**

(b) (i) Given that $sin(A+B) - sin(A-B) = 2\sin B \cos A$, **1**

express $2\sin\theta\cos 6\theta$ as the difference of two sines.

(ii) Show that $2\sin\theta\,(\cos 6\theta + \cos 4\theta + \cos 2\theta) = \sin 7\theta - \sin\theta$. **2**

(iii) Hence, by letting $\theta = \frac{12\pi}{7}$, **3**

show that $cos\frac{12\pi}{7} + cos\frac{8\pi}{7} + cos\frac{4\pi}{7} = \frac{-1}{2}$

(c) (i) Find the sum of n terms of the geometric series **2**

$1 + x + x^2 + x^3 + \cdots + x^{n-1}$.

(ii) Hence, show that

$$1 + 2x + 3x^2 + \cdots + (n-1)x^{n-2} = \frac{(n-1)x^n - nx^{n-1} + 1}{(x-1)^2}.$$ **3**

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Differentiate $sin^{-1}(u) - \sqrt{1-u^2} \quad (-1 \le u \le 1)$ **3**

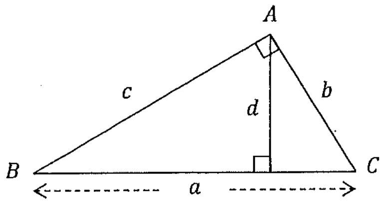
(ii) Hence show that **2**

$$\int_0^{\alpha} \left(\frac{1+u}{1-u}\right)^{\frac{1}{2}} du = sin^{-1}\alpha + 1 - \sqrt{1-\alpha^2} \quad \text{for } 0 < \alpha < 1$$

(b) Prove using mathematical induction that **4**

$10^n + 3 \times 4^{n+2} + 5$ is divisible by 9, for all integers $n \ge 1$

(c) (i) **2**



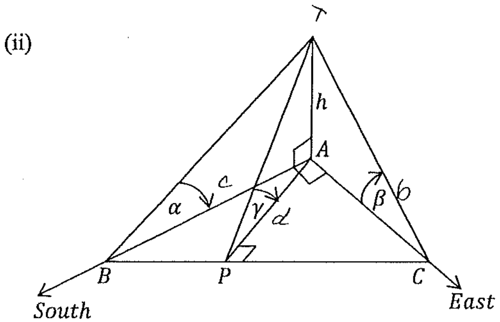
The triangle ABC is right-angled at A, with side lengths a, b and c, as shown in the diagram. The perpendicular distance from A to BC is d.

Show that $b^2c^2 = d^2(b^2 + c^2)$.

Question 16 continues on page 13

Question 16 (continued)

(ii) **4**



The points A, B and C lie on a horizontal plane.

The points B and C are due South and East of A, respectively.

There is a vertical tower, AT of height h at A.

The point P lies on BC and is chosen such that AP is perpendicular to BC.

Let α, β and γ denote the angles of elevation to the top of the tower from B, C and P respectively.

Using the result from part (i), or otherwise, show that $tan^2\gamma = tan^2\alpha + tan^2\beta$.

End of paper

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MATHEMATICS EXTENSION 2 TRIAL 2014

SECTION I – MULTIPLE CHOICE ANSWER SHEET

Circle the correct answer

1	A	B	C	D
2	A	B	C	D
3	A	B	C	D
4	A	B	C	D
5	A	B	C	D
6	A	B	C	D
7	A	B	C	D
8	A	B	C	D
9	A	B	C	D
10	A	B	C	D

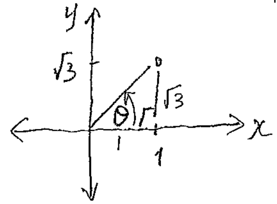
FAMILY NAME ANSWERS **FIRST NAME** ________

$\boxed{\text{QU.1}}$ (B)

$z = 1 + i\sqrt{3}$

$\arg(z) = \theta$ $\left\{ \tan\theta = \sqrt{3} \right.$

$\theta = \frac{\pi}{3}$



$\boxed{\text{QU.2}}$ (D)

$xy + \sin x + \cos y = 0$

$1.y + x.1.y' + \cos x - \sin y . y' = 0$

$y'(x - \sin y) = -\cos x - y$

$y' = \frac{-\cos x - y}{x - \sin y}$

$= \frac{y + \cos x}{-x + \sin y}$

$\boxed{\text{QU.3}}$ (A)

$\frac{x^2}{9} - \frac{y^2}{4} = 1$

$a = 3 \quad b = 2$

$b^2 = a^2(e^2 - 1)$

$e^2 - 1 = \frac{4}{9}$

$e^2 = \frac{13}{9}$

$e = \frac{\sqrt{13}}{3}$ $\left\{ \begin{array}{l} ae = \sqrt{13} \\ \therefore \text{foci } (\pm\sqrt{13}, 0) \end{array} \right.$

$\boxed{\text{QU.4}}$ (A)

$y = \frac{1}{\sqrt{4 - x^2}}$

vertical asymptotes:

