South Sydney High School
Conic Sections Worksheet
4 Unit - PAST HSC QUESTIONS

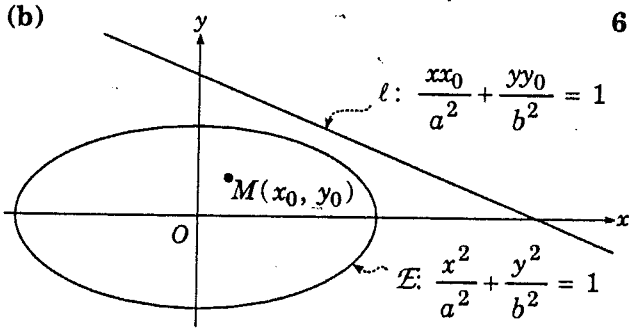
HSC '95

QUESTION EIGHT

(a) Suppose that p and q are real numbers. **1**

Show that $pq \leq \dfrac{p^2+q^2}{2}$.

(b) **6**



The ellipse $\mathcal{E}$ is given by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The point $M(x_0, y_0)$ lies inside $\mathcal{E}$, so that

$$\frac{x_0^{\,2}}{a^2}+\frac{y_0^{\,2}}{b^2}<1.$$

The line ℓ is given by the equation

$$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1.$$

(i) Using the result of part (a), or otherwise, show that the line ℓ lies entirely outside $\mathcal{E}$. That is, show that if $P(x_1, y_1)$ is any point on ℓ, then

$$\frac{x_1^{\,2}}{a^2}+\frac{y_1^{\,2}}{b^2}>1.$$

(ii) The chord of contact to $\mathcal{E}$ from any point $Q(x_2, y_2)$ outside $\mathcal{E}$ has equation

$$\frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1.$$

Show that M lies on the chord of contact to $\mathcal{E}$ from any point on ℓ.

HSC '94

QUESTION THREE

(a) The hyperbola $\mathcal{H}$: $16x^2-9y^2=144$ has foci $S(5,0)$ and $S'(-5,0)$.

The directrices are $x=\dfrac{9}{5}$ and $x=-\dfrac{9}{5}$.

(i) Find the equation of each asymptote of $\mathcal{H}$.

(ii) Sketch $\mathcal{H}$ and indicate on your diagram the foci, directrices, and asymptotes.

(iii) By differentiation, find the gradient of the tangent to $\mathcal{H}$ at $P(3\sec\theta, 4\tan\theta)$.

(iv) Show that the tangent to $\mathcal{H}$ at P has equation $4x=(3\sin\theta)y+12\cos\theta$.

(v) Given that $0<\theta<\dfrac{\pi}{2}$, show that Q, the point of intersection of the tangent to $\mathcal{H}$ at P and the nearer directrix, has y coordinate $\dfrac{12-20\cos\theta}{5\sin\theta}$.

(vi) Calculate the gradients of SP and SQ.

(vii) Determine whether $\angle PSQ$ is a right angle.

HSC '93

QUESTION THREE

(a) Consider the point $P\left(ct, \frac{c}{t}\right)$, where $t \neq \pm 1$, which lies on the rectangular hyperbola $xy = c^2$.

(i) Show that the equation of the tangent to the hyperbola at P is $x + t^2 y = 2ct$.

(ii) Let the tangent to the hyperbola at P intersect the coordinate axes at A and B. Show that $PA = PB$.

(iii) Let the normal to the hyperbola at P meet the axes of symmetry of the hyperbola at C and D. Show that $PC = PD = PA$. [You may assume that the equation of the normal is $t^3 x - ty = c(t^4 - 1)$.]

(iv) Sketch a graph of the hyperbola showing the results proved so far.

(v) Explain why A, B, C, and D must be the vertices of a square.

(b) Let $R(x_0, y_0)$, $P(x_1, y_1)$, and $Q(x_2, y_2)$ be points on the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

(i) If d is the distance between the points R and P, show that

$$-\frac{d^2}{2} = x_0 x_1 + y_0 y_1 + g(x_0 + x_1) + f(y_0 + y_1) + c.$$

(ii) Suppose Q is also distance d from R. Explain why the equation of the chord PQ is

$$-\frac{d^2}{2} = x_0 x + y_0 y + g(x_0 + x) + f(y_0 + y) + c.$$

HSC '92

QUESTION THREE

(a) The ellipse $\mathcal{E}$ has equation

$$\frac{x^2}{100} + \frac{y^2}{75} = 1.$$

(i) Sketch the curve $\mathcal{E}$, showing on your diagram the coordinates of the foci and the equation of each directrix.

