QUESTION 1 (14 marks) **MARKS**

(a) Evaluate $\int_2^3 \frac{x+1}{x^2+2x-6}\,dx$ **2**

(b) Find $\int \frac{2x}{\sqrt{x^2+2}}\,dx$ **2**

(c) Evaluate $\int_{\frac{\pi}{2}}^{\pi} \frac{1}{\sqrt{9-x^2}}\,dx$ **3**

(d) Find $\int \frac{\sin x}{\cos^n x}\,dx$ **2**

(e) Find $\int \frac{x^2-2}{x^2-1}\,dx$ **3**

(f) Find $\int \sin^2 3x\,dx$ **2**

QUESTION 2 : (12 marks)

(a) Find $\frac{d}{du}\left[\log_e\left(u+\sqrt{1+u^2}\right)\right] = \frac{1}{\sqrt{1+u^2}}$. **5**

Hence, or otherwise, evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sqrt{1+\sin^2\theta}}\,d\theta$

(b) Find $\int \frac{3x+4}{x(x+1)}\,dx$ **3**

(c) Using the result : $\cos 2\theta = 2\cos^2\theta - 1$, or otherwise **4**

evaluate $\int_{\frac{\pi}{2}}^{\pi} \cos^4\theta\,d\theta$

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QUESTION 3 (13 marks)

(a) Solve the equation $4|x| = |x-1|$. **3**

On the diagram, sketch the graphs of $y = 4|x|$ and $y = |x-1|$ and hence, or otherwise, solve the inequality

$$4|x| > |x-1|.$$

(b) (i) Sketch the graph of $y = x^2 - 9$ showing the intercepts on the axes. **7**

(ii) Without using calculus, use the graph in (i) to sketch on separate axes the graphs of ;

(α) $y = \frac{1}{x^2-9}$ (β) $y^2 = \frac{1}{x^2-9}$ (α) $y^2 = x^2 - 9$

(c) Prove that the tangent at a point (x_1, y_1) to $xy = c^2$ is given by **3**

$$xy_1 + x_1 y = 2c^2.$$

QUESTION 4 (14 marks)

(a) The complex number $z = \cos\theta + i\sin\theta$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. **8**

Show that $|1+z| = 2\cos\frac{\theta}{2}$ and $\arg(1+z) = \frac{\theta}{2}$.

Hence, or otherwise, show that $\frac{1}{1+z} = \frac{1}{2}\left[1 - i\tan\frac{\theta}{2}\right]$.

Describe the locus of the point representing z and $\frac{1}{1+z}$ in an Argand diagram as θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

(b) By using De Moivre's theorem, or otherwise, show that **6**

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1.$$

Hence find all acute angles θ for which $\cos 4\theta + 2\cos^2\theta = 0$.

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QUESTION 5 (15 marks)

(a) Draw a careful sketch of the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$, showing the vertices the directrices and the asymptotes. Write on your diagram the equations of both directrices and of both asymptotes. **4**

(b) Let $P=(4\sec\theta, 3\tan\theta)$ be any point on this hyperbola. Find the equation of: **3**

(i) the tangent at P (ii) the normal at P.

(c) The tangent and normal at P meet the y–axis at T and N respectively. Show that $T=(0,-3\cot\theta)$ and $N=\left(0,\frac{25}{3}\tan\theta\right)$. **2**

(d) Show that the circle with diameter NT passes through both foci. **4**

(e) Find the points P on the hyperbola which make the diameter of the circle in part (d) minimum. **2**

End of Assessment Task

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n\neq -1;\quad x\neq 0,\ \text{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\quad a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\quad x>0$

1

81%

MOSES YOON.

11

Q1.

