Sydney Girls High School

2014

YEAR 12 HSC ASSESSMENT TASK 2

MATHEMATICS EXTENSION 2

Time Allowed: 60 minutes + 5 minutes reading time

Topic: Complex Numbers

Total: 48 marks

General Instructions:

- There are Four (4) Questions which are of equal value.
- Attempt all questions.
- Show all necessary working. Marks may be deducted for badly arranged work or incomplete working.
- Start each Question on a new page.
- Write on one side of the paper only.
- Diagrams are NOT to scale.
- Board-approved calculators may be used.
- Write your name clearly at the top of each question and clearly number each question.

Student Name :________________ **Teacher Name :**________________

Question 1 **12 Marks**

a) Given the complex numbers $A=3+4i$ and $B=2-3i$, write the following in the form $x+iy$:

i) $A+B$ **1**

ii) $\overline{AB}$ **1**

iii) $\dfrac{A}{B}$ **1**

iv) $\sqrt{A}$ **3**

b) For the complex number $z=1+\sqrt{3}i$:

i) Find $|z|$ and $\arg z$ **2**

ii) Write z in modulus-argument form **1**

iii) By using your answer in part ii) or otherwise, write the complex number z^4 in the form $a+ib$. **1**

c) If $a=x+iy$ and $b=u+iv$, show that $|a+b|^2=|a|^2+|b|^2+2\operatorname{Re}\left(a\bar{b}\right)$. **2**

Question 2 (Start a New Page) 12 Marks

a) Evaluate i^{2014}. **1**

b)

i) Given that ω is a complex root of the equation $z^3 = 1$, show that ω^2 is also a root of this equation. **1**

ii) Show that $1+\omega+\omega^2 = 0$. **1**

iii) Evaluate $(1-\omega)(1-\omega^2)\left(1-\frac{1}{\omega^4}\right)\left(1-\frac{1}{\omega^8}\right)$. **2**

c)

i) Sketch and describe the locus of $|z-1+2i| = |z+3|$. **2**

ii) Find the Cartesian equation of the locus of z. **2**

d) Shade the region where $|z-1-i| \le 1$ and $0 < \arg(z-i) < \frac{\pi}{4}$ both hold. **3**

Question 3 (Start a New Page) 12 Marks

a) If $z = \frac{1+i}{1-i}$ and $w = \frac{2}{1-\sqrt{3}i}$:

i) Express z and w in modulus argument form. **3**

ii) Plot z, w and $z+w$ on an Argand diagram. **1**

iii) Show that $\tan\frac{5\pi}{12} = \sqrt{3}+2$. **2**

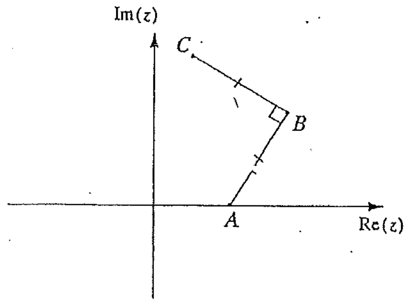
b)

i) Sketch the locus of a point z which moves such that: **1**

$$\arg\left(\frac{z-2i}{z+2i}\right) = -\frac{\pi}{2}$$

ii) Find the Cartesian equation of the locus. **2**

c)



The diagram above shows the fixed points A, B and C in the Argand plane, where $AB = BC$, $\angle ABC = \frac{\pi}{2}$, and A, B and C are in anticlockwise order. The point A represents the complex number $z_1 = 2$ and the point B represents the complex number $z_2 = 3+\sqrt{5}\,i$.

i) Find the complex number z_3 represented by the point C. **2**

ii) D is the point such that $ABCD$ is a square. Find the complex number z_4, that represents D. **1**

Question 4 (Start a New Page) **12 Marks**

a) The complex number $z = x + iy$ where x and y are real, satisfies the relation $|z+4| = 2\sqrt{3}$.

