SCEGGS Darlinghurst

Student Number

Wednesday 24th November, 2014

HIGHER SCHOOL CERTIFICATE 2015

ASSESSMENT TASK 1
Task Weighting: 15%

Extension 2 Mathematics

Outcomes Assessed: E2, E3 and E9

General Instructions

- Time allowed – 70 minutes
- Write using black or blue pen
  Black pen is preferred
- Board approved calculators, mathematical templates and geometrical instruments may be used
- Draw diagrams in pencil using at least one-third of a page
- In Questions 6-7, show relevant mathematical reasoning and/or calculations

Total marks – 40 Marks

Section I Multiple Choice

5 marks

- Attempt Questions 1 – 5
- Allow at most 10 minutes for this section

Section II Extended response

35 marks

- Attempt Questions 6-7
- Allow about 60 minutes for this section

Question	Marks
Multiple choice 1-5	/5
6	/18
7	/17
Total	/40

Average: .... **Standard Deviation:** .... **Rank:** ....

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Section I
5 marks
Attempt Questions 1 – 5
Allow about 10 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 5.

1. If $z = 2 - 5i$ find z^{-1} expressed with a real denominator.

(A) $\frac{2+5i}{24}$

(B) $\frac{2-5i}{29}$

(C) $\frac{2+5i}{29}$

(D) $\frac{-2+5i}{24}$

2. What is $z = -3 - \sqrt{3}i$ in modulus-argument form?

(A) $2\sqrt{3}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

(B) $2\sqrt{3}\left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right)$

(C) $12\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

(D) $12\left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right)$

3. If $\bar{z}$ is the conjugate of z then which of the following is a true statement.

(A) $z\bar{z}$ is real

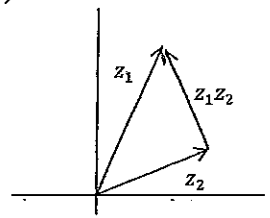
(B) $\arg z = \arg \bar{z}$

(C) $\frac{z}{\bar{z}}$ is real

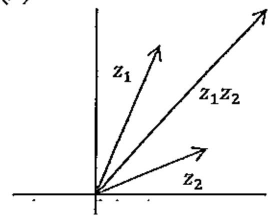
(D) $\bar{z}$ is the mirror image of z in the y-axis

4. If $z_1 = 2 + i$ and $z_2 = 1 + 3i$ then $z_1 z_2$ is represented by:

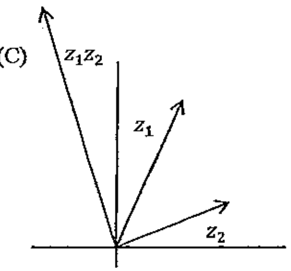
(A)



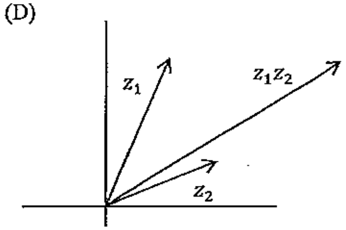
(B)



(C)

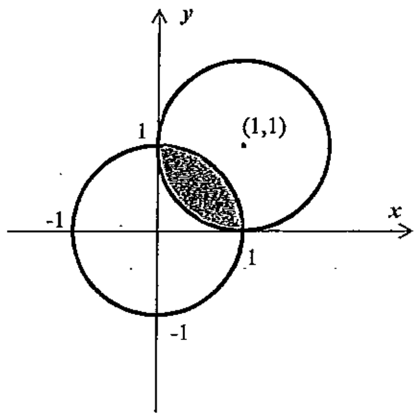


(D)



Section I continued.

5. Consider the Argand diagram below:



Which inequality could define the shaded region:

(A) $|z| \leq 1$ and $|z-(1-i)| \geq 1$

(B) $|z| \leq 1$ and $|z-(1+i)| \geq 1$

(C) $|z| \leq 1$ and $|z-(1-i)| \leq 1$

(D) $|z| \leq 1$ and $|z-(1+i)| \leq 1$

End of Section I.