(ii) Find the equation of the normal to the ellipse at the point $P(5, 7{\cdot}5)$.

(iii) Find the equation of the circle that is tangential to the ellipse at P and $Q(5, -7{\cdot}5)$.

South Sydney High School
4 Unit - Conic Sections Worksheet
SOLUTIONS

HSC '95

QUESTION EIGHT

(a) $(p-q)^2 \geq 0$ where p and q are real numbers.

$\therefore \quad p^2 - 2pq + q^2 \geq 0$

$\therefore \quad 2pq \leq p^2 + q^2$

$$pq \leq \frac{p^2+q^2}{2}.$$

(b) ***Method 1***:

Suppose $P(x_1, y_1)$ is any point on the line ℓ. Then, using the equation of ℓ,

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1,$$

we see that

$$\frac{x_1 x_0}{a^2} + \frac{y_1 y_0}{b^2} = 1$$

$$= \frac{1}{a^2}(x_1 x_0) + \frac{1}{b^2}(y_1 y_0)$$

Using the result of (a), and rearranging,

$$1 \leq \frac{1}{a^2}\left(\frac{x_1^2 + x_0^2}{2}\right) + \frac{1}{b^2}\left(\frac{y_1^2 + y_0^2}{2}\right)$$

$$= \frac{x_1^2}{2a^2} + \frac{x_0^2}{2a^2} + \frac{y_1^2}{2b^2} + \frac{y_0^2}{2b^2}$$

$$= \frac{1}{2}\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right) + \frac{1}{2}\left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}\right)$$

$$< \frac{1}{2}\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right) + \frac{1}{2},$$

since $\quad \dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} < 1.$

$$\therefore \quad \frac{1}{2} < \frac{1}{2}\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right)$$

$$\therefore \quad 1 < \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}.$$

Hence $P(x_1, y_1)$ lies outside the ellipse, and hence so does ℓ.

Method 2:

To show that ℓ lies outside the ellipse $\mathcal{E}$, it suffices to show that ℓ and $\mathcal{E}$ never meet.

So consider the simultaneous equations:

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1 \qquad \text{—①}$$

and

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{—②}$$

From ①, $\quad \dfrac{x_0 x}{a^2} = 1 - \dfrac{y_0 y}{b^2}$

$$\therefore \quad x = \frac{a^2}{x_0}\left(1 - \frac{y_0 y}{b^2}\right)$$

Substituting in ②:

$$\frac{1}{a^2}\left[\frac{a^2}{x_0}\left(1 - \frac{y_0 y}{b^2}\right)\right]^2 + \frac{y^2}{b^2} = 1$$

$$\therefore \quad \frac{a^2}{x_0}\left(1 - \frac{2y_0 y}{b^2} + \frac{y_0^2 y^2}{b^4}\right) + \frac{y^2}{b^2} = 1$$

$$\therefore \quad a^2\left(b^4 - 2b^2 y_0 y + y_0^2 y^2\right) + x_0^2 b^2 y^2 = x_0^2 b^4$$

$$\therefore \quad y^2\left[a^2 y_0^2 + x_0^2 b^2\right] + y\left[-2a^2 b^2 y_0\right] + a^2 b^4 - x_0^2 b^4 = 0 \qquad \text{—③}$$

