South Sydney High School

FUNCTIONS

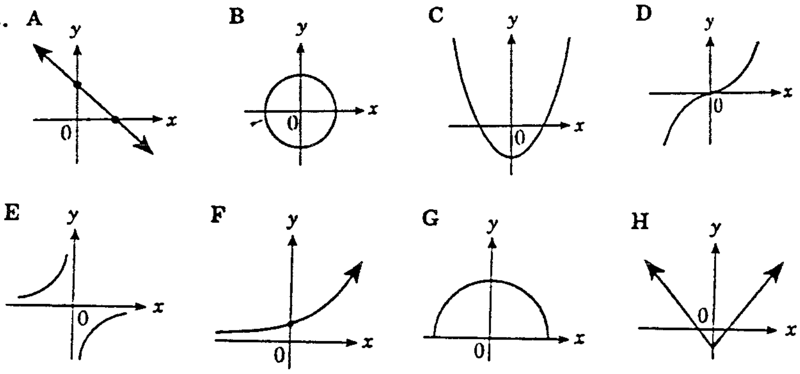
3 Unit Worksheet

EXERCISE 2: EXAMINATION-TYPE QUESTIONS

1. For the function $f(x) = 5 - 2x^2$, find:
   (a) $f(0)$ (b) $f(1) + f(-2)$
   (c) $f(b+1)$ (d) For what values of k is $f(k) = -11$?

2. A function $g(x)$ is given by $g(x) = 2x^2 + 3x - 2$.
   (a) Evaluate: (i) $g(3)$ (ii) $g(-2)$
   (iii) $g(a)$ (iv) $g(x + 1)$

   (b) For what values of x is: (i) $g(x) = 0$? (ii) $g(x) < 2x^2 + 1$?

3. 

   (a) Match each of the above graphs **A** to **H** with its equation.
   (i) $y = 2x^2 - 1$ (ii) $y = 5^x$ (iii) $y = x^3$
   (iv) $y = \frac{-2}{x}$ (v) $y = \sqrt{5 - x^2}$ (vi) $x^2 + y^2 = 4$
   (vii) $2x + y = 3$ (viii) $y = |x - 2|$

   (b) Which of the above graphs *do not* represent functions?

4. Sketch the following functions, showing all critical points.
   State the domain and range in each case.
   (a) $y = \sqrt{4 - x^2}$ (b) $y = \frac{2}{x+1}$ (c) $y = |x + 2|$
   (d) $y = -\sqrt{9 - x^2}$ (e) $y = \sqrt{4 - x}$

5. (a) A function, $g(x)$ is defined by $g(x) = \begin{cases} x+2 & \text{for } x \le -1 \\ 1 & \text{for } -1 < x \le 1 \\ 2 - x & \text{for } x > 1 \end{cases}$

   Evaluate: (i) $g(-4) + 2g\left(\frac{1}{2}\right) - g(5)$ (ii) $g(\alpha)$ if $-1 < \alpha \le 1$

   (b) A function, $f(x)$ is defined by $f(x) = \begin{cases} x^2 & \text{for } -2 \le x \le 1 \\ 4 - 3x & \text{for } 1 < x \le 2 \end{cases}$

   (i) Sketch the function. (ii) Evaluate $f(-1) + f\left(1\frac{1}{2}\right)$
   (iii) State the domain and range of the function.
   (iv) In what domain is the function increasing?

6. A function is defined as $f(x) = \begin{cases} x^2 & \text{for } -2 \le x \le 1 \\ 3 - 2x & \text{for } 1 < x \le 4 \end{cases}$
   (a) Sketch the function $y = f(x)$. (b) Evaluate $f(-1) + f(3)$
   (c) What is the **domain** and **range** of the function?
   (d) In the domain $-2 \le x < 0$ is the function increasing or decreasing?