Section II

35 marks
Attempt Questions 6 and 7
Allow about 60 minutes for this section

Answer each question in a SEPARATE writing booklet.

In Questions 6 and 7, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (18 marks) Use a SEPARATE writing booklet. **Marks**

(a) Given that $z = 1 + 4i$ and $w = 2 - i$, find in the form $a + ib$,

(i) $\bar{z} - w$ **1**

(ii) $Re(zw)$ **1**

(b) i) Express $\sqrt{3} - 3i$ in modulus-argument form **1**

ii) Hence evaluate $(\sqrt{3} - 3i)^{12}$ **2**

(c) On an Argand diagram sketch $Re(z + iz) \geq 4$ **2**

(d) Describe, in geometric terms, the locus in the Argand plane represented by **2**

$$2|z| = z + \bar{z} + 4$$

Question 6 continues on the next page.

Question 6 continued **Marks**

(e) i) Find the square roots of $-3-4i$ **3**

ii) Hence, or otherwise, solve the quadratic equation: **2**

$$z^2 - 3z + (3+i) = 0$$

(f) (i) Use De Moivre's Theorem to show that: **2**

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

(ii) By using the substitution, $x = \cos\theta$, or otherwise solve the equation: **2**

$$4x^3 - 3x - 1 = 0$$

End of Question 6

Question 7 (17marks) Use a SEPARATE writing booklet **Marks**

(a) Let $P(z) = z^4 + 3z^2 - 6z + 10$

(i) If $z = 1+i$ is a root of $P(z) = 0$, find another root of $P(z) = 0$ **1**

(ii) Factorise $P(z) = z^4 + 3z^2 - 6z + 10$ over the real numbers $\mathbb{R}$ **2**

(iii) Hence find all the roots of $P(z) = 0$. **2**

(b) If $z = -1 + i$ and $w = \sqrt{3} - i$,

(i) Find wz and express your answer in the form $x + iy$. **1**

(ii) Evaluate wz in modulus-argument form. **2**

(iii) Hence, find the exact value of $\sin\frac{7\pi}{12}$. **2**

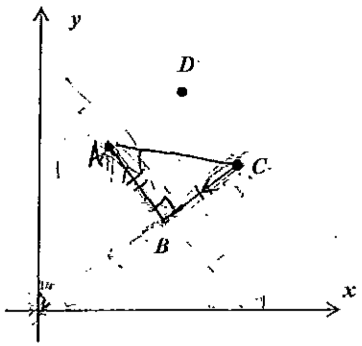
(c) i) On an Argand diagram shade the region where the inequalities **2**

$|z| \leq 2$ and $\frac{\pi}{4} \leq \arg(z+2) \leq \frac{\pi}{2}$ hold simultaneously.

ii) For the values of z that satisfy the region, what are the possible values of $|z|$ and $\arg z$? **2**

Question 7 continues on the next page.

(d) In the diagram the vertices of a triangle ABC are represented by the numbers z_1, z_2 and z_3 respectively. The triangle is isosceles and right-angled at B.



(i) Explain why $(z_1 - z_2)^2 = -(z_3 - z_2)^2$ **2**

(ii) Suppose D is the point such that $ABCD$ is a square. Find the complex number in terms of z_1, z_2 and z_3 that represents D **1**

End of assessment.

Extension 2 Mathematics Assessment Task 1 - 2015

Section I - Multiple Choice

1. $\frac{1}{2-5i} \times \frac{2+5i}{2+5i}$

$= \frac{2+5i}{4+25}$

$= \frac{2+5i}{29}$ (C) ✓

2. $|z| = \sqrt{(-3)^2+(-\sqrt{3})^2}$

$= \sqrt{9+3}$

$= \sqrt{12} = 2\sqrt{3}$

$\arg z = \tan^{-1}\left(\frac{-\sqrt{3}}{-3}\right)$

$= -\frac{5\pi}{6}$

$\therefore 2\sqrt{3}\left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right)$ (B) ✓