$x^2 = 4$

$x = \pm 2$

Domain: $4 - x^2 > 0$

$\therefore \underline{-2 < x < 2}$

$\boxed{\text{QU.5}}$ $\boxed{\text{C}}$

$x^3 + px^2 + qx + r = 0$ $\left\{ \begin{array}{l} \Sigma\alpha = -\frac{b}{a} \\ \Sigma\alpha\beta = \frac{c}{a} \\ \alpha\beta\gamma = -\frac{d}{a} \end{array} \right.$

$(\alpha + \beta + \gamma)^2 = [(\alpha + \beta) + \gamma]^2$

$= \alpha^2 + 2\alpha\beta + \beta^2 + 2\gamma(\alpha + \beta) + \gamma^2$

$= \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$\therefore \alpha^2 + \beta^2 + \gamma^2 = \left(-\frac{p}{1}\right)^2 - 2\left(\frac{q}{1}\right)$

$= p^2 - 2q$

$\boxed{\text{QU.6}}$ $\boxed{\text{B}}$

$\int \frac{(\cos^3 x + \sin^3 x)}{(\cos x + \sin x)} \, dx$

$= \int \frac{\cancel{(\cos x + \sin x)}(\cos^2 x - \cos x . \sin x + \sin^2 x)}{\cancel{(\cos x + \sin x)}} \, dx$

$= \int 1 - \cos x . \sin x \, dx$

$= \int 1 - \frac{1}{2}(2\sin x \cos x) \, dx$

$= x - \frac{1}{2} \sin^2 x + c$

QU.7 D

$$\int \cos^n x\,dx = \frac{1}{n}\left[\cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx\right]$$

$$\int \cos^4 x\,dx$$

$$= \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\int \cos^2 x\,dx$$

$$= \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\left[\frac{1}{2}\cos x\sin x + \frac{1}{2}\int 1\,dx\right]$$

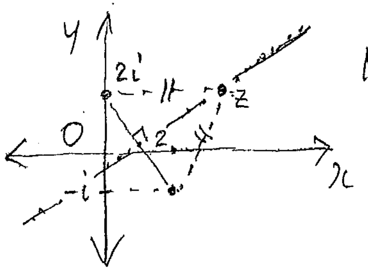
$$= \frac{1}{4}\cos^3 x\sin x + \frac{3}{8}\cos x\sin x + \frac{3}{8}x + C$$

$$= \frac{1}{4}\cos^3 x\sin x + \frac{3}{8}\left[\cos x\sin x + x\right] + C$$

QU.8 C

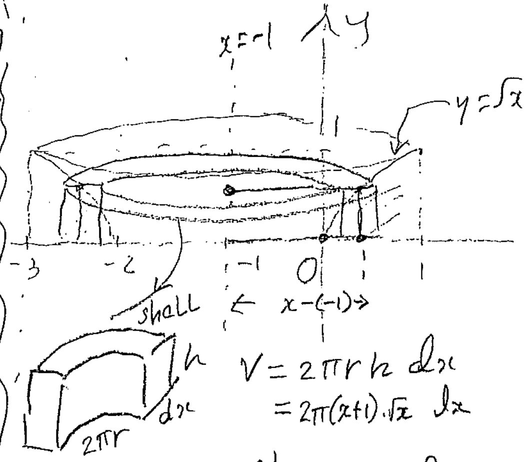
$$|z-2i| = |z-2+i|$$

$$|z-2i| = |z-(2-i)|$$



perpendicular bisector

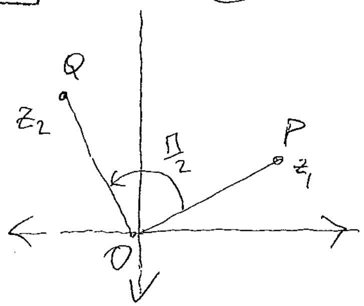
QU.9 B



$$V = 2\pi r h\,dx$$
$$= 2\pi(x+1)\sqrt{x}\,dx$$

$$\therefore V = 2\pi\int_0^1 (x+1)\sqrt{x}\,dx$$

QU.10 D



$z_2 = i z_1$ (Anti-clockwise Rotation by $\frac{\pi}{2}$)

$z_2^2 = i^2 z_1^2$ (squaring)

$z_2^2 = -z_1^2$ ($i^2 = -1$)

$$\therefore z_2^2 + z_1^2 = 0$$

✓ = 1 Mark

QU.11

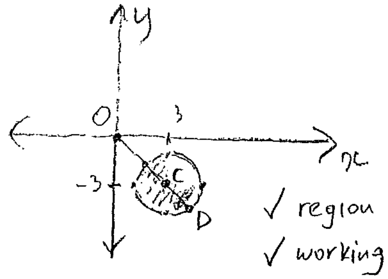
(a) $z = 1+3i$, $w = 2-i$

(i) $\bar{z} - w = (1-3i) - (2-i)$
$= 1 - 3i - 2 + i$
$= -1 - 2i$ ✓

(ii) $zw = (1+3i)(2-i)$
$= 2 - i + 6i - 3i^2$
$= 5 + 5i$ ✓

(b)