7. A function $f(x)$ is **even** if $f(-x)=f(x)$ for all values of x in the domain. A function $f(x)$ is **odd** if $f(-x)=-f(x)$ for all values of x in the domain. For each of the following functions state whether they are even, odd, or neither:
(a) $f(x)=2x^2$ (b) $f(x)=x^3-3$ (c) $f(x)=\frac{x^2}{x^2+1}$
(d) $f(x)=2x^3$ (e) $f(x)=\frac{x}{x^2-1}$

8. A function $f(x)$ is defined by $f(x)=\frac{2}{x^2+1}$.
(a) Evaluate $f(0)$.
(b) What value does $y=f(x)$ approach as $x\to\infty$?
(c) Show that $y=f(x)$ is an even function.
(d) Draw a neat sketch of the function $f(x)=\frac{2}{x^2+1}$.

9. (a) Sketch the function $y=x^3$, $0\le x\le 2$, and write down its range in the given domain.
(b) If y in (a) is part of an even function $f(x)$ defined for $-2\le x\le 2$, sketch on a new diagram the function $f(x)$.

10. (a) Sketch the function $y=x^2+1$, $0\le x\le 2$, and write down its range in the given domain.
(b) If y in (a) is part of an even function $g(x)$ defined for $-2\le x\le 2$, sketch on a new diagram the function $g(x)$.

11. On the same graph, sketch the region where the inequalities $x+3y<-6$, $x\le 1$ and $y\ge -3$ hold simultaneously.

12. For the parabola $y=x^2-4x$:
(a) Find the x and y intercepts.
(b) Find the coordinates of the vertex.
(c) Draw a neat sketch, showing all critical points.
(d) State its domain and range.
(e) State if curve represents a function.
(f) Over what domain is it decreasing?
(g) State the range of $y=x^2-4x$ in the domain $1\le x\le 5$.

13. (a) On the same diagram sketch $y=x^2$ and $y=|2x|$.
(b) Show the points of intersection of the two curves on your sketch.
(c) Shade the region satisfied by $y\ge x^2$ and $y\le |2x|$.

14. (a) On same diagram sketch the relationships $y=2x^2$ and $y=2-3x$.
(b) Find where the above curves intersect and label these points on your sketch.
(c) Indicate on your diagram by shading, the region of the number plane determined by those points which satisfy all the inequalities $y<2-3x$, $y\ge 2x^2$ and $x\ge -1$.

15. (a) What is the equation of the locus of a point which moves such that its distance from the origin is 3 units? What does it represent geometrically?
(b) Write down the equation of a circle with centre $(2,-3)$ and radius 5 units.
(c) Draw a clear sketch of the region whose points (x,y) satisfy all three inequalities: $x^2+y^2\le 9$, $y>x$, $x\le 0$

16. A point $P(x, y)$ moves so that it is always 4 units from the point $(1,-2)$. What is the equation of the locus of P?

17. Find the centre and radius of the following circles:
(a) $x^2+(y-2)^2=9$ (b) $(x-2)^2+y^2=4$ (c) $(x+1)^2+(y-2)^2=5$

18. By completing the square, show that $x^2+y^2+2x-6y-1=0$ represents a circle Find its centre and radius.

EXERCISE 2: WORKED SOLUTIONS

1. $f(x) = 5 - 2x^2$

(a) $f(0) = 5 - 2(0)^2 = 5$

(b) $f(1) + f(-2)$
$= (5-2) + (5 - 2 \times 4)$
$= (3) + (-3)$
$= 0$

(c) $f(b+1)$
$= 5 - 2(b+1)^2$
$= 5 - 2(b^2 + 2b + 1)$
$= 5 - 2b^2 - 4b - 2$
$= 3 - 2b^2 - 4b$

(d) $f(k) = -11$
$\Rightarrow 5 - 2k^2 = -11$
$\Rightarrow -2k^2 = -16$
$\Rightarrow k^2 = 8$
$\Rightarrow k = \pm\sqrt{8}$
$\therefore k = \pm 2\sqrt{2}$

2. (a) $g(x) = 2x^2 + 3x - 2$

(i) $g(3) = 2(3^2) + 3(3) - 2$
$= 18 + 9 - 2$
$= 25$

(ii) $g(-2)$
$= 2(-2)^2 + 3(-2) - 2$
$= 8 - 6 - 2$
$= 0$

(iii) $g(a) = 2a^2 + 3a - 2$

(iv) $g(x+1)$
$= 2(x+1)^2 + 3(x+1) - 2$
$= 2(x^2 + 2x + 1) + 3x + 3 - 2$
$= 2x^2 + 4x + 2 + 3x + 3 - 2$
$= 2x^2 + 7x + 3$