(a) $\int_2^3 \frac{x+1}{x^2+2x-6}\,dx = \frac{1}{2}\int_2^3 \frac{2x+2}{x^2+2x-6}\,dx = \frac{1}{2}\left[\ln|x^2+2x-6|\right]_2^3$

$= \frac{1}{2}\left(\ln|3^2+2\cdot 3-6| - \ln|2^2+2\cdot 2-6|\right)$

$= \frac{1}{2}(\ln 9 - \ln 2) = \frac{1}{2}\ln\frac{9}{2}$

(b) Let $u = x^2+2$ $\quad \frac{du}{dx} = 2x$ $\quad \therefore dx = \frac{du}{2x}$

$\therefore = \int \frac{2x}{\sqrt{u}} \times \frac{du}{2x} = \int u^{\frac{1}{2}}\,du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}u^{\frac{3}{2}} = \frac{2}{3}(x^2+2)\sqrt{x} + C$

(c) $= \int_{\frac{\pi}{3}}^{\pi} \frac{dx}{\sqrt{3^2-x^2}} = \left[\sin^{-1}\frac{x}{3}\right]_{\frac{\pi}{2}}^{\pi} = \sin^{-1}\frac{\pi}{3} - \sin^{-1}\frac{\pi}{6}$

$=$

(d) Let $u = \cos x$ $\quad \frac{du}{dx} = -\sin x$ $\quad \therefore dx = -\frac{du}{\sin x}$

$\therefore = \int \frac{\cancel{\sin x}}{u^n} \cdot \frac{du}{(-\cancel{\sin x})} = -\int u^{-n}\,du = -\left(\frac{u^{-n+1}}{-n+1}\right) + c$

$= \frac{(\cos x)^{-n+1}}{n-1} + c$

(e) $\int \frac{x^2-2}{x^2-1}\,dx = \int \frac{(x^2-1)-1}{x^2-1}\,dx = \int 1 - \frac{1}{x^2-1}\,dx$

$\left(\frac{1}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1}\right.$

$\therefore A(x-1) + B(x+1) = 1$

$x=1: \ 2B = 1 \ \therefore B = \frac{1}{2}$

$\left. x=-1: \ -2A = 1 \ \therefore A = -\frac{1}{2}\right)$

$= \int 1 - \left(\frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1}\right)dx$

$= x + \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| + c$

$= \frac{1}{2}\left(\ln\frac{x+1}{x-1}\right) + x + c$

$\left(\because \cos^2 x = \frac{\cos 2x + 1}{2}\right)$

(f) $\int \sin^2 3x\,dx = \int 1 - \cos^2 3x\,dx = \int 1 - \frac{\cos 6x + 1}{2}\,dx$

$= (2)\int 2 - \cos 6x - 1\,dx = 2\int 1 - \cos 6x\,dx = 2\left(x - \frac{\sin 6x}{6}\right) + c$

$= 2x - \frac{\sin 6x}{3} + c$

Q2.

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(a) $\frac{d}{du}\left(\ln\left(u + \sqrt{1+u^2}\right)\right) = \frac{\left(u+\sqrt{1+u^2}\right)'}{u+\sqrt{1+u^2}} = \frac{1 + \frac{2u}{2\sqrt{1+u^2}}}{u+\sqrt{1+u^2}}$

$= \frac{\frac{2\sqrt{1+u^2}+2u}{2\sqrt{1+u^2}}}{u+\sqrt{1+u^2}} = \frac{\left(u+\sqrt{1+u^2}\right)}{\sqrt{1+u^2}\left(u+\sqrt{1+u^2}\right)} = \frac{1}{\sqrt{1+u^2}} = \text{R.H.S}$

Let $u = \sin\theta$ $\quad \therefore \frac{du}{dx} = \cos\theta$ $\quad dx = \frac{du}{\cos\theta}$

$\theta = \frac{\pi}{2} \quad \therefore u = \sin\frac{\pi}{2} = 1, \quad \theta = -\frac{\pi}{2} \quad \therefore u = \sin\left(-\frac{\pi}{2}\right) = -1$

$\therefore \int_{-1}^{1} \frac{\cos\theta}{\sqrt{1+u^2}} \times \frac{du}{\cos\theta} = \left[\ln\left(u+\sqrt{1+u^2}\right)\right]_{-1}^{1}$

