Student's Name: ____________________

SANTA SABINA COLLEGE

Mathematics Extension 1

H.S.C. Assessment Task 2

16 June 2006

Time Allowed: 60 Minutes

Directions to Candidates

- Attempt all questions.
- Start a new page for each question.
- Use blue or black pen to write your answers. Pencil may be used to draw diagrams.
- The marks indicated are a guide only.
- All necessary working must be shown.
- Full marks may not be awarded for careless or badly arranged work.
- Approved scientific calculators may be used.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n\neq -1;\quad x\neq 0,\ \text{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\quad a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a>0,\quad -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\quad x>0$

QUESTION 1: Further Methods of Integration (5 Marks)

Marks

(a) Find $\int x(1+x^2)^4\,dx$ using the substitution $u = 1 + x^2$ or otherwise. 2

(b) Find the exact value of $\int_0^{\pi/4} 4\sin^2 x\,dx$ 3

QUESTION 2: Further Trigonometry (13 Marks)

(a) Find the exact value of $\dfrac{\tan 47^0 - \tan 17^0}{1 + \tan 47^0 . \tan 17^0}$ 2

(b) Simplify $\cos(x-y) - \cos(x+y)$ showing all working. 2

(c) Find the general solution for $\tan 2\theta = \tan\theta$ if $t = \tan\theta$. 3

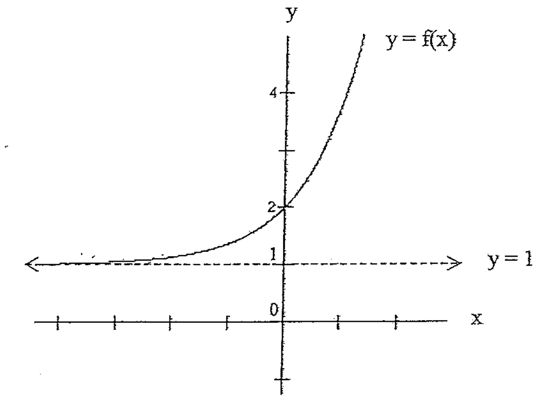
(d) (i) Express $\sqrt{3}\sin\theta + \cos\theta$ in the form $r\sin(\theta + \alpha)$ 3

(ii) Hence solve $\sqrt{3}\sin\theta + \cos\theta = 1$ for $0 \le \theta \le 2\pi$ 3

QUESTION 3: Inverse Trigonometric Functions (12 marks)

Marks

(a)



(i) Copy the sketch of $y = f(x)$ onto your own paper and add a sketch of $y = f^{-1}(x)$ 1

(ii) If $y = f(x)$ is given by the function $y = e^x + 1$, find the equation of the inverse function which you have sketched 1

(b) Sketch the graph of the function $y = \sin^{-1} 3x$, clearly showing the domain and range. 3

(c) Find $\int \dfrac{dx}{9+x^2}$ 1

(d) Find the derivative of $y = \cos^{-1}(\dfrac{x}{2})$ 1

(e) Find the gradient of the tangent to the curve $y = \tan^{-1}\sqrt{x}$ when $x = 1$. 2

(f) The area bounded by the curve $y = \dfrac{1}{(1-x^2)^{\frac{1}{4}}}$, the x - axis and the lines $x = 0.2$ and $x = 0.6$ is rotated about the $x -$ axis.

Find the volume of the solid formed, correct to 2 decimal places. 3

Extension 1 HSC Task 2, 2006

1. (a) Let $u = 1 + x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$\therefore \int x(1+x^2)^4\,dx$

$= \int \frac{1}{2} u^4\,du$ (1)

$= \frac{1}{2}\frac{u^5}{5} + C$ C

$= \frac{(1+x^2)^5}{10} + C$ (1)

(b) $\cos 2x = 1 - 2\sin^2 x$

$\therefore \sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\therefore \int_0^{\pi/4} 4\sin^2 x\,dx$

$= \int_0^{\pi/4} 2 - 2\cos 2x\,dx$ (1) C

$= \left[2x - \frac{\cancel{2}\sin 2x}{\cancel{2}}\right]_0^{\pi/4}$ (1)

$= \left[\frac{\pi}{2} - \sin\frac{\pi}{2}\right] - \left[0 - 0\right]$

$= \frac{\pi}{2} - 1$ (1)

2 (a) $\frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \tan(\alpha - \beta)$

$\therefore \frac{\tan 47^\circ - \tan 17^\circ}{1 + \tan 47^\circ \tan 17^\circ} = \tan(47^\circ - 17^\circ)$ (1)

$= \tan 30^\circ$

$= \frac{1}{\sqrt{3}}$ (1)

(b) $\cos(x-y) - \cos(x+y)$

$= \cos x\cos y + \sin x\sin y - [\cos x\cos y - \sin x\sin y]$ (1)

$= 2\sin x\sin y$ (1)

(c) $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ $t = \tan\theta$

$\therefore \frac{2t}{1-t^2} = t$ (1)