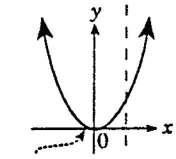
(b) (i) $g(x) = 0$
$\Rightarrow 2x^2 + 3x - 2 = 0$
$\Rightarrow (2x-1)(x+2) = 0$
$\Rightarrow x = \frac{1}{2}$ or -2

(ii) $g(x) < 2x^2 + 1$
$\Rightarrow 2x^2 + 3x - 2 < 2x^2 + 1$
$\Rightarrow 3x < 3$
$\Rightarrow x < 1$

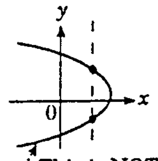
3. (a) (i) C (ii) F (iii) D
(iv) E (v) G (vi) B
(vii) A (viii) H

(b) *Note* A graph of a relationship represents a function if any vertical line only cuts the graph once (and once only).

For example:

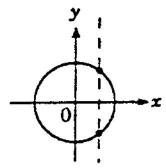


This represents a function

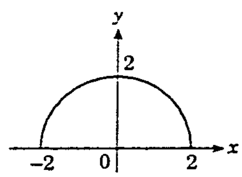


This is NOT a function

$\therefore$ from all the graphs in 3(a), only B doesn't represent a function.



4. (a) $y = \sqrt{4 - x^2}$ represents a semi-circle, above the x axis, of radius 2 units and centre 0.



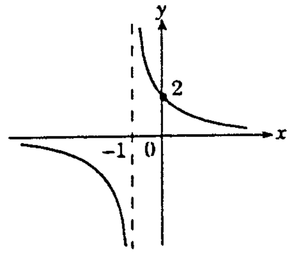
Domain $= \{x : -2 \le x \le 2\}$
Range $= \{y : 0 \le y \le 2\}$

> *Note* The **domain** of a function is the possible x values and the range is the possible y values.

$\therefore$ for $y = \sqrt{4 - x^2}$ there are possible real values for y only when $4 - x^2 \ge 0$, that is, $-2 \le x \le 2$,
$\therefore$ the range for $y = \sqrt{4 - x^2}$ is $-2 \le x \le 2$.

The possible real values for y in the domain $-2 \le x \le 2$ are $0 \le y \le 2$ for $y = \sqrt{4 - x^2}$. This is called the range.
$\therefore$ the domain for $y = \sqrt{4 - x^2}$ is $0 \le y \le 2$.

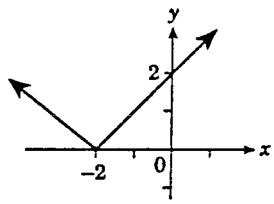
(b) $y = \dfrac{2}{x+1}$ represents a hyperbola with asymptote at $x = -1$.



Domain $= \{x : x \in \mathbf{R},\ x \ne -1\}$
Range $= \{y : y \in \mathbf{R},\ y \ne 0\}$

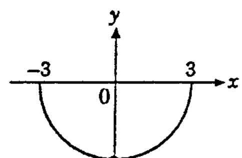
(c) $y = |x+2|$ implies $y = x + 2$ for $x + 2 > 0$ and $y = -(x+2)$ for $x + 2 < 0$, that is,

$$y = \begin{cases} x + 2 & \text{for } x \ge -2 \\ -x - 2 & \text{for } x < -2 \end{cases}$$



Domain $= \{x : x \in \mathbf{R}\}$
Range $= \{y : y \ge 0\}$

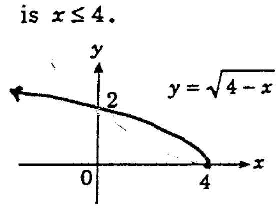
(d) $y = -\sqrt{9 - x^2}$ represents a semi-circle, below the x axis, of radius 3 and centre 0.



Domain $= \{x : -3 \le x \le 3\}$
Range $= \{y : -3 \le y \le 0\}$

(e) $y = \sqrt{4 - x}$

For this function there are possible values for y only when $4 - x \ge 0$, that is $x \le 4$.