$= \ln\left(1+\sqrt{1+1}\right) - \ln\left(-1+\sqrt{1+1}\right)$

$= \ln\left(\frac{1+\sqrt{2}}{\sqrt{2}-1}\right) = \ln\left(\frac{(1+\sqrt{2})(1+\sqrt{2})}{2-1}\right) = \ln\left(1+2+2\sqrt{2}\right) = \ln\left(3+2\sqrt{2}\right)$

(b) $\frac{3x+4}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$ $\quad \therefore A(x+1) + Bx = 3x+4$

$x=0 \quad A=4 \qquad x=-1 \quad -B = 1 \quad \therefore B=-1$

$\therefore = \int \frac{3x+4}{x(x+1)}\,dx = \int \frac{4}{x} + \frac{-1}{x+1}\,dx$

$= 4\ln|x| - \ln|x+1| + c$

(c) $\int_{\frac{\pi}{2}}^{\pi} \cos^4\theta \, d\theta = \int_{\frac{\pi}{2}}^{\pi} \left(\frac{\cos 2\theta + 1}{2}\right)^2 d\theta = \int_{\frac{\pi}{2}}^{\pi} \frac{\cos^2 2\theta + 2\cos 2\theta + 1}{4} d\theta$

$= \frac{1}{4}\int \frac{\cos 4\theta + 1}{2} + 2\cos 2\theta + 1 \, d\theta = \frac{1}{8}\int_{\frac{\pi}{2}}^{\pi} \cos 4\theta + 1 + 4\cos 2\theta + 2 \, d\theta$

$= \frac{1}{8}\left[\frac{\sin 4\theta}{4} + \frac{\cancel{4}^{2}\sin 2\theta}{\cancel{2}} + 3\theta\right]_{\frac{\pi}{2}}^{\pi} = \frac{1}{8}\left[\frac{\sin 4\pi}{4} + 2\sin 2\pi + 3\pi - \left(\frac{\sin 2\pi}{4} + 2\sin \pi + \frac{3\pi}{2}\right)\right]$

$= \frac{1}{8}\left[0 + 0 + 3\pi - \left(0 + 0 + \frac{3}{2}\pi\right)\right] = \frac{1}{8} \cdot \frac{3}{2}\pi = \frac{3}{16}\pi$

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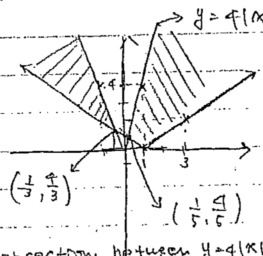
Q3

(a) $(4|x|)^2 = (|x-1|)^2 \Rightarrow 16x^2 = x^2 - 2x + 1 \quad \therefore 15x^2 + 2x - 1 = 0$

$\therefore (5x-1)(3x+1) = 0 \quad \therefore x = \frac{1}{5}, -\frac{1}{3}$



$4|x| > |x-1|$

$\therefore$ Solution : $x < -\frac{1}{3}$, $x > \frac{1}{5}$

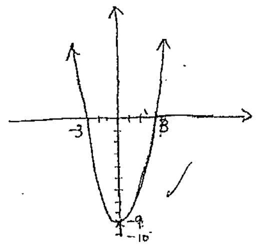
Intersection between $y = 4|x|$ and $y = |x-1|$

① $-4x = -x + 1$
$-3x = 1 \quad \therefore x = -\frac{1}{3} \quad y = \frac{4}{3}$

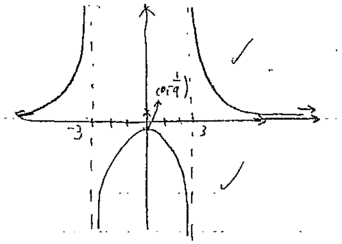
②
$4x = -x + 1$
$5x = 1 \quad \therefore \left(x = \frac{1}{5} \quad y = \frac{4}{5}\right)$

(b)

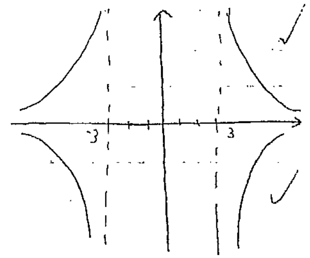
(i)