i) Sketch the locus of the point P representing z. **1**

ii) Find the maximum and minimum values of $\arg z$, where $-\pi < \arg z \leq \pi$. **2**

iii) Find the value of z in the form $x+iy$ when $\arg z$ takes its minimum value. **2**

b)

Given that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$:

i) Solve the equation $16x^5 - 20x^3 + 5x - 1 = 0$. **2**

ii) Deduce that $\cos\frac{2\pi}{5}$ and $\cos\frac{4\pi}{5}$ have a product of $-\frac{1}{4}$ and a sum of $-\frac{1}{2}$. **3**

iii) Hence or otherwise, find as surds, the exact values of $\cos\frac{2\pi}{5}$ and $\cos\frac{4\pi}{5}$. **2**

End of Task

Sydney Girls High School

Mathematics Faculty

Years 12 HSC Mathematics Extension 2

2014 Task 2

Question	Marker's Comment
1	(a) This question was well done. (b) This question was well done. However a number of students did not answer the question as requested in (iii) which asked for the Cartesian form of z^4. (c) The most likely downfall for this question was the failure by some students to clearly show $a\overline{b} = (x+iy)(u-iv) = ux+vy+i(uy-vx)$. This was expected in order to receive full marks.
2	a) i^{2013} was in last year's paper. This should have been an easy mark. b) iii) This type of question can fallout (if you are lucky or perceptive) or waste a lot of time (if you are unlucky). Most students sensibly moved on if they were getting nowhere. 1 mark was awarded for some sensible working c) i) The description was poorly done and many students simply gave the equation of the locus which is requested in ii)
3	(b) i) The semi-circle is on the right hand side of the origin. ii) The equation should reflect the above. (c) This question was better done by those who shifted the points.
4	a) ii) Many students rushed to give an incorrect range of arg z, when their working was on the right track. iii) Many students gave mod z as the radius of the circle and this is incorrect. b) ii) The sum and product of the roots had to be carefully derived for full marks iii) It was good to see that students were successful in attempting this part of the question when earlier parts did not work for them.

Question 1 Solutions

(a)(i) $A+B=5+i$

(a)(ii)
$$\overline{AB}=\overline{(3+4i)(2-3i)}$$
$$=\overline{6+8i-9i+12}$$
$$=\overline{18-i}$$
$$=18+i$$

(a)(iii)
$$\frac{A}{B}=\frac{3+4i}{2-3i}\times\frac{2+3i}{2+3i}$$
$$=\frac{6+8i+9i-12}{4+9}$$
$$=-\frac{6}{13}+\frac{17}{13}i$$

(a)(iv) let $\sqrt{3+4i}=x+iy$
$$3+4i=x^2+2xyi-y^2$$
Equate real and imaginary parts.
$$x^2-y^2=3$$
$$2xy=4 \qquad y=\frac{2}{x}$$
$$x^2-\frac{4}{x^2}=3$$
$$x^4-3x^2-4=0$$
$$(x^2-4)(x^2+1)=0$$
$$x=\pm 2 \ (x \text{ is real}) \qquad y=\pm 1$$
$$\therefore \sqrt{3+4i}=\pm(2+i)$$

(b)(i)
$$|z|=\sqrt{1^2+\left(\sqrt{3}\right)^2}=2$$
$$\arg(z)=\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)=\frac{\pi}{3}$$

(b)(ii)
$$z=2\,cis\frac{\pi}{3}$$

(b)(iii)
$$z^4=\left(2cis\frac{\pi}{3}\right)^4$$
$$=2^4 cis\frac{4\pi}{3} \qquad \text{by DeMoivre's Theorem}$$
$$=16\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$$
$$=16\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$$
$$=-8-8\sqrt{3}i$$

(c)
$$LHS=|a+b|^2$$
$$=|x+iy+u+iv|^2$$
$$=|x+u+i(y+v)|^2$$
$$=(x+u)^2+(y+v)^2$$
$$RHS=|a|^2+|b|^2+2\,\mathrm{Re}\left(a\overline{b}\right)$$
$$=|x+iy|^2+|u+iv|^2+2\,\mathrm{Re}\big((x+iy)(u-iv)\big)$$
$$=x^2+y^2+u^2+v^2+2\,\mathrm{Re}\big(ux+vy+i(uy-xv)\big)$$
$$=x^2+y^2+u^2+v^2+2(ux+vy)$$
$$=x^2+2ux+u^2+y^2+2vy+v^2$$
$$=(x+u)^2+(y+v)^2$$
i.e. $LHS=RHS$