For this equation, the discriminant Δ is given by:

$$\Delta = \left[-2a^2 b^2 y_0\right]^2 - 4\left[a^2 y_0^2 + x_0^2 b^2\right] \times \left[a^2 b^4 - x_0^2 b^4\right]$$

$$= 4a^4 b^4 y_0^2 - 4b^4\left(a^2 y_0^2 + x_0^2 b^2\right) \times \left(a^2 - x_0^2\right)$$

$$= 4b^4\left[a^4 y_0^2 - \left(a^2 y_0^2 + x_0^2 b^2\right) \times \left(a^2 - x_0^2\right)\right]$$

$$= 4b^4\left[a^4 y_0^2 - a^4 y_0^2 + a^2 y_0^2 x_0^2 - a^2 b^2 x_0^2 + x_0^4 b^2\right]$$

$$= 4b^4\left[a^2 y_0^2 x_0^2 + x_0^4 b^2 - a^2 x_0^2 b^2\right]$$

$$= 4a^2b^6x_0^{\,2}\left[\frac{y_0^{\,2}}{b^2}+\frac{x_0^{\,2}}{a^2}-1\right],$$

and $\Delta < 0$ since $\dfrac{x_0^{\,2}}{a^2}+\dfrac{y_0^{\,2}}{b^2} < 1$.

This means that the equation ③ has no real roots, so the simultaneous equations ① and ② have no real solutions, that is, ℓ and $\mathcal{E}$ do not meet, as required.

(ii) If $Q(x_2, y_2)$ lies on the line ℓ, then its coordinates satisfy the equation of ℓ,

that is, $\dfrac{x_2x_0}{a^2}+\dfrac{y_2y_0}{b^2}=1.$

If this is true, then it also means that the point (x_0, y_0) lies on the line

$$\frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1.$$

That is, $M(x_0, y_0)$ lies on the chord of contact to $\mathcal{E}$ from any point on ℓ.

Note: In this part, it is essential not to confuse the coordinates of the fixed point $M(x_0, y_0)$, the coordinates of the point $Q(x_2, y_2)$, and the variables x and y, which can take infinitely many values in both the equation of ℓ and the equation of the chords of contact.

HSC '94

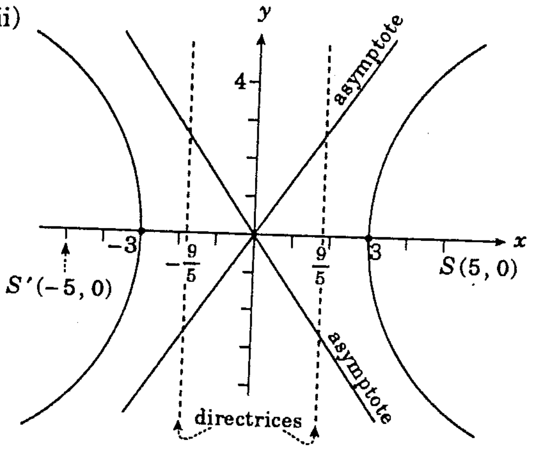
QUESTION THREE

(a) (i) $16x^2-9y^2=144$

$$\frac{x^2}{9}-\frac{y^2}{16}=1.$$

Hence the asymptotes are $y=\pm\frac{4}{3}x$.

(ii)



foci S, S'

(iii) $x = 3\sec\theta$, $y = 4\tan\theta$

$$\frac{dx}{d\theta}=\sec\theta\tan\theta$$

$$\left(\text{OR } 3(\cos\theta)^{-2}\sin\theta = 3\sin\theta\sec^2\theta\right)$$

$$\frac{dy}{d\theta}=4\sec^2\theta$$

$$\therefore\quad \frac{dy}{dx}=\frac{4}{3\sin\theta}.$$

OR

Using implicit differentiation,

$$32x-18y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{16x}{9y}$$

$$=\frac{16\times3\sec\theta}{9\times4\tan\theta}$$

$$=\frac{4}{3\sin\theta}.$$

(iv) Eqn. of tangent at P is:

$$y-4\tan\theta=\frac{4}{3\sin\theta}(x-3\sec\theta)$$

$$(3\sin\theta)y-12\tan\theta\sin\theta = 4x-12\sec\theta$$

$$(3\sin\theta)y-12\sin^2\theta\sec\theta = 4x-12\sec\theta$$

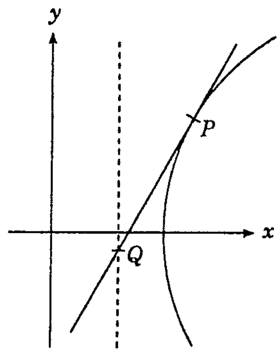
$$4x=(3\sin\theta)y+12\sec\theta(1-\sin^2\theta)$$

$$=(3\sin\theta)y+12\sec\theta\cos^2\theta.$$

That is, eqn. of tangent is

$$4x=(3\sin\theta)y+12\cos\theta.$$

(v)



For $0<\theta<\frac{\pi}{2}$, P is in the first quadrant, so the nearer directrix is $x=\frac{9}{5}$.