(i)



✓ region
✓ working

$|z-3+3i| \le 2$

$|z-(3-3i)| \le 2$ — region inside/on circle at $(3,-3)$, $r=2$

$\max |z| = OC + CD$
$= \sqrt{3^2+3^2} + 2$
$= 3\sqrt{2} + 2$ ✓

(c) $x^2 - 2x + 5 = (x-1)^2 + 4$
$= (x-1)^2 + 2^2$ ✓

$$\therefore \int_{-1}^{1} \frac{1}{x^2-2x+5}\,dx = \int_{-1}^{1} \frac{1}{(x-1)^2+2^2}\,dx$$

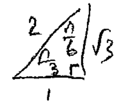
$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{x-1}{2}\right)\right]_{-1}^{1}$$ ✓

$$= \frac{1}{2}\left[(\tan^{-1}(0)) - \tan^{-1}(-1)\right]$$

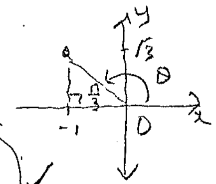
$$= \frac{1}{2}\left[0 - \left(-\frac{\pi}{4}\right)\right]$$

$$= +\frac{\pi}{8}$$ ✓

(d) $z = -1 + i\sqrt{3}$ (i)



$\theta = \frac{2\pi}{3}$



$|z| = \sqrt{(-1)^2 + (\sqrt{3})^2}$ ✓

$|z| = 2$

$\therefore z = 2\operatorname{cis}\left(\frac{2\pi}{3}\right)$ ✓

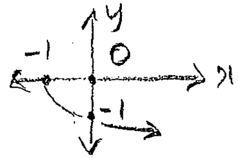
(ii) $z^8 - 16z^4 = 2^8 \operatorname{cis}\left(\frac{16\pi}{3}\right) - 16 \times 2^4 \operatorname{cis}\left(\frac{8\pi}{3}\right)$ $5\frac{1}{3}\pi$ $2\frac{2}{3}\pi$

✓ (working) $= 256\left[-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right] - 256\left[-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right]$

$= \cancel{-128} - 128\sqrt{3}i + \cancel{128} - 128\sqrt{3}i$

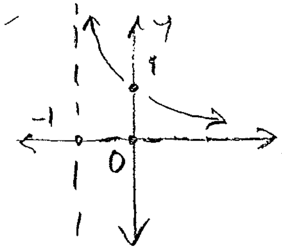
$= -256\sqrt{3}\,i$ ✓

(e) (i)



✓ shape, x-intercept, y-intercept

(ii)



✓ asymptotes, intercept

✓ shape

QV.12

a) (i) $\frac{7x+1}{(x+1)(x-2)} \equiv \frac{A}{x+1}+\frac{B}{x-2}$

$\therefore 7x+1 = A(x-2)+B(x+1)$ ✓ working

Let $x=2$

$\therefore 15 = 3B$

$\underline{B = 5}$ ✓ A, B

Let $x=-1$

$\therefore -6 = -3A$

$\underline{A = 2}$

$\int \frac{7x+1}{(x+1)(x-2)}\,dx = \int \frac{2}{x+1} + \frac{5}{x-2}\,dx$

✓ $= 2\log_e|x+1| + 5\log_e|x-2| + c$

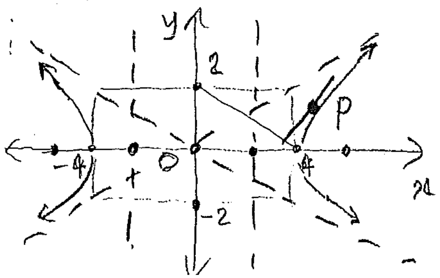
$= \log_e|x+1|^2 + \log_e(x-2)^5 + c$

✓ $= \log_e\left[|x+1|^2\,|x-2|^5\right] + c$

b) $4x^2-16y^2=64$

$\frac{x^2}{16} - \frac{y^2}{4} = 1$

$a=4 \quad b=2$



(i) foci: $(\pm 2\sqrt{5}, 0)$ $\quad c^2=a^2+b^2$

$a^2e^2 = 16+4$

$ae = 2\sqrt{5}$

$e = \frac{\sqrt{5}}{2}$

(ii) $x = \pm \frac{4}{\frac{\sqrt{5}}{2}} = \pm\frac{8}{\sqrt{5}}$

(iii) (below left)

(iv) $x = a\sec\theta = 4\sec\theta$

$y = b\tan\theta = 2\tan\theta$

$4x^2-16y^2=64$

$\therefore y^2 = \frac{1}{4}x^2 - 4$

$2yy' = \frac{1}{2}x$

$y' = \frac{1}{4}\cdot\frac{x}{y}$

at P: $y' = \frac{1}{4}\cdot\frac{4\sec\theta}{2\tan\theta}$

$= \frac{\sec\theta}{2\tan\theta} = \frac{\frac{1}{\cos\theta}}{2\cdot\frac{\sin\theta}{\cos\theta}}$