3. $z\bar{z}$ is real (A) ✓

4. (C) ✓

5. (D) ✓

/5

Section II

Q6

a) i) $\bar{z} - w = 1-4i-(2-i)$

$= -1-3i$ ✓

ii) $zw = (1+4i)(2-i)$

$= 2-i+8i+4$

$= 6+7i$

$Re(zw) = 6$ ✓ /2

b) i) $|z| = \sqrt{12}$

$\arg z = -\frac{\pi}{3}$

$\therefore z = \sqrt{12}\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$ ✓ /1

ii) $(\sqrt{3}-3i)^{12} = \left(\sqrt{12}\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)\right)^{12}$

$= 12^6(\cos 4\pi - i\sin 4\pi)$ ✓

$= 12^6 = 2985984$ /2

c) $Re(z+iz) \geq 4$

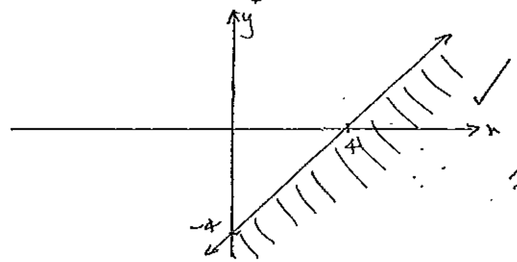
let $z = x+iy$

$z+iz = x+iy+i(x+iy)$

$= x-y+(x+y)i$

$\therefore Re(z+iz) = x-y$ ✓

$\therefore x-y \geq 4$



/2

d) $2|z| = z + \bar{z} + 4$

let $z = x + iy$

$2\sqrt{x^2+y^2} = x + iy + x - iy + 4$

$2\sqrt{x^2+y^2} = 2x + 4$

$4(x^2+y^2) = 4x^2 + 16x + 16$

$\cancel{4x^2} + 4y^2 = \cancel{4x^2} + 16x + 16$

$4y^2 = 16x + 16$

$y^2 = 4x + 4$

$y^2 = 4(x+1)$ ✓

sideways parabola vertex $(-1, 0)$ ✓

focus at $(0,0)$ /2

e) i) let $x + iy = \sqrt{-3-4i}$

$x^2 + 2xyi - y^2 = -3 - 4i$

$x^2 - y^2 = -3$ ...① $\quad 2xy = -4$

$\therefore xy = -2$

$y = -\frac{2}{x}$ ...② ✓

sub ② into ①

$x^2 - \left(-\frac{2}{x}\right)^2 = -3$

$x^2 - \frac{4}{x^2} = -3$

$x^4 - 4 = -3x^2$ ✓

$x^4 + 3x^2 - 4 = 0$

$(x^2+4)(x^2-1) = 0$

no real solution $\quad x = \pm 1$

$y = \mp 2$

$\therefore z = 1 - 2i, \; z = -1 + 2i$ ✓ /3

ii) $z = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 1 \times (3+i)}}{2}$

$= \frac{3 \pm \sqrt{9 - 12 - 4i}}{2}$

$= \frac{3 \pm \sqrt{-3-4i}}{2}$ ✓

$\therefore z = \frac{3 \pm (1-2i)}{2}$

$z = \frac{4-2i}{2} = 2 - i \qquad z = \frac{2+2i}{2} = 1 + i$ ✓ /2

f) i) $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

$\cos^3\theta + 3\cos^2\theta\, i\sin\theta + 3\cos\theta\, i^2\sin^2\theta + i^3\sin^3\theta = \cos 3\theta + i\sin 3\theta$ ✓

equating real parts

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)$ ✓

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$ /2

ii) $4x^3 - 3x - 1 = 0$

let $x = \cos\theta$

$4\cos^3\theta - 3\cos\theta - 1 = 0$

$4\cos^3\theta - 3\cos\theta = 1$

$\cos 3\theta = 1$ ✓

$3\theta = 2k\pi \qquad k = 0, \pm 1, \ldots$

$\theta = \frac{2k\pi}{3}$

$\theta = 0, -\frac{2\pi}{3}, \frac{2\pi}{3}, \ldots$

$x = \cos 0 = 1 \qquad x = \cos\frac{2\pi}{3}, \; x = -\frac{1}{2}$ ✓ /2

Q7

a) i) as the coefficients of $P(z)$ are real, another root is $z = 1-i$ ✓ /1

ii) if $z = 1+i$ and $z = 1-i$ are roots then $z^2 - 2\text{Re}(z)z + z\bar{z}$ is a factor