Question Two

a) Now $i^4 = 1$, $\frac{2014}{4} = 503$ Remainder 2

$\therefore i^{2012} \times i^2 = 1 \times -1$

$= -1$ ①

b) i) $z^3 = 1$, w is a root

$w^3 = 1$

$\therefore w^6 = 1$

$(w^2)^3 = 1$

$w^2 = 1$ or other methods ①

ii) $w^3 = 1$

$\therefore w^3 - 1 = 0$

but $w^3 - 1 = (w-1)(w^2 + w + 1)$

$\therefore (w-1)(w^2 + w + 1) = 0$

$w - 1 = 0$, $\underline{w^2 + w + 1} = 0$ ①

iii) $(1-w)(1-w^2)(1-\frac{1}{w})(1-\frac{1}{w^2})$

$= (w-1)(w^2-1) \left(\frac{w-1}{w}\right)\left(\frac{w^2-1}{w^2}\right)$

$= (w^3 - w^2 - w + 1)\left(\frac{w^3 - w^2 - w + 1}{w^3}\right)$

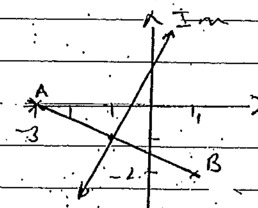
$= (2+1-(1+w+w^2))((2+1-(1+w+w^2))$

$= (3) \times (3)$ ②

$= 9$

c) i) Perpendicular bisector of the interval joining $A(-3, 0)$ and $B(1, -2)$

$|z-(1-2i)| = |z-(-3)|$



②

ii) $\left[\sqrt{(x+3)^2 + (y-0)^2}\right]^2 = \left[\sqrt{(x-1)^2 + (y+2)^2}\right]^2$

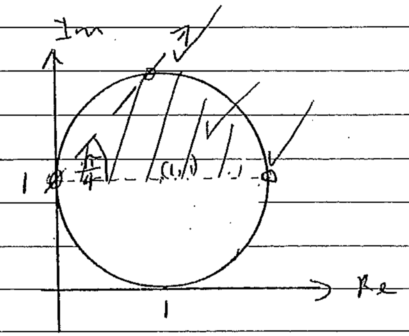
$\cancel{x^2} + 6x + 9 + \cancel{y^2} = \cancel{x^2} - 2x + 1 + \cancel{y^2} + 4y + 4$

$8x - 4y + 4 = 0$

$2x - y + 1 = 0$ ②

or $y = 2x + 1$

d)

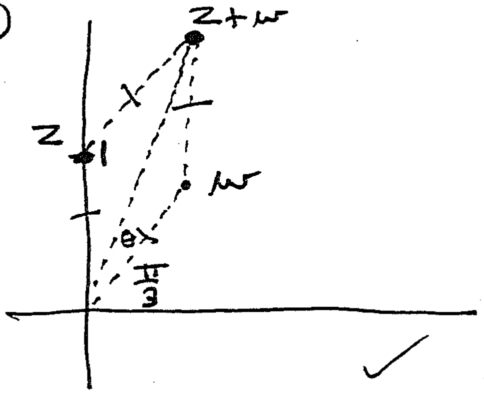


1	2	3	4	5	6
3	4	5			4

3 a) i) $Z = \dfrac{\text{cis}\,\frac{\pi}{4}}{\text{cis}(-\frac{\pi}{4})}$ $\qquad w = \dfrac{2\,\text{cis}\,0}{2\,\text{cis}(-\frac{\pi}{3})}$