$= \frac{1}{2\sin\theta}$

(v) $y-y_1 = m(x-x_1)$

$y - 2\tan\theta = \frac{1}{2\sin\theta}(x - 4\sec\theta)$

$2\sin\theta\cdot y - 4\cdot\frac{\sin\theta}{\cos\theta}\cdot\sin\theta = x - \frac{4}{\cos\theta}$

$2\sin\theta\cdot y - \frac{4\sin^2\theta}{\cos\theta} = x - \frac{4}{\cos\theta}$

$x = 2\sin\theta\, y - 4\frac{\sin^2\theta}{\cos\theta} + \frac{4}{\cos\theta}$

$= 2\sin\theta\, y - \frac{4}{\cos\theta}\left[\sin^2\theta - 1\right]$

$\underline{x = (2\sin\theta)y + 4\cos\theta}$

QV.12 - continued

(c) $I_n = \int_1^e (1-\log_e x)^n\,dx \qquad n = 0, 1, 2, \ldots$

i) Let $u = (1-\log_e(x))^n$, $v' = 1$

$\therefore u' = n(1-\log_e(x))^{n-1}\left(-\frac{1}{x}\right)$, $v = x$

By parts: $\int uv' = uv - \int u'v$

$\therefore I_n = \left[x(1-\log_e(x))^n\right]_1^e + n\int_1^e (1-\log_e(x))^{n-1}\,dx$

$I_n = \left[e(1-\log_e(e))^n\right] - [1] + n I_{n-1}$

$= 0 - 1 + nI_{n-1}$

$\therefore \underline{I_n = -1 + nI_{n-1}} \qquad n = 1, 2, 3, \ldots$

ii) $I_3 = -1 + 3I_2$

$= -1 + 3[-1 + 2I]$

$= -1 - 3 + 6I_1$

$= -4 + 6[-1 + 1\cdot I_0]$

$= -4 - 6 + 6I_0$

$= -10 + 6\int_1^e 1\,dx$

$= -10 + 6[x]_1^e$

$= -10 + 6(e-1)$

$= -10 + 6e - 6$

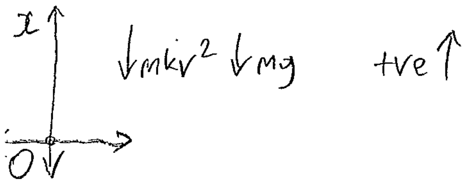
$= \underline{-16 + 6e}$

√ = 1 mark

QU. 13

(i) Equation of motion is:

$m\ddot{x} = -mg - mkv^2$ ✓



∴ $v\frac{dv}{dx} = -g - kv^2$

$\frac{-v}{g+kv^2}\,dv = 1\,dx$

$-\frac{1}{2k}\int \frac{2v}{g+kv^2}\,dv = \int 1\,dx$

$-\frac{1}{2k}\log_e(g+kv^2) = x + c$

$x = 0,\ v = V_0$ ✓

∴ $c = -\frac{1}{2k}\log_e(g+kV_0^2)$

$-\frac{1}{2k}\log_e(g+kv^2) = x - \frac{1}{2k}\log_e(g+kV_0^2)$

$x = \frac{1}{2k}\log_e\left[\frac{g+kV_0^2}{g+kV^2}\right]$ ✓

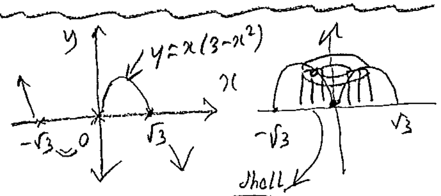
∴ $2kx = \log_e\left(\frac{g+kV_0^2}{g+kV^2}\right)$

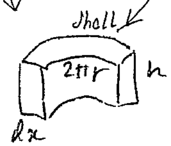
(ii) At maximum height $v=0$ ✓

∴ $2kx = \log_e\left[\frac{g+kV_0^2}{g}\right]$

$x_{max} = \frac{1}{2k}\log_e\left[\frac{g+kV_0^2}{g}\right]$ ✓

(c)





$V = 2\pi r h\,dr$

$V = 2\pi x \cdot x(3-x^2)\,dx$
$= 2\pi(3x^2 - x^4)\,dx$

∴ $V = 2\pi\int_0^{\sqrt3} 3x^2 - x^4\,dx$

$= 2\pi\left[x^3 - \frac{1}{5}x^5\right]_0^{\sqrt3}$

$= 2\pi\left[3\sqrt3 - \frac{1}{5}\cdot 9\sqrt3\right]$

$= 2\pi\left[\frac{15\sqrt3 - 9\sqrt3}{5}\right]$

$= \frac{12\pi\sqrt3}{5}$ units³

$(\sqrt3)^5 = (\sqrt3)^4\sqrt3 = 9\sqrt3$

√ = 1 mark

QU. 13 - continued

(b)

(i) In Δ's ARP, RBP:

1. ∠RPB is common
2. ∠RAB = ∠PRB (∠ between tangent PR and chord BR equals ∠ in alternate seg)
3. ∠RBP = ∠PRA (∠ sum of a Δ)