Domain $= \{x : x \le 4\}$
Range $= \{y : y \ge 0\}$

5. (a) $g(x)$

$$= \begin{cases} x+2 & \text{for } x \le -1 \\ 1 & \text{for } -1 < x \le 1 \\ 2-x & \text{for } x > 1 \end{cases}$$

(i) $g(-4) + 2g\left(\frac{1}{2}\right) - g(5)$

$= (-4+2) + 2[1] - (2-5)$

$\boxed{x+2}$ $\boxed{2-x}$

$= -2 + 2 - (-3)$
$= 3$

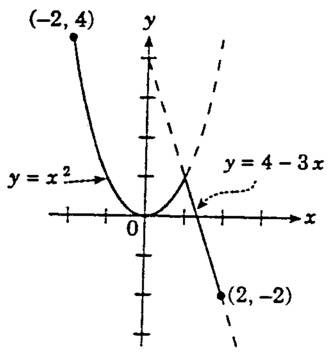
(ii) Now $-1 < \alpha \le 1$

$\therefore\ g(\alpha) = 1$

(b) $f(x)$

$$= \begin{cases} x^2 & \text{for } -2 \le x \le 1 \\ 4-3x & \text{for } 1 < x \le 2 \end{cases}$$

(i) This function has domain $-2 \le x \le 2$ but 2 different relationships, that is $f(x) = x^2$ for $-2 \le x \le 1$ and $f(x) = 4 - 3x$ for $1 < x \le 2$.



(ii) $f(-1) + f\left(1\frac{1}{2}\right)$

$= (-1)^2 + \left(4 - 3 \times 1\frac{1}{2}\right)$

$= 1 - \frac{1}{2} = \frac{1}{2}$

(iii) Domain $= \{x : -2 \le x \le 2\}$
Range $= \{y : -2 \le y \le 4\}$

(iv) *Note* A function is increasing from a to b if as x increases from a to b, $f(x)$ also increases from a to b (where graph rises).

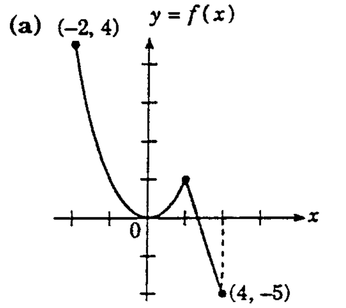
From the graph in (i), the function $f(x)$

$$= \begin{cases} x^2 & \text{for } -2 \le x \le 1 \\ 4-3x & \text{for } 1 < x \le 2 \end{cases}$$

increases in the domain $0 < x < 1$.

6. $f(x) = \begin{cases} x^2 & \text{for } -2 \le x \le 1 \\ 3-2x & \text{for } 1 < x \le 4 \end{cases}$

(a)



(b) $f(-1) + f(3)$
$= (-1)^2 + (3 - 2 \times 3)$
$= 1 - 3$
$= -2$

(c) Domain $= \{x : -2 \le x \le 4\}$
Range $= \{y : -5 \le y \le 4\}$

(d) In the domain $-2 \le x < 0$ the function is **decreasing** because as x increases from -2 to 0, $f(x)$ decreases (from 4 to 0).

7. (a)
$f(x) = 2x^2$
$f(-x) = 2(-x)^2$
$= 2x^2$
$f(x) = f(-x)$
$\therefore$ even function

(b)
$f(x) = x^3 - 3$
$f(-x) = (-x)^3 - 3$
$= -x^3 - 3$
$-f(x) = -[x^3 - 3]$
$= -x^3 + 3$
$\therefore$ neither

(c)
$f(x) = \dfrac{x^2}{x^2+1}$

$f(-x) = \dfrac{(-x)^2}{(-x)^2+1}$

$= \dfrac{x^2}{x^2+1}$

$f(x) = f(-x)$
$\therefore$ even function

(d)
$f(x) = 2x^3$
$f(-x) = 2(-x)^3$
$= -2x^3$
$-f(x) = -(2x^3)$
$= -2x^3$
$f(-x) = -f(x)$
$\therefore$ odd function