Substituting: $4\times\frac{9}{5}=(3\sin\theta)y+12\cos\theta$

$$y=\frac{\frac{36}{5}-12\cos\theta}{3\sin\theta}$$

$$=\frac{12-20\cos\theta}{5\sin\theta}.$$

(vi) $m_{SP}=\dfrac{4\tan\theta-0}{3\sec\theta-5}$

$$=\frac{4\sin\theta}{3-5\cos\theta}.$$

$$m_{SQ}=\frac{\dfrac{12-20\cos\theta}{5\sin\theta}-0}{\dfrac{9}{5}-5}$$

$$=\frac{12-20\cos\theta}{9\sin\theta-25\sin\theta}$$

$$=\frac{12-20\cos\theta}{-16\sin\theta}$$

$$=\frac{3-5\cos\theta}{-4\sin\theta}.$$

(vii) $m_{SP}\cdot m_{SQ}=\left(\dfrac{4\sin\theta}{3-5\cos\theta}\right)\left(\dfrac{3-5\cos\theta}{-4\sin\theta}\right)$

$=-1.$

$\therefore\ \angle PSQ$ is a right angle.

HSC '93

QUESTION THREE

(a) (i) $x=ct \qquad y=\dfrac{c}{t}$

$\dfrac{dx}{dt}=c \qquad \dfrac{dy}{dt}=-\dfrac{c}{t^2}$

$\therefore\ \dfrac{dy}{dx}=-\dfrac{1}{t^2}.$

At P, $\dfrac{dy}{dx}=-\dfrac{1}{t^2}$,

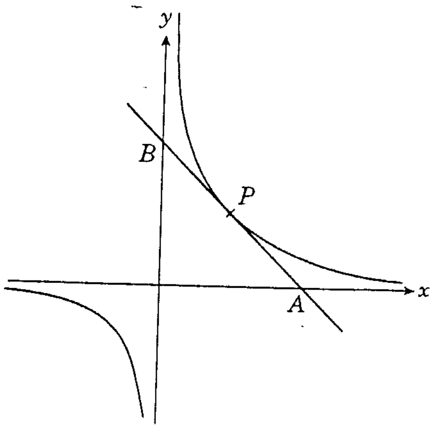
$\therefore$ Eqn. of tangent at P is

$$\left(y-\frac{c}{t}\right)=-\frac{1}{t^2}(x-ct)$$

$$t^2y-ct=-x+ct$$

$$x+t^2y=2ct.$$

(ii)



At A, $y=0$, $\therefore\ x=2ct.$

At B, $x=0$, $\therefore\ t^2y=2ct$

$y=\dfrac{2c}{t}.$

$\therefore\ PA^2=(ct-2ct)^2+\left(\dfrac{c}{t}-0\right)^2$

$=c^2t^2+\dfrac{c^2}{t^2}.$

$PB^2=(ct-0)^2+\left(\dfrac{c}{t}-\dfrac{2c}{t}\right)^2$

$=c^2t^2+\dfrac{c^2}{t^2},$

$\therefore\ PA=PB.$

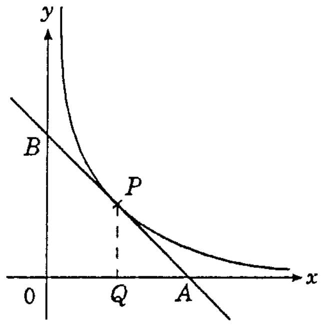
OR

The abscissae of B, P, A are $O, ct, 2ct$,

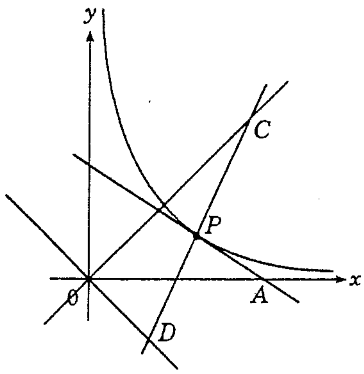
$\therefore\ AQ=QO,$

so $AP=PB.$

[Equal Intercept Theorem]



