# South Sydney High School
# EXPONENTIAL RATES OF CHANGE
# 2/3 Unit Worksheet

1. $$P = P_0 e^{kt},$$
where $P$ and $t$ are variables and $P_0$ and $k$ are constants.

(a) Prove that $\dfrac{dP}{dt} = kP$

(b) If, when $t = 0$, $P = 100$, find the value of $P_0$

(c) If also, $P = 120$ when $t = 2$, prove that
$$e^{2k} = 1.2$$
The value of $k$ determined by this equation is called the growth rate of the function. Show that the growth rate in this case is approximately 9.1%.

(d) By writing $e^{5k}$ as $(e^{2k})^{5/2}$, or otherwise, find $P$ to the nearest integer when $t = 5$.

(e) Show that, when $P = 300$, $t = \dfrac{\ln 3}{k}$ and evaluate this to the nearest integer.

(f) Draw a graph of $P = P_0 e^{kt}$ for $0 \le t \le 12$.

(g) Find, correct to the nearest integer, the values of $\dfrac{dP}{dt}$ when
(i) $t = 0$ (ii) $P = 250$
and illustrate these results on your graph.

2. The population of an urban area is increasing at a rate proportional to the population.
i.e., $\dfrac{dP}{dt} = kP$, where $k$ is a constant.

(a) Verify that this condition is satisfied by the function
$P = P_0 e^{kt}$, where $P_0$ is a constant.

(b) At the beginning of 1980 the population was 15 000 and, at the beginning of 1990, 21 000.Estimate

(i) The population at the beginning of 1995 to the nearest 50

(ii) the year in which the population can be expected to reach 30 000.

3. The mass of a radio-active element present in a substance decreases at a rate proportional to the mass present.
i.e. $\dfrac{dM}{dt} = kM$, where $k$ is a constant.

(a) Write a relation between $M$ and $t$

(b) Measurements show that, in 1 year, the mass of the element has fallen from 10g to 8.5g. Estimate how long it will take for the 10g to reduce to 5g.

(c) Show further that the same time will be required for any given mass of this element to be reduced by half. (This is the *half-life* period of the element)

(d) Sketch a graph of $M$ (in g) against $t$ (in years) for $0 \le t \le 5$

4. The number of bacteria, $N$, in a colony is increasing at a rate proportional to the number present.

$$\text{i.e. } \frac{dN}{dt} = kN, \text{ where } k \text{ is constant.}$$

Initially, the number present was $2 \times 10^6$ and, after 3 days, $3.2 \times 10^6$. Find

(a) the number present after a further 4 days, correct to 2 significant figures.

(b) the number of days, correct to 1 decimal place, for the number in such a colony to double.

5. A radio-active element has a half-life period of 10 days. Find, to the nearest day where necessary, the time taken for a given mass of this element to be reduced to (a) 25% (b) 10% of that mass.

6. The population of a town is assumed to be increasing at a rate proportional to the population. At the end of 1980 it was 7 500 and, at the end of 1985, 8 100. Estimate

(a) the population at the end of 1995

(b) the rate at which the population will be increasing at the end of 1995.

7. From the time when the engines of a ship are cut off, its acceleration is proportional to its velocity.

$$\text{i.e. } \frac{dv}{dt} = kv$$

The engines are cut at a velocity of 8 $ms^{-1}$ and, after 6 s, the velocity is 5 $ms^{-1}$. Find

(a) the velocity 12 seconds after the engines are cut.

(b) the acceleration, correct to 2 decimal places, when the velocity is 4 $ms^{-1}$

(c) the velocity, correct to 2 decimal places, when the acceleration is $-0.5$ $ms^{-2}$.

8. The mass, M g, of a radio-active element present in a substance after $t$ years is given by

$$M = M_o e^{kt}, \text{ where } M_o \text{ and } k \text{ are constants.}$$

Initially, the mass was 25 g and, at the time, it was known to be decreasing at 0.2 g/year (i.e. $\frac{dM}{dt} = -0.2$)

(a) Find the values of $M_o$ and $k$

(b) Estimate the mass present after 10 years, to the nearest 0.1 g.

(c) Estimate the half-life period of the element to the nearest 0.1 year.

† 9. Two functions, P($t$) and Q($t$), measured from the same time origin, are changing so that

$$P'(t) = k_1 P \quad \text{and} \quad Q'(t) = k_2 Q$$

Also P(0) = 15 and Q(0) = 5.

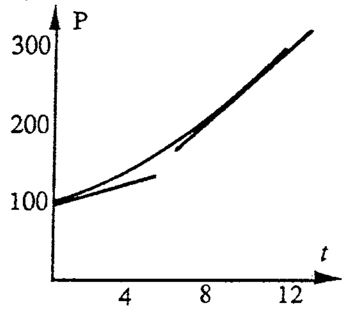
(a) Prove that, when P($t$) = Q($t$),

$$e^{(k_2 - k_1)t} = 3$$

and find the value of $t$, correct to 1 decimal place, if $k_1 = -0.11$ and $k_2 = 0.09$.

(b) Illustrate these results by drawing, on the same number plane, graphs of P($t$) and Q($t$) for $0 \le t \le 6$

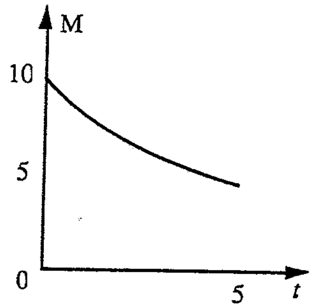
1. (b) 100 (d) 158 (e) 12
(f)




(g) (i) 9 (ii) 23

2. (b) (i) 24850 (ii) 2000

3. (a) $M = M_0e^{kt}$ (b) Approx 4.27 years
(d)




4. (a) $6.0 \times 10^6$ (b) 4.4 days 5. (a) 20 (b) 33

6. (a) 9450 (b) 145 persons per year

7. (a) 3.125 $ms^{-1}$ (b) – 0.31 $ms^{-2}$ (c) 6.38 $ms^{-1}$

8. (a) $M_0 = 25$, $k = -0.008$ (b) 23.1 g (c) 86.6 yrs

9. (a) 5.5
(b)