$= \text{cis}\,\frac{\pi}{2}$ $\qquad = \text{cis}\,\frac{\pi}{3}$

ii)



iii) $\theta = \frac{1}{2} \times \left(\frac{\pi}{2} - \frac{\pi}{3}\right)$

$= \frac{\pi}{12}$

$\arg(z+w) = \frac{\pi}{12} + \frac{\pi}{3}$

$= \frac{5\pi}{12}$

$z+w = \text{cis}\,\frac{\pi}{2} + \text{cis}\,\frac{\pi}{3}$

$= i + \frac{1}{2} + i \times \frac{\sqrt{3}}{2}$

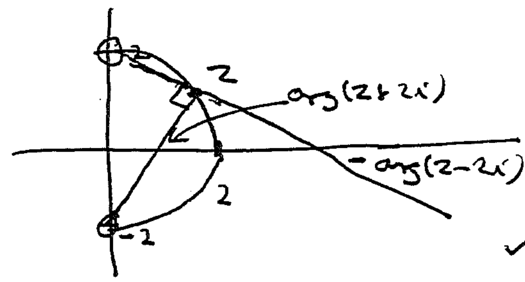
$= \dfrac{1 + i(2+\sqrt{3})}{2}$

$\therefore \tan \frac{5\pi}{12} = \dfrac{2+\sqrt{3}}{1}$

$= 2+\sqrt{3}$

b) i) $\arg(z-2i) - \arg(z+2i) = -\frac{\pi}{2}$

$\frac{\pi}{2} = \arg(z+2i) - \arg(z-2i)$



ii) $x^2 + y^2 = 4, \quad x > 0$

c) i) $z_3 = (z_2 - 2)\sqrt{2}\,\text{cis}\,\frac{\pi}{4} + 2$

$= (3+\sqrt{5}i - 2)\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) + 2$

$= (1+\sqrt{5}i)(1+i) + 2$

$= 1 + i + \sqrt{5}i - \sqrt{5} + 2$

$= 3 - \sqrt{5} + i(1+\sqrt{5})$

ii) $z_4 = i(z_2 - 2) + 2$

$= i(3+\sqrt{5}i - 2) + 2$

$= i - \sqrt{5} + 2$

$= 2 - \sqrt{5} + i$

Mathematics Extension 2
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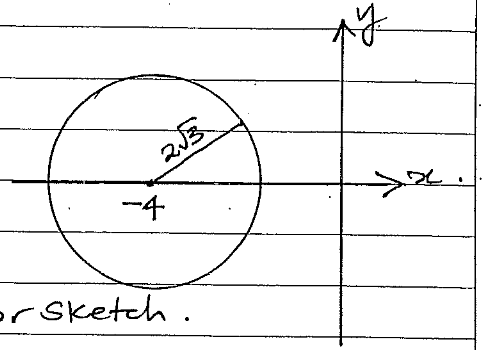
Question 4 — REMARKED!!

a) i) $|z+4| = 2\sqrt{3}$

$|x+iy+4| = 2\sqrt{3}$

$(x+4)^2 + y^2 = (2\sqrt{3})^2$

$(x+4)^2 + y^2 = 12$



* CORRECTION: 1 Mark for sketch.

ii) Max $\arg z = \pi$ 1 Mark.

* DUE TO AMBIGUOUS QUESTION RELATING TO MINIMUM $\arg z$, FULL MARKS WERE ALLOCATED FROM THIS POINT, FOR ALL STUDENTS *.

Min $\arg z \Rightarrow$ 1 Mark ALL STUDENTS.

iii) 2 Marks ALL STUDENTS.