∴ ΔARP ||| ΔRBP (equiangular) ✓

Hence $\frac{PR}{PA} = \frac{PB}{PR}$ (corresponding sides in similar Δ's) ✓ ✓

∴ $PR^2 = PA \cdot PB$

(iii) Since ΔATP ||| ΔTBP

$PT^2 = PA \cdot PB$ ✓

∴ $PT^2 = PR^2$ (both equal to $PA \cdot PB$)

or $PT = PR$

* SEE ATTACHMENT – page 14 (INCLUDES DIAGRAM) } QUESTIONS (i), (iii)

(iv) In Δ's PRQ, PTQ:

1. PQ is common
2. PT = PR (proved)
3. ∠RPQ = ∠TPQ (data) ✓

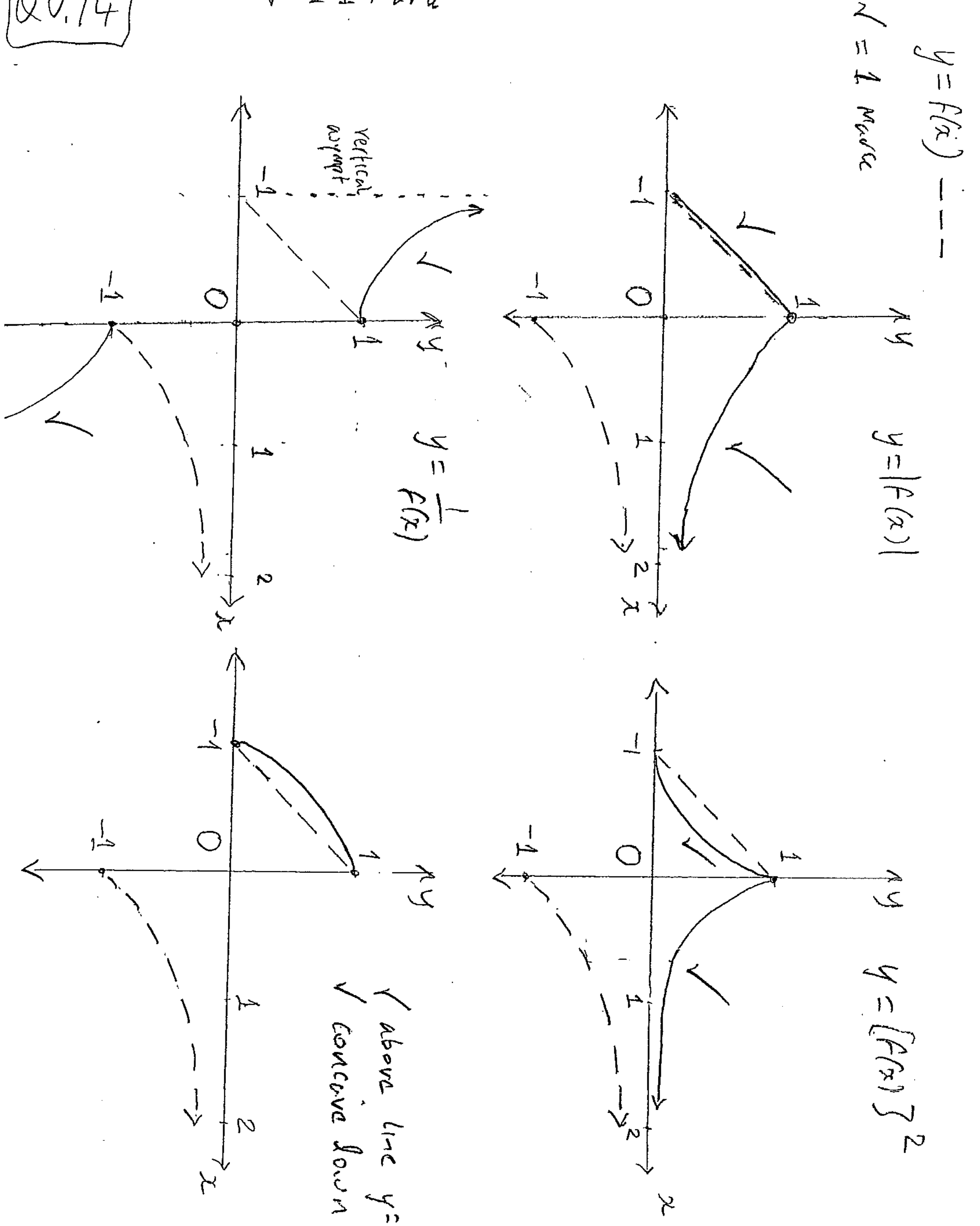
∴ ΔPRQ ≡ ΔPTQ (SAS)

Hence ∠PTQ = ∠PRQ (corresponding sides in congruent Δ's)
= 90° (∠PRQ = 90°, data)

Since PT is a tangent and ∠PTQ = 90° then QT must pass through the centre of C_2

(Converse of theorem "angle between tangent and radius of a circle at point of contact is 90°"). ✓

QV.14 ✓ = 1 mark



✓ = 1 mark

QV.14 - continued

b) $y = \frac{1-x^3}{x}$

i) $y = 0$ (x-intercept)

$\therefore 0 = \frac{1-x^3}{x}$

$1 - x^3 = 0$

$x = 1$

Coordinates: $(1,0)$ ✓

(No 'y' intercept as 'x' cannot take a value of zero - see below)

ii) Vertical asymptotes occur where denominator (only) is equal to zero.