(iii)



Eqn. of PC is $t^3x-ty=c(t^4-1)$ [given]

C lies on $y=x$,

$\therefore\ t^3x-tx=c(t^4-1)$

$$x=\frac{c(t^4-1)}{t(t^2-1)}$$

$$=\frac{c(t^2-1)}{t}$$

$\therefore\ c=\left(\dfrac{c(t^2+1)}{t},\ \dfrac{c(t^2+1)}{t}\right).$

D lies on $y=-x$,

$\therefore\ t^3x+tx=c(t^4-1)$

$$x=\frac{c(t^2-1)}{t}$$

$\therefore\ D=\left(\dfrac{c(t^2-1)}{t},\ \dfrac{c(t^2-1)}{t}\right),$

$$\therefore\ PC^2 = \left[ct - \frac{c(t^2+1)}{t}\right]^2 + \left[\frac{c}{t} - \frac{c(t^2+1)}{t}\right]^2$$

$$= \left[\frac{ct^2 - ct^2 - c)}{t}\right]^2 + \left[\frac{c - ct^2 - c)}{t}\right]^2$$

$$= \frac{c^2}{t^2} + c^2t^2.$$

$$PD^2 = \left[ct - \frac{c(t^2-1)}{t}\right]^2 + \left[\frac{c}{t} - \frac{c(t^2-1)}{t}\right]^2$$

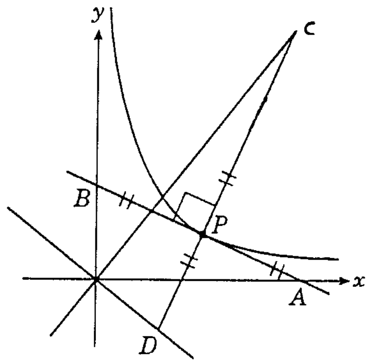
$$= \left[\frac{ct^2 - ct^2 + c)}{t}\right]^2 + \left[\frac{c - ct^2 + c)}{t}\right]^2$$

$$= \frac{c^2}{t^2} + c^2t^2,$$

$$\therefore\ PA^2 = c^2t^2 + \frac{c^2}{t^2},$$

$$\therefore\ PC = PD = PA.$$

(iv)



(v) The diagonals AB, CD are equal and bisect each other at right angles. Hence $ABCD$ is a square.

(b) (i) $d^2 = (x_0 - x_1)^2 + (y_0 - y_1)^2$.

Since $R(x_0, y_0)$ and $P(x_1, y_1)$ both lie on the circle,

$$x_0^2 + y_0^2 + 2gx_0 + 2fy_0 + c = 0 \qquad —①$$

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0. \qquad —②$$

Adding and rearranging ① and ②:

$$x_0^2 + x_1^2 + y_0^2 + y_1^2 = -2g(x_0 + x_1) - 2f(y_0 + y_1) - 2c,$$

$$\therefore\ (x_0 - x_1)^2 + (y_0 - y_1)^2 = -2g(x_0 + x_1) - 2f(y_0 + y_1) - 2c - 2x_0x_1 - 2y_0y_1,$$

i.e. $d^2 = -2g(x_0 + x_1) - 2f(y_0 + y_1) - 2c - 2x_0x_1 - 2y_0y_1$

$$\therefore\ \frac{-d^2}{2} = x_0x_1 + y_0y_1 + g(x_0 + x_1) + f(y_0 + y_1) + c.$$

(ii) It follows from (i) that $P(x_1, y_1)$ lies on the line

$$\frac{-d^2}{2} = x_0x + y_0y + g(x_0 + x) + f(y_0 + y) + c. \qquad —③$$

If Q is also distant d from R, then

$$\frac{-d^2}{2} = x_0x_2 + y_0y_2 + g(x_0 + x_2) + f(y_0 + y_2) + c,$$

and hence $Q(x_2, y_2)$ also lies on the line ③,
$\therefore$ ③ is the eqn. of the chord PQ.