$2t = t - t^3$

$t^3 + t = 0$ C

$t(t^2+1) = 0$

$t = 0$ or $t^2 = -1$ (no solution) (1)

$\therefore \tan\theta = 0$

$\therefore \theta = 0 + \pi n$

$= \pi n$ (where n is an integer) (1)

(d) (i) $r = \sqrt{(\sqrt{3})^2 + 1^2}$

$= \sqrt{4}$

$= 2$ (1)

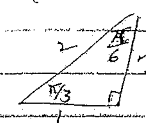
$\sqrt{3}\sin\theta + \cos\theta$

$= 2\left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right)$

$= 2(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$

$r\sin(\theta + \alpha)$ (1)

$= r(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$



$\therefore \cos\alpha = \frac{\sqrt{3}}{2}$ and $\sin\alpha = \frac{1}{2}$

(1st Quad)

$\therefore \alpha = \frac{\pi}{6}$ (1) $\therefore \sqrt{3}\sin\theta + \cos\theta = 2\sin\left(\theta + \frac{\pi}{6}\right)$

(ii). If $\sqrt{3}\sin\theta + \cos\theta = 1$ $\quad 0 \le \theta \le 2\pi$

then $2\sin(\theta + \pi/6) = 1$ $\quad \pi/6 \le \theta + \pi/6 \le \frac{13}{6}\pi$ (1)

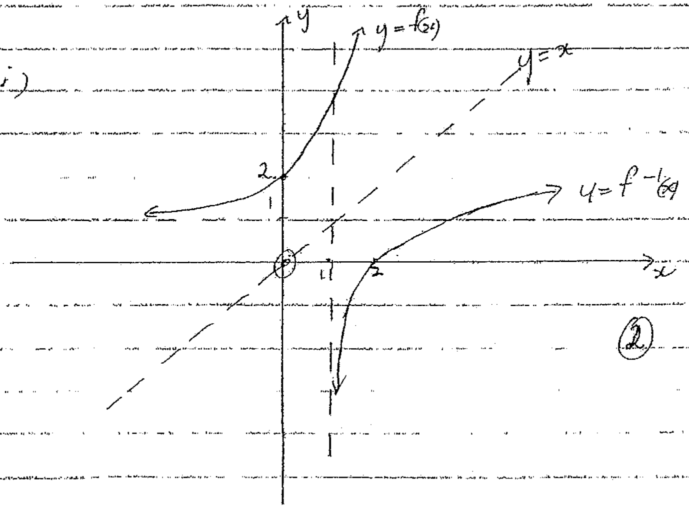
$\sin(\theta + \pi/6) = \frac{1}{2}$ (1)

(acute ∠ = $\pi/6$) 1st, 2nd, 5th Quads

$\therefore \theta + \pi/6 = \pi/6, \frac{5}{6}\pi, \frac{13}{6}\pi$

so $\theta = 0, \frac{2}{3}\pi, 2\pi$ (1)

3. (a)(i)



(2)

(ii) $y = e^x + 1$

Inverse: $x = e^y + 1$

$e^y = x - 1$

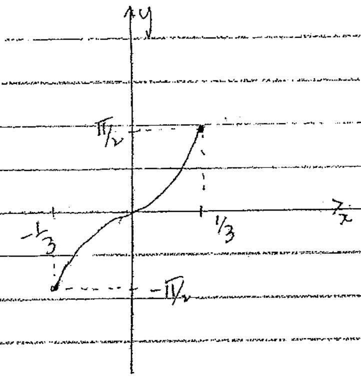
$\ln(x-1) = y$

or $y = \log_e(x-1)$ (1)

(b) $y = \sin^{-1} 3x$ $\quad -1 \le 3x \le 1$

so Domain is: $-\frac{1}{3} \le x \le \frac{1}{3}$

Range: $-\pi/2 \le y \le \pi/2$



(3)

(c) $\int \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c$ (1)

(d) $\frac{d}{dx}\left(\cos^{-1}\left(\frac{x}{2}\right)\right) = \frac{-1}{\sqrt{4-x^2}}$ (1)

(e) gradient of tangent $= \frac{dy}{dx} = \frac{1}{1+(\sqrt{x})^2} \times \frac{1}{2}x^{-\frac{1}{2}}$ (2)

$= \frac{1}{2\sqrt{x}(1+x)}$

when $x = 1$, gradient $= \frac{1}{2\times1(2)}$

$= \frac{1}{4}$

(f) $V = \pi \int y^2 \, dx$

$= \pi \int_{0.2}^{0.6} \frac{1}{(1-x^2)^{\frac{1}{2}}} \, dx$ (1)

$= \pi \int_{0.2}^{0.6} \frac{1}{\sqrt{1-x^2}} \, dx$

$= \pi \left[\sin^{-1} x \right]_{0.2}^{0.6}$ (1)

$= \pi \left[\left(\sin^{-1} 0.6 \right) - \left(\sin^{-1} 0.2 \right) \right]$

$= 1.389033291$ (1)

$\doteq 1.39 \text{ units}^3$