$\therefore x = 0$ ✓

iii) $y = \frac{1-x^3}{x}$

$= \frac{1}{x} - x^2$

a) $x \to \pm\infty$, $y \to -x^2$

$\therefore$ quadratic asymptote is $y = -x^2$ ✓

NOTE: when $x<0$, $\frac{1}{x} - x^2 < -x^2$ hence the graph is below the asymptote.

(iv) $y' = \frac{-3x^2(x) - (1-x^3)(1)}{x^2}$

$= \frac{-2x^3 - 1}{x^2}$ ✓ working ✓ points

$y' = 0$ when $2x^3 = -1$, $x^3 = -\frac{1}{2}$

$\therefore x = \frac{-1}{\sqrt[3]{2}} \doteqdot -0.8$

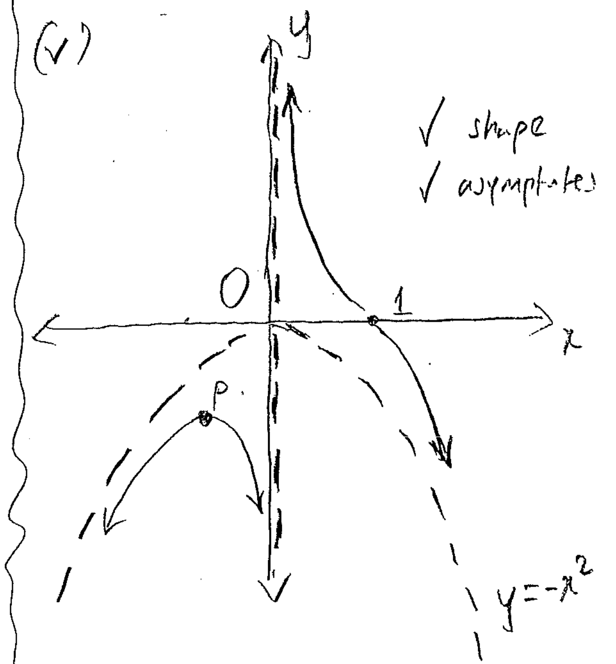
also... $y = \frac{1+\frac{1}{2}}{-\frac{1}{\sqrt[3]{2}}} = -\frac{3}{2}\sqrt[3]{2} \doteqdot -1.9$

Nature

x	-1	$\frac{-1}{\sqrt[3]{2}}$	-0.5
y'	+ve	0	-ve

$\therefore$ local max $P\left(\frac{-1}{\sqrt[3]{2}}, \frac{-3}{2}\sqrt[3]{2}\right)$

(v)



✓ shape
✓ asymptotes

$\boxed{\text{QU.15}}$ $\checkmark = 1$ mark

a) $3x^3 - 9x^2 + 6x + 2 = 0$ (1)

(i) Roots: α, β, γ

Let $M = \alpha^2 \quad \therefore \alpha = \pm\sqrt{M}$

but 'α' is a root and hence satisfies equation (1)

$\therefore 3(\pm\sqrt{M})^3 - 9M \pm 6\sqrt{M} + 2 = 0$

$\pm 3M\sqrt{M} \pm 6\sqrt{M} = 9M - 2$

$\pm 3\sqrt{M}(M+2) = 9M - 2$

working $9M(M+2)^2 = (9M-2)^2$

$9M(M^2+4M+4) = 81M^2 - 36M + 4$

$9M^3 + 36M^2 + 36M = 81M^2 - 36M + 4$

$9M^3 - 45M^2 + 72M - 4 = 0$ ✓

OR $9x^3 - 45x^2 + 72x - 4 = 0$

ii) ✓ working

$\alpha^3\beta\gamma + \alpha\beta^3\gamma + \alpha\beta\gamma^3$

$= \alpha\beta\gamma(\alpha^2+\beta^2+\gamma^2)$

$= \alpha\beta\gamma\left[(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)\right]$

$= -\frac{2}{3}\left[3^2 - 2(2)\right]$

$= -\frac{2}{3}[5]$

$= -\frac{10}{3}$ ✓

$\Sigma\alpha = -\frac{b}{a} = 3 \qquad \Sigma\alpha\beta = \frac{c}{a} = 2 \qquad \alpha\beta\gamma = -\frac{d}{a} = -\frac{2}{3}$

(b) (i) $\sin(A+B) - \sin(A-B) = 2\sin B\cos A$

$\therefore 2\sin(\theta)\cos(6\theta)$

$= \sin(6\theta+\theta) - \sin(6\theta-\theta)$

$= \sin 7\theta - \sin 5\theta$ ✓

(ii)