(e)
$f(x) = \dfrac{x}{x^2-1}$

$f(-x) = \dfrac{-x}{(-x)^2-1}$

$= \dfrac{-x}{x^2-1}$

$-f(x) = -\left[\dfrac{x}{x^2-1}\right]$

$= \dfrac{-x}{x^2-1}$

$f(-x) = -f(x)$
$\therefore$ odd function

8. $f(x) = \dfrac{2}{x^2+1}$

(a) $f(0) = \dfrac{2}{0^2+1} = 2$

(b) As $x \to \infty$,

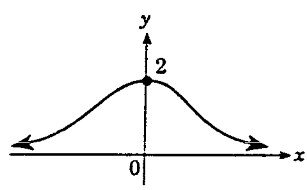
$f(x) = \dfrac{2}{x^2+1} \to 0$

(c) $f(-x) = \dfrac{2}{(-x)^2+1} = \dfrac{2}{x^2+1}$

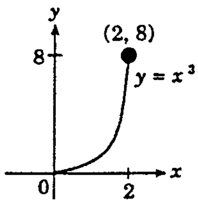
$f(x) = f(-x)$

$\therefore\ y = f(x)$ is even

(d) *Note* $x \to \infty$, $f(x) \to 0$
$\therefore\ f(x) = 0$
is an asymptote for the graph $y = f(x)$.



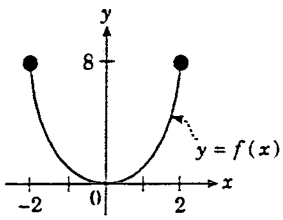
9. (a)



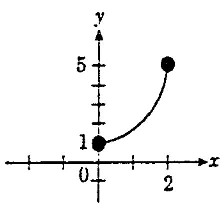
Domain = $\{x : 0 \le x \le 2\}$
Range = $\{y : 0 \le y \le 8\}$

(b) *Note* The graph of an even function is symmetric with respect to reflection in the y axis, that is, it has line symmetry about the y axis.

Thus, to complete the sketch, reflect the curve in (a) about the y axis.



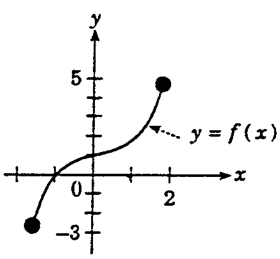
10. (a)



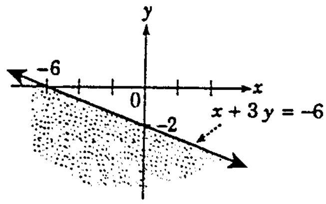
Domain = $\{x : 0 \le x \le 2\}$
Range = $\{y : 1 \le y \le 5\}$

(b) *Note* The graph of an odd function is symmetric with respect to reflection in the point 0 (the origin), that is, it has point symmetry about the origin.

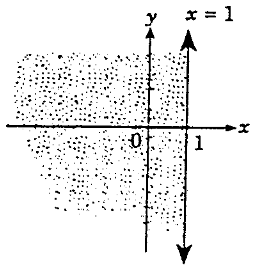
Thus, to complete the sketch, rotate the curve in (a) about 0 by 180°.



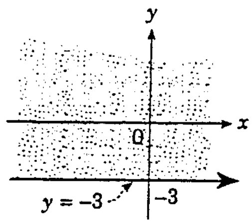
11.



The region $x + 3y < -6$ is the set of all points to the left of but not on the line $x + 3y = -6$.

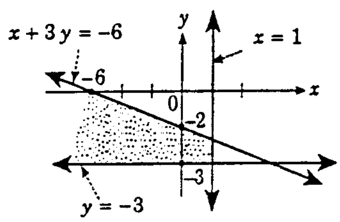


The region $x \le 1$ is the set of all points to the left of and on the line $x = 1$.



The region $y \ge -3$ is the set of all points to the left of and on the line $y = -3$.

The graph of the region where all three inequalities $x + 3y < -6$, $x \le 1$ and $y \ge -3$ hold simultaneously is the intersection of all the above three regions, as shown in the diagram following:



Note
(1) If the inequality sign is < or > then the line or curve is broken: ←- - - - - - →
(2) If the inequality sign is ≤ or ≥ then the line or curve is continuous: ←——→

12. $y = x^2 - 4x$ represents a parabola.

Note Any equation of the form $y = ax^2 + bx + c$, $a \ne 0$ represents a parabola.