HSC '92

QUESTION THREE

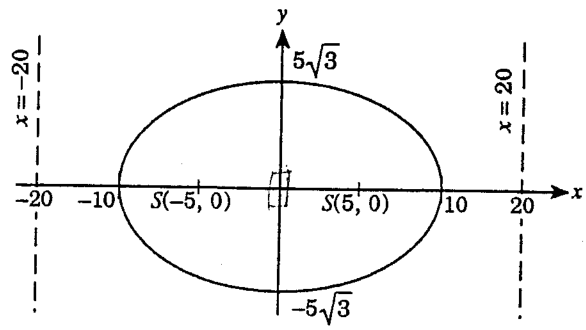
(a)(i) $\dfrac{x^2}{100} + \dfrac{y^2}{75} = 1$

$\therefore\ a = 10, \quad b = \sqrt{75} = 5\sqrt{3}$

and $b^2 = a^2(1 - e^2)$

$75 = 100(1 - e^2)$

$e = \dfrac{1}{2}$



$\therefore$ Coordinates of foci are

$(\pm ae,\ 0) = (\pm 5,\ 0)$.

Equations of directrices are

$x = \pm a / e = \pm 20$.

(ii) Differentiating w.r.t. x:

$$\frac{2x}{100} + \frac{2y}{75}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{50}\cdot\frac{75}{2y} = -\frac{3x}{4y}$$

At $P(5,\ 7{\cdot}5)$, $\dfrac{dy}{dx} = -\dfrac{15}{30} = -\dfrac{1}{2}$

$\therefore$ The gradient of the normal is 2.
Equation of normal at P is

$y - 7{\cdot}5 = 2(x - 5)$

$y = 2x - 2{\cdot}5$

or $4x - 2y - 5 = 0 \qquad \ldots (1)$

(iii) By symmetry, the intersection of the normals at P and Q will lie on the x-axis. The centre of the required circle is the point of intersection of these normals.

Putting $y = 0$ in (1) gives $x = \frac{5}{4}$.

Let C be the point $\left(\frac{5}{4},\ 0\right)$.

$$CP^2 = \left(5 - \frac{5}{4}\right)^2 + (7{\cdot}5 - 0)^2$$
$$= \left(\frac{15}{4}\right)^2 + \left(\frac{15}{2}\right)^2$$
$$= \frac{225}{16} + \frac{225}{4}$$
$$= \frac{1125}{16}$$

$\therefore$ Equation of circle is

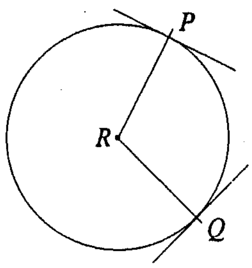
$$\left(x - \frac{5}{4}\right)^2 + y^2 = \frac{1125}{16}$$

Alternative solution to (iii):

$P(5,\ 7{\cdot}5)$; $Q(5,\ -7{\cdot}5)$

are on the ellipse ξ.

The normals meet at R, where R is the centre of the circle.



Normal at Q: $(y - -7{\cdot}5) = -2(x - 5)$ *

$$y + 7{\cdot}5 = -2x + 10$$
$$y = -2x + 2{\cdot}5$$

Normal at P: $y = 2x - 2{\cdot}5$

Solving: $R(1{\cdot}25,\ 0)$

* Normal at P has grad. 2.

$\therefore$ By symmetry normal at Q has grad. -2.

$$\text{Radius} = PR = \sqrt{(5 - 1{\cdot}25)^2 + (7{\cdot}5 - 0)^2}$$
$$= \sqrt{\left\{\left(3\frac{3}{4}\right)^2 + \left(7\frac{1}{2}\right)^2\right\}}$$
$$= \sqrt{\frac{225}{16} + \frac{225}{4}} = \frac{15\sqrt{5}}{4}.$$

Circle: $\left(x - 1\frac{1}{4}\right)^2 + y^2 = \left(\frac{15\sqrt{5}}{4}\right)^2$

$$\frac{1}{16}(4x - 5)^2 + y^2 = \frac{1}{16} \times 1125$$
$$(4x - 5)^2 + 16y^2 = 1125.$$
$$16x^2 + 16y^2 - 40x = 1100$$
$$4x^2 + 4y^2 - 10x = 275.$$