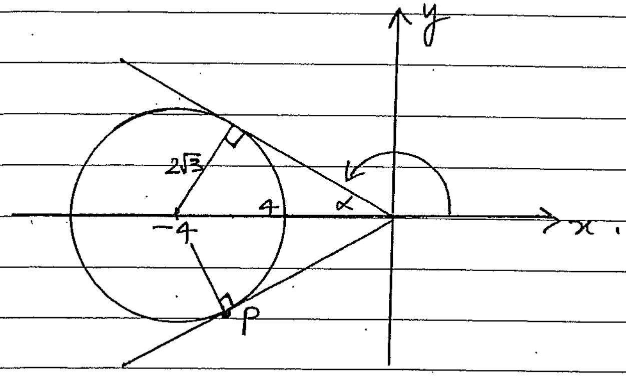
Question 4

a) i) $|z+4| = 2\sqrt{3}$

$|x+iy+4| = 2\sqrt{3}$

$(x+4)^2 + y^2 = (2\sqrt{3})^2$

$(x+4)^2 + y^2 = 12$



2

ii) $\sin\alpha = \frac{2\sqrt{3}}{4}$

$\sin\alpha = \frac{\sqrt{3}}{2} \quad \therefore \alpha = \frac{\pi}{3} \ (60^\circ)$

$\therefore \quad -\frac{120\pi}{180} \le \arg z \le \frac{120\pi}{180}$

$-\frac{2\pi}{3} \le \arg z \le \frac{2\pi}{3}$ 2

iii) At minimum argument, z is at point P.

$P = |z| \operatorname{cis}\left(-\frac{2\pi}{3}\right)$

$= \sqrt{4^2 - (2\sqrt{3})^2}\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$

$= \sqrt{4}\left(-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right)$.

$= -1 - \sqrt{3}\,i$ 1

Question 4

b) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

i) Let $x = \cos\theta$

$\therefore 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta - 1 = 0$

$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 1$

That is: $\cos 5\theta = 1$

$5\theta = 2n\pi, n = 0, 1, 2, 3, 4$

$\theta = \frac{2n\pi}{5}$ (2)

$\therefore$ Solutions are:

$x = \cos 0, \cos\frac{2\pi}{5}, \cos\frac{4\pi}{5}, \cos\frac{6\pi}{5}, \cos\frac{8\pi}{5}$

or $x = 1, \cos\frac{2\pi}{5}, \cos\frac{4\pi}{5}, \cos\left(-\frac{4\pi}{5}\right), \cos\left(-\frac{2\pi}{5}\right)$

or $x = 1, \cos\frac{2\pi}{5}, \cos\frac{4\pi}{5}, \cos\frac{4\pi}{5}, \cos\frac{2\pi}{5}$

ii) Sum of roots:

$a = 16$	$\cos 0 + 2\left(\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5}\right) = 0$
$-b = 0$	$2\left(\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5}\right) = -1$
$c = -20$	$\therefore \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$ (2)
$-d = 0$	
$e = 5$	Product of roots:
$-f = -1$	$\cos 0 \cdot \cos^2\frac{2\pi}{5} \cdot \cos^2\frac{4\pi}{5} = -\left(\frac{-1}{16}\right)$

$\left(\cos\frac{2\pi}{5} \cdot \cos\frac{4\pi}{5}\right)^2 = \frac{1}{16}$

$\cos\frac{2\pi}{5} \cdot \cos\frac{4\pi}{5} = \pm\frac{1}{4}$

But $\cos\frac{2\pi}{5} > 0$ and $\cos\frac{4\pi}{5} < 0$

$\therefore \cos\frac{2\pi}{5} \cdot \cos\frac{4\pi}{5} = -\frac{1}{4}$ (1)

Question 4

b) iii) $x^2 - (\text{sum})x + (\text{product}) = 0$

$x^2 - \left(-\frac{1}{2}\right)x + \left(-\frac{1}{4}\right) = 0$

$4x^2 + 2x - 1 = 0$

$x = \frac{-2 \pm \sqrt{2^2 - 4(4)(-1)}}{8}$

$x = \frac{-2 \pm \sqrt{20}}{8}$

$x = \frac{-1 \pm \sqrt{5}}{4}$

$\cos\frac{2\pi}{5} > 0 \quad \therefore \cos\frac{2\pi}{5} = \frac{-1 + \sqrt{5}}{4}$

$\cos\frac{4\pi}{5} < 0 \quad \therefore \cos\frac{4\pi}{5} = \frac{-1 - \sqrt{5}}{4}$