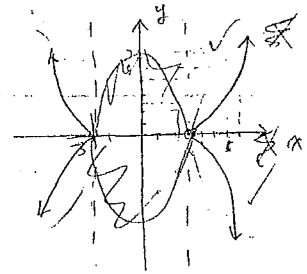
(ii) (a)



(b) $y^2 = \frac{1}{x^2 - 9} \quad \therefore y = \pm\sqrt{\frac{1}{x^2-9}}$



(c) $y^2 = x^2 - 9 \quad \therefore y = \pm\sqrt{x^2 - 9}$



(c) $xy = c^2$

Diff $xy = c^2 \Rightarrow x \cdot \frac{dy}{dx} + y = 0 \quad \therefore \frac{dy}{dx} = -\frac{y}{x}$

$\therefore$ Gradient of tangent $= -\frac{y}{x} = -\frac{y_1}{x_1}$

$\therefore$ Eq : $y - y_1 = -\frac{y_1}{x_1}(x - x_1) \Rightarrow x_1 y - y_1 x_1 = -y_1 x + x_1 y_1$

$x_1 y + x y_1 = x_1 y_1 + x_1 y_1 = 2x_1 y_1 = 2c^2$ (Since P is on $xy = c^2$)

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Q 4.

(a) $z = \cos\theta + i\sin\theta.$

$$z+1 = \cos\theta + i\sin\theta + \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$
$$= (\cos\theta + \cos\frac{\pi}{2}) + i(\sin\theta + \sin\frac{\pi}{2})$$
$$=$$

1 1
1 2 1
1 3 3 1
1 4 6 4 1 → 4

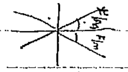
(b)

$$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$$
$$= \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i\sin\theta)^2 + 4\cos\theta\sin^3\theta + (i\sin\theta)^4$$
$$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + i(\dots)$$

$$\therefore \cos 4\theta = \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$
$$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + \cos^4\theta - 2\cos^2\theta + 1$$
$$= 8\cos^4\theta - 8\cos^2\theta + 1$$

$$\cos 4\theta + 2\cos^2\theta = 8\cos^4\theta - 8\cos^2\theta + 1 + 2\cos^2\theta$$
$$= 8\cos^4\theta - 6\cos^2\theta + 1 = 0.$$

$$\therefore (4\cos^2 - 1)(2\cos^2 - 1) = 0$$
$$(2\cos\theta - 1)(2\cos\theta + 1)(\sqrt{2}\cos\theta - 1)(\sqrt{2}\cos\theta + 1) = 0$$
$$\cos\theta = \frac{1}{2}, -\frac{1}{2}, \pm\frac{\sqrt{2}}{2}$$



$$\therefore \theta = \frac{\pi}{3}, \frac{2}{3}\pi, \frac{4}{3}\pi, -\frac{\pi}{3}, \frac{\pi}{4}, \frac{3}{4}\pi, \frac{5}{4}\pi, -\frac{\pi}{4}.$$

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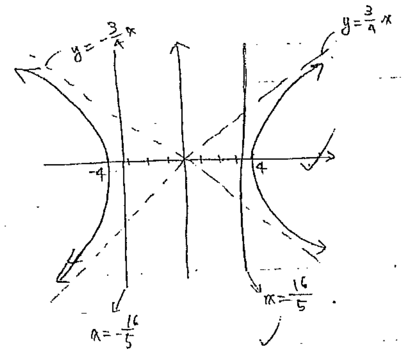
Q5.

(a) $\frac{x^2}{16} - \frac{y^2}{9} = 1$. $a=4$ $b=3$ $b^2 = a^2(e^2-1)$ $9 = 16(e^2-1)$ $\therefore e^2 = \frac{9}{16}+1 = \frac{25}{16}$ $\therefore e = \frac{5}{4}$.

$\therefore$ Directrices $x = \pm\frac{a}{e} = \pm\frac{4}{\frac{5}{4}} = \pm 4 \times \frac{4}{5} = \pm\frac{16}{5}$.