$\therefore z^2 - 2z + 2$ is a factor of $P(z)$ ✓

$$\begin{array}{r} z^2+2z+5 \\ z^2-2z+2 \,\overline{)\, z^4+0z^3+3z^2-6z+10} \\ \underline{z^4-2z^3+2z^2} \\ 2z^3+z^2-6z \\ \underline{2z^3-4z^2+4z} \\ 5z^2-10z+10 \end{array}$$

$\therefore P(z) = (z^2-2z+2)(z^2+2z+5)$ ✓ /2

iii) $z^2+2z+5=0$

$$z = \frac{-2 \pm \sqrt{2^2 - 4\times1\times5}}{2}$$

$$= \frac{-2 \pm \sqrt{4-20}}{2} \quad ✓$$

$$= \frac{-2 \pm \sqrt{-16}}{2}$$

$$= \frac{-2 \pm 4i}{2}$$

$$= -1 \pm 2i \quad ✓ \quad /2$$

$\therefore$ roots are $1+i,\ 1-i,\ -1+2i,\ -1-2i$

b) i) $wz = (-1+i)(\sqrt{3}-i)$

$= -\sqrt{3} + i + \sqrt{3}i + 1$

$= (1-\sqrt{3}) + (1+\sqrt{3})i$ ✓

/1

ii) $z = \sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)$ $\quad w = 2\,\text{cis}\left(-\frac{\pi}{6}\right)$ ✓

$zw = 2\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4} + -\frac{\pi}{6}\right)$

$= 2\sqrt{2}\,\text{cis}\left(\frac{7\pi}{12}\right)$ ✓ /2

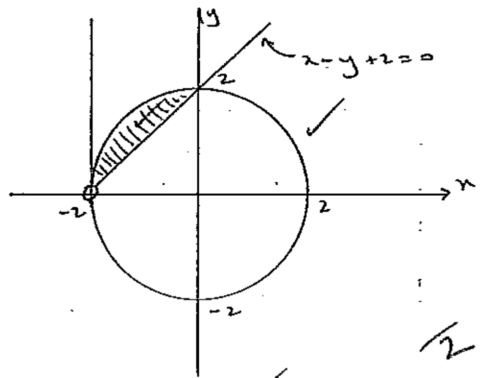
iii) $2\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right) = (1-\sqrt{3}) + (1+\sqrt{3})i$

equating imaginary parts ✓

$2\sqrt{2}\sin\frac{7\pi}{12} = 1+\sqrt{3}$

$\sin\frac{7\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}}$ ✓ /2

c) i)



✓ /2

ii) $\frac{\pi}{2} \le \arg z < \pi$ ✓

shortest distance from the origin to $x-y+2=0$ is $= \frac{|0-0+2|}{\sqrt{1^2+1^2}}$

$= \sqrt{2}$

$\therefore \sqrt{2} \le |z| \le 2$ ✓

d) i) $\vec{BC} = \vec{BO} + \vec{OC}$

$= -z_2 + z_3$

$= z_3 - z_2$

$\vec{BA} = i \times \vec{BC}$

$= i(z_3 - z_2)$ ✓

$\vec{BA} = \vec{BO} + \vec{OA}$

$= -z_2 + z_1$

$= z_1 - z_2$

$\therefore z_1 - z_2 = i(z_3 - z_2)$ ✓

$(z_1 - z_2)^2 = -(z_3 - z_2)^2$

2/2

ii) $\vec{OD} = \vec{OC} + \vec{CD}$

$= \vec{OC} + \vec{BA}$ as ABCD is a square

$= z_3 + z_1 - z_2$

$= z_1 - z_2 + z_3$ ✓

1/1

Section I

5 marks
Attempt Questions 1–5
Allow about 10 minutes for this section

Use the multiple-choice answer sheet.

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

correct →

A ⊗ B ⊗ C ○ D ○