$LHS = 2\sin\theta.(\cos 6\theta + \cos 4\theta + \cos 2\theta)$

$= 2\sin\theta.\cos 6\theta + 2\sin\theta\cos 4\theta + 2\sin\theta\cos 2\theta$

$= (\sin 7\theta - \sin 5\theta) + (\sin 5\theta - \sin 3\theta) + (\sin 3\theta - \sin\theta)$

$= \sin 7\theta - \sin\theta$

(iii) Let $\theta = \frac{2\pi}{7}$

$\therefore 2\sin(\frac{2\pi}{7})\left[\cos\frac{12\pi}{7} + \cos\frac{8\pi}{7} + \cos\frac{4\pi}{7}\right]$

$= \sin(2\pi) - \sin(\frac{2\pi}{7})$

$= 0 - \sin(\frac{2\pi}{7})$

$\therefore \cos\frac{12\pi}{7} + \cos\frac{8\pi}{7} + \cos\frac{4\pi}{7} = -\frac{1}{2}$

(dividing through by $2\sin(\frac{2\pi}{7})$)

$\boxed{\text{QU.15}}$-continued $\checkmark = 1$ mark

(c)

(i) For a geometric series the sum is

$S_n = \frac{a(r^n-1)}{(r-1)}$ ✓

$\therefore 1 + x + x^2 + x^3 + \dots + x^{n-1}$

$= \frac{1.(x^n-1)}{(x-1)} \quad \begin{cases} a=1 \\ r=x \end{cases}$

$= \frac{x^n-1}{(x-1)}$ ✓

(ii) Differentiating both sides of

$1 + x + x^2 + x^3 + \dots + x^{n-1} = \frac{x^n-1}{(x-1)}$

$1 + 2x + 3x^2 + \dots + (n-1)x^{n-2}$

$= \frac{(nx^{n-1})(x-1) - (x^n-1)(1)}{(x-1)^2}$ ✓

$= \frac{nx^n - nx^{n-1} - x^n + 1}{(x-1)^2}$ ✓

$= \frac{(n-1)x^n - nx^{n-1} + 1}{(x-1)^2}$ ✓

QV.16 $\checkmark = 1$ mark

a)

(i)

$$\frac{d}{dx}\left[\sin^{-1}(u) - \sqrt{1-u^2}\right] \quad (-1 \le u \le 1)$$

$$= \frac{1}{\sqrt{1-u^2}} - \frac{1}{2}\cdot\frac{1}{\sqrt{1-u^2}}\cdot(-2u) \quad \checkmark$$

$$= \frac{1}{\sqrt{1-u^2}} + \frac{u}{\sqrt{1-u^2}}$$

$$= \frac{1+u}{\sqrt{1-u^2}} \quad \checkmark$$

$$= \sqrt{\frac{(1+u)^2}{(1-u^2)}}$$

$$= \sqrt{\frac{1+u}{1-u}} \quad \checkmark \quad (-1 < u < 1)$$

(ii) $\displaystyle\int_0^{\alpha}\left(\frac{1+u}{1-u}\right)^{\frac{1}{2}} du$ {using result from (i)}

$$= \left[\sin^{-1}(u) - \sqrt{1-u^2}\right]_0^{\alpha} \quad \checkmark$$

$$= \left(\sin^{-1}(\alpha) - \sqrt{1-\alpha^2}\right) - \left(\sin^{-1}(0) - \sqrt{1-0}\right)$$

$$= \sin^{-1}(\alpha) - \sqrt{1-\alpha^2} + 1$$

Note: $1-\alpha^2 \geq 0$ or $\alpha^2 \le 1$

$-1 \le \alpha \le 1$ $\checkmark$

'α' satisfies $0 < \alpha < 1$

(b)

$P(n) = 10^n + 3\times 4^{n+2} + 5$ {divisible by 9 where $n \geq 1$ (n - integer)}

$P(1)$:

$10^1 + 3\times 4^3 + 5 = 207 = 9\times 23$ $\checkmark$

$\therefore$ expression is divisible by 9

$\therefore$ $P(1)$ is true

Assume expression is divisible by 9 for some +ve integer k ($k \geq 1$)

i.e. $10^k + 3\times 4^{k+2} + 5 = 9p$ ($p \geq 1$) $\checkmark$

Assume $P(k)$ is true

Test $P(k+1)$:

$$10^{k+1} + 3\times 4^{k+1+2} + 5$$

$$= 10^k \cdot 10^1 + 3\times 4^1 \times 4^{k+2} + 5$$

$$= 10(10^k) + 12(4^{k+2}) + 5$$

$$= (9+1)(10^k) + (9+3)(4^{k+2}) + 5$$

$$= (10^k) + 9(10^k) + 3(4^{k+2}) + 9(4^{k+2}) + 5$$

$$= (10^k) + 3(4^{k+2}) + 5 + 9(10^k) + 9(4^{k+2}) \quad \checkmark$$

$$= 9p + 9(10^k) + 9(4^{k+2}) \quad \text{\{using assumption\}}$$

$$= 9(p + 10^k + 4^{k+2})$$

{$k \geq 1$, $p \geq 1$ {integers}; $\therefore$ 10^k +ve integer, 4^{k+2} +ve integer; $\therefore$ $p + 10^k + 4^{k+2}$ +ve integer}

$\therefore$ $P(k+1)$ is true.