(a) ***x* intercept(s)** is the point(s) where the parabola cuts (intersects) the x axis. This occurs when $y = 0$.

$y = x^2 - 4x$
If $y = 0 \Rightarrow x^2 - 4x = 0$
$\Rightarrow x(x-4) = 0$
$\Rightarrow x = 0$ or 4

***y* intercept(s)** is the point(s) where the parabola cuts (intersects) the y axis. This occurs when $x = 0$.

$y = x^2 - 4x$
If $x = 0 \Rightarrow y = 0^2 - 4 \times 0$
$\Rightarrow \quad = 0$

(b) The vertex (or turning point) of any parabola is the point where $x = -\dfrac{b}{2a}$ (on the axis of the parabola).

For $y = x^2 - 4$
$a = 1, \; b = -4, \; c = 0$

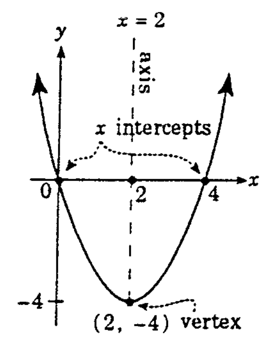
$$\therefore \quad x = -\frac{b}{2a} = -\frac{-4}{2 \times 1} = --2$$

i.e. $x = 2$.

When $x = 2$, $y = x^2 - 4x$
$= 2^2 - 4(2)$
$= 4 - 8$
$= -4$

∴ the vertex is the point $(2, -4)$.

(c)

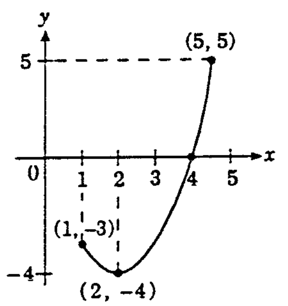


(d) For the parabola
$y = x^2 - 4x$
Domain $= \{x : x \in \mathbf{R}\}$
Range $= \{y : y \geq -4\}$

(e) From the graph of $y = x^2 - 4x$ it can be seen that any vertical line can only cut the graph once. Therefore, $y = x^2 - 4x$ represents a function.

(f) The function $y = x^2 - 4x$ is decreasing in the domain $\{x : x < 2\}$ since $y = f(x)$ decreases as x increases in that interval.

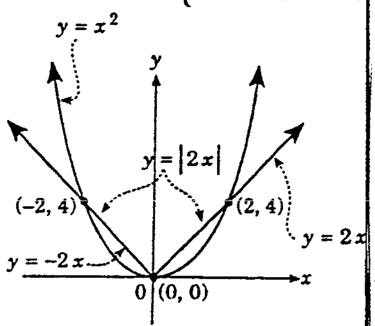
(g)



The range of $f(x) = x^2 - 4x$ in the domain $1 \leq x \leq 5$ is $\{y : -4 \leq y \leq 5\}$ as seen from the above diagram.

13. (a) $y = x^2$ represents a parabola.

$$y = |2x| \Leftrightarrow y = \begin{cases} 2x & \text{for } x \geq 0 \\ -2x & \text{for } x < 0 \end{cases}$$



(b) For $x \geq 0$, $y = 2x$ —①
and $y = x^2$ —②
Substitute $y = x^2$ into ①:

$$x^2 = 2x$$
$$x^2 - 2x = 0$$
$$x(x-2) = 0$$
$$\therefore \quad x = 0 \text{ or } 2$$

But $x \geq 0$, so $x = 0$ and $x = 2$ are both solutions.

Substitute $x = 0$ into ①:
$y = 2 \times 0 = 0$
so (0, 0) is one point of intersection.

Substitute $x = 2$ into ①:
$y = 2 \times 2 = 4$
so (2, 4) is another point of intersection.

For $x < 0$, $y = -2x$ —①
and $y = x^2$ —②

Substitute $y = x^2$ into ①:

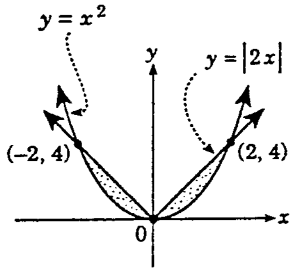
$$x^2 = -2x$$
$$x^2 + 2x = 0$$
$$x(x+2) = 0$$
$$\therefore \quad x = 0 \text{ or } -2$$

But $x < 0$, so $x = -2$ is the required solution.