$\therefore$ Asymptotes $y = \pm\frac{b}{a}x = \pm\frac{3}{4}x$



(b) $x = 4\sec\theta$ $x^2 = 16\sec^2\theta = 16(\tan^2\theta+1)$ $y^2 = 9\tan^2\theta$ $\therefore \tan^2\theta = \frac{y^2}{9}$

$\therefore x^2 = 16\left(\frac{y^2}{9}+1\right)$ $\therefore \frac{x^2}{16} - \frac{y^2}{9} = 1$

(i) Diff $\frac{x^2}{16} - \frac{y^2}{9} = 1 \Rightarrow \frac{2x}{16} - \frac{2y}{9}\cdot\frac{dy}{dx} = 0$ $\therefore \frac{dy}{dx} = \frac{2x}{16} \times \frac{9}{2y} = \frac{9x}{16y}$

$\therefore$ m of tangent $= \frac{9 \cdot 4\sec\theta}{16 \cdot 3\tan\theta} = \frac{3\sec\theta}{4\tan\theta}$

$\sec^2 = 1+\tan^2$

$\therefore$ Eq $\Rightarrow y - 3\tan\theta = \frac{3\sec\theta}{4\tan\theta}(x - 4\sec\theta)$

$4\tan\theta\, y - 12\tan^2\theta = 3\sec\theta \cdot x - 12\sec^2\theta$.

$4\tan\theta\, y - 3\sec\theta\, x = 12(\tan^2\theta - \sec^2\theta) = -12$

$\therefore 4y\tan\theta - 3x\sec\theta = -12$

(ii) Gradient of normal $= -\frac{4\tan\theta}{3\sec\theta}$

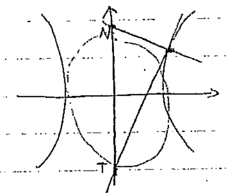
$\therefore$ Eq $y - 3\tan\theta = -\frac{4\tan\theta}{3\sec\theta}(x - 4\sec\theta)$

$3\sec\theta \cdot y - 9\sec\theta \cdot \tan\theta = -4\tan\theta \cdot x + 16\sec\theta \cdot \tan\theta$

$\cos\theta \times (3\sec\theta \cdot y + 4\tan\theta \cdot x) = (25\sec\theta \cdot \tan\theta) \times \cos\theta$

$3y + 4\sin\theta \cdot x = 25\tan\theta$

(c)



Tangent $\Rightarrow 4y\tan\theta - 3x\sec\theta = -12$

$x=0$

~~y intercept~~ $\Rightarrow \cancel{4}y\tan\theta = -\cancel{12}\ -3$

$\therefore y = -\frac{3}{\tan\theta} = -3\cot\theta$

$\therefore T(0, -3\cot\theta)$

Normal $\Rightarrow 3y + 4\sin\theta \cdot x = 13\tan\theta$

$x=0$ $3y = 13\tan\theta$

$y = \frac{25}{3}\tan\theta$

$\therefore N(0, \frac{25}{3}\tan\theta)$

(d) Foci $(\pm ae, 0) \Rightarrow (\pm 5, 0)$

$NT = \frac{25}{3}\tan\theta - (-3\cot\theta) = \frac{25}{3}\tan\theta + 3\cot\theta = 2r$.

$\therefore$ centre $= (0, -3\cot\theta + r)$ $\left(r = \frac{\frac{25}{3}\tan\theta + 3\cot\theta}{2}\right)$

$\therefore$ Eq $x^2 + (y - (-3\cot\theta + r))^2 = r^2$ (r is ...)

$\therefore x = \pm 5$.

Eq $= 25 + (-3\cot\theta + r)^2 = r^2$

$= 25 + 9\cot^2\theta - 6r\cot\theta + \cancel{r^2} = \cancel{r^2}$

$$25 + 9\cot^2\theta - 6\left(\frac{\left(\frac{25}{3}\tan\theta + 3\cot\theta\right)\cot\theta}{\cancel{2}}\right) \checkmark$$

$$= \cancel{25 + 9\cot^2\theta} - \left(\cancel{25 + 9\cot^2\theta}\right) = 0$$

$\therefore$ true

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