Since $P(1)$ is true $\checkmark$ and we proved $P(k+1)$ is true then $P(1+1) = P(2)$, $P(2+1) = P(3)$, and so on, $P(n)$ is true for ...

$\checkmark = 1$ mark

QV.16 - continued

(c)

(i)

Area $\Delta ABC = \frac{1}{2}ad$ (1)

Area $\Delta ABC = \frac{1}{2}bc$ (2)

but $a^2 = b^2 + c^2$ (3) (Pythagoras) $\checkmark$

$\therefore$ $\frac{1}{2}ad = \frac{1}{2}bc$ {from (1),(2)}

or $ad = bc$ $\checkmark$

$a^2d^2 = b^2c^2$ (squaring)

$b^2c^2 = d^2(b^2+c^2)$ {using (3)}

(ii) Using ΔABC from (i) and the result $b^2c^2 = d^2(b^2+c^2)$ we arrive at:

$$AC^2 \cdot AB^2 = AP^2(AC^2 + AB^2) \quad \checkmark$$

$$\therefore AP^2 = \frac{AC^2 \cdot AB^2}{AC^2 + AB^2}$$

or $$\frac{1}{AP^2} = \frac{AC^2 + AB^2}{AC^2 \cdot AB^2}$$

Also $\tan\gamma = \dfrac{h}{AP}$ $\checkmark$

$$\therefore \tan^2\gamma = \frac{h^2}{AP^2} = \frac{h^2[AC^2 + AB^2]}{[AC^2 \cdot AB^2]}$$

Similarly $\tan^2\beta = \dfrac{h^2}{AC^2}$, $\tan^2\alpha = \dfrac{h^2}{AB^2}$ $\checkmark$

$$\therefore \tan^2\beta + \tan^2\alpha = \frac{h^2}{AC^2} + \frac{h^2}{AB^2}$$

$$= \frac{h^2[AB^2 + AC^2]}{AC^2 AB^2} \quad \checkmark$$

$$= \tan^2\gamma$$

MATH EXT 2 TRIAL 2014 ATTACHMENT

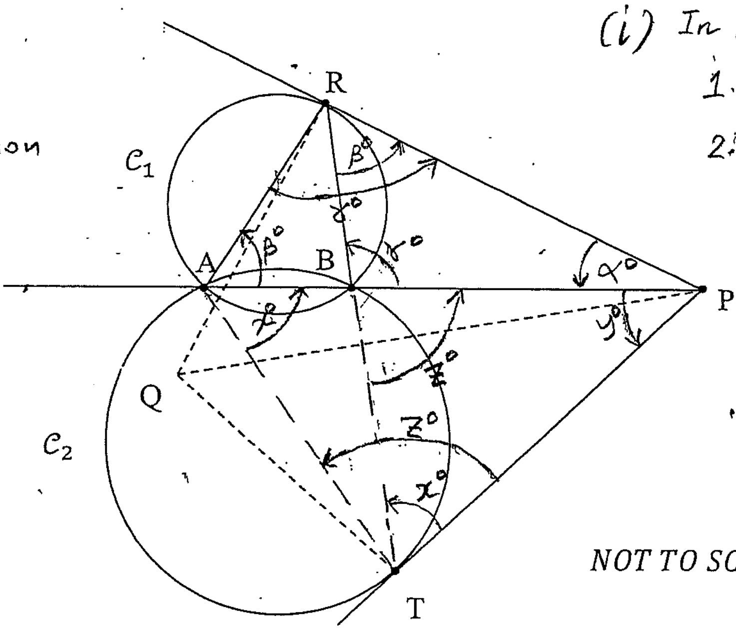
SECTION II

Qu.13 (b)

p.14 of 14

Qu.13 (b) Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at the points A and B. Let P be a point on AB produced and let PR and PT be tangents to $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively.

Also, $\angle PRQ = 90°$ and $\angle RPQ = \angle TPQ$.



NOT TO SCALE

(i) In Δ's ARP, RBP :

1. $\angle RPB = \alpha$ is common

2. $\angle RAB = \beta = \angle PRB$ ($\angle$ between tangent PR and chord BR equals $\angle$ in alternate segm)

3. $\therefore \angle ARP = \gamma = \angle RBP$ ($\angle$ sum Δ)

$\therefore \Delta ARP \,|||\, \Delta RBP$ (equiangular)

(iii) (OPTIONAL)

In Δ's ATP, TPB :

1. $\angle APT = y$ is common

2. $\angle PTA = x = \angle PAT$ ($\angle$ between tangent PT and chord TB is...)

3. $\angle ATP = z = \angle PBT$ ($\angle$ sum Δ)

$\therefore \Delta ATP \,|||\, \Delta TPB$ (equiangular)