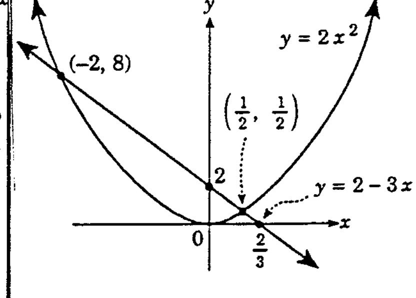
Substitute $x = -2$ into ①:
$y = -2 \times -2 = 4$
so (–2, 4) is is also a point of intersection.

Therefore $y = x^2$ and $y = |2x|$ intersect at three points, namely, (2, 4), (0, 0) and (–2, 4).

(c)



14. (a) The graph of $y = 2x^2$ is a parabola and $y = 2 - 3x$ a straight line.



(b) To find the points of intersection of $y = 2x^2$ and $y = 2 - 3x$, solve the equations simultaneously.

$y = 2x^2$ —①
$y = 2 - 3x$ —②

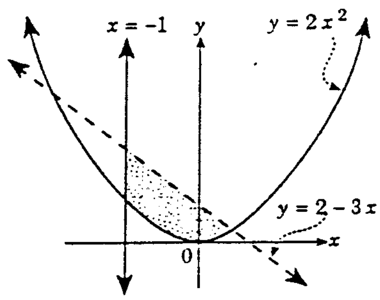
Substitute $y = 2x^2$ into ②:

$$2x^2 = 2 - 3x$$
$$2x^2 + 3x - 2 = 0$$
$$(2x-1)(x+2) = 0$$
$$\therefore \quad x = \frac{1}{2} \text{ or } -2$$

Substitute $x = \frac{1}{2}$ into ②:
$y = 2 - 3 \times \frac{1}{2} = \frac{1}{2}$,
so $\left(\frac{1}{2}, \frac{1}{2}\right)$ is one point of intersection.

Substitute $x = -2$ into ②:
$y = 2 - 3 \times -2 = 8$
so (–2, 8) is the other point of intersection.
$\therefore$ $y = 2x^2$ and $y = 2 - 3x$ intersect at the points $\left(\frac{1}{2}, \frac{1}{2}\right)$ and (–2, 8).

(c) $y < 2 - 3x$, $y \geq 2x^2$ and $x \geq -1$.



15. (a) The equation of the locus of a point which moves so that its distance from the origin is 3 units is

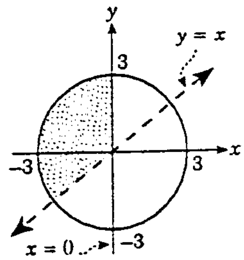
$$x^2 + y^2 = 9.$$

Geometrically it represents a circle, (0, 0) and radius 3 units.

(b) *Note* The general equation of a circle is $(x-a)^2+(y-b)^2=r^2$ where (a, b) is the centre and r is the radius.

Therefore, the equation of a circle with centre (2, –3) and radius of 5 units, is $(x-2)^2+(y+3)^2=25$.

(c) $x^2+y^2=9$ represents a circle, centre (0, 0) and radius 3 units.



16. *Note* The locus of a point which moves so that its distance from a fixed point is a constant represents a circle.

Therefore, if a point moves so that it is always 4 units from the point (1, –2) its equation is
$$(x-1)^2+(y+2)^2=16.$$

17. (a) centre (0, 2), radius = 3 units
(b) centre (2, 0), radius = 2 units
(c) centre (–1, 2), radius = $\sqrt{5}$ units

18. $x^2+y^2+2x-6y-1=0$
$$\underbrace{x^2+2x}+\underbrace{y^2-6y}=1$$

By completing the square on both x^2+2x and y^2-6y:
$$x^2+2x+y^2-6y=1$$
$$x^2+2x+(1)^2+y^2-6y+(-3)^2 = 1+(1)^2+(-3)^2$$
$$\therefore (x+1)^2+(y-3)^2=1+1+9 = 11$$

which represents a circle centre (–1, 3) and radius $\sqrt{11}$ units.