South Sydney High School
EXTENSION I MATHEMATICS
2008
HSC Assessment Task 2

Name:____________________ **Teacher:**____________________

Question 1:

(a) Solve the equation $2\ln(3x+1) - \ln(x+1) = \ln(7x+4)$

(b) Evaluate

(i) $\int_0^1 \frac{2x}{x^2+1}\,dx$

(ii) $\int_0^{\pi} \sin^2 x\,dx$

(iii) $\int \frac{2x}{\sqrt{x^2-4}}\,dx$ Use $u = x^2 - 4$

(c) If $y = \cos(\ln x)$ find:

i. $\frac{dy}{dx}$

ii. $\frac{d^2y}{dx^2}$

(d) Differentiate $y = \frac{x\sqrt{x^2+1}}{x-1}$ with respect to x.

Question 2:

(a)

(i) Draw a neat sketch showing the graphs of $y = e^{2x}$ and $y = e^x + 2$ on the same diagram.

(ii) Find the coordinates of the point(s) of intersection of $y = e^{2x}$ and $y = e^x + 2$.

(ii) Find the area bounded by the y-axis and the two curves $y = e^{2x}$ and $y = e^x + 2$. Give your final answer correct to 2 decimal places.

(b)

(i) Prove that $8\cos^4 x = 3 + 4\cos 2x + \cos 4x$.

(ii) Sketch, on the same diagram, the curves $y = \cos x$, $y = \cos^2 x$, for $0 \le x \le \frac{\pi}{2}$.

(iii) Find the area enclosed between these curves

(iv) Find the volume generated when this area is rotated about the x-axis.

(c) Ordinates are drawn from $P(1, 0)$ and $Q(2, 0)$ to intersect the curve $y = \frac{1}{x}$ at R and S respectively.

(i) Find the exact value for the area bounded by the curve $y = \frac{1}{x}$, the x axis and the lines PR and QS.

(ii) Use the trapezoidal rule with one interval to find a rational approximation for this area.

(iii) A tangent is drawn to touch the curve $y = \frac{1}{x}$ at the point T on it where $x = 1\frac{1}{2}$. This tangent cuts PR and QS at L and M respectively. Find the area of the trapezium $PQML$.

(iv) Hence show that $\frac{2}{3} < \log_e 2 < \frac{3}{4}$.

Question 3:

(a) How many arrangements of the letters of the word *HOCKEYROO* are possible?

(b) Annie is to celebrate her *18*th birthday by having a dinner party for herself and *11* other people. Annie is to sit at the head of the table.

(i) In how may ways can the people be seated round the table?

(ii) If there are six men and six women at the party, and Annie decides to seat the men and women alternately, in how many ways can this be done?

(c) Five travellers arrive in a town where there are five hotels.

(i) How many different accommodation arrangements are there if there are no restrictions on where the travellers stay?

(ii) How many different accommodation arrangements are there if each traveller stays at a different hotel?

(iii) Suppose two of the travellers are husband and wife and must go to the same hotel. How many different accommodation arrangements are there if the other three can go to any of the other hotels?

(d) The diagram above shows the circle $x^2 + y^2 = 36$. The point $S(x, y)$ lies on the circle in the first quadrant. O is the origin, $R(4, 0)$ lies on the x-axis and $T(0, 2)$ lies on the y-axis. The size of $\angle ROS$ is α radians, where $0 < \alpha < \dfrac{\pi}{2}$.

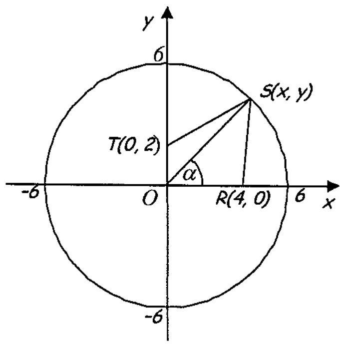
(i) Show that the area of triangle SOR is $12\sin\alpha$.

(ii) Hence show that the area, A, of the quadrilateral $ORST$ is given by:

$$A = 6\cos\alpha\,(2\tan\alpha + 1)$$

(iii) Find the value of $\tan\alpha$ for which the area A is a maximum.

(iii) Hence, show that for this maximum area, the coordinates of point S are $\left(\dfrac{6}{5}\sqrt{5}, \dfrac{12}{5}\sqrt{5}\right)$



STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a \neq 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln(x+\sqrt{x^2-a^2}),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln(x+\sqrt{x^2+a^2})$$

NOTE: $\ln x = \log_e x,\ x > 0$

EXTENSION I HSC ASS II
SOLUTIONS (S.S.H.S)

QUESTION 1:

(a) $2\ln(3x+1) - \ln(x+1) = \ln(7x+4)$

$\ln(3x+1)^2 - \ln(x+1) = \ln(7x+4)$

ie $\ln\left[\frac{(3x+1)^2}{(x+1)}\right] = \ln(7x+4)$

ie $\frac{(3x+1)^2}{(x+1)} = (7x+4)$

$9x^2+6x+1 = 7x^2+11x+4$

$2x^2-5x-3=0$ $P=-6$, $S=-5$, $F=-6,+1$

$\frac{(2x-6)(2x+1)}{2}=0$

$(x-3)(2x+1)=0$

$\therefore x=3$ or $x=-\frac{1}{2}$ NOT POSSIBLE

$\therefore x=3$ #

(b) (i) $\int_0^1 \frac{2x}{x^2+1}\,dx = \left[\ln(x^2+1)\right]_0^1 = \ln 2 - \ln 1 = \ln 2$ #

(ii) $\int_0^\pi \sin^2 x\,dx = \frac{1}{2}\int_0^\pi 1-\cos 2x\,dx$

$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^\pi$

$= \frac{1}{2}\left[(\pi - \frac{1}{2}\sin 2\pi) - (0 - \frac{1}{2}\sin 0)\right]$

$= \frac{\pi}{2}$ #

(iii) $\int \frac{2x}{(x^2-4)^{\frac{1}{2}}}\,dx$

$u = x^2-4$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$\therefore I = \int \frac{du}{u^{\frac{1}{2}}} = \int u^{-\frac{1}{2}}\,du = +2u^{\frac{1}{2}} + c$

$= +2\sqrt{x^2-4} + c$ #

(c) $y = \cos(\ln x)$

(i) $\frac{dy}{dx} = -\sin(\ln x) \times \frac{1}{x}$

$= \frac{-\sin(\ln x)}{x}$ #

(ii) $\frac{dy}{dx} = \frac{-\sin(\ln x)}{x} \Rightarrow -\frac{1}{x}\,[\sin \ln(x)]$

f g

$f = -x^{-1}$ $g = \sin(\ln x)$

$f' = x^{-2}$ $g' = \frac{1}{x}\cos(\ln x)$

$\frac{d^2y}{dx^2} = f'g+g'f = x^{-2}\sin(\ln x) - \frac{1}{x^2}\cos(\ln x)$

$= \frac{\sin(\ln x) - \cos(\ln x)}{x^2}$

(d) $y = \frac{x\sqrt{x^2+1}}{x-1}$

$y = \frac{(x^4+x^2)^{\frac{1}{2}}}{x-1}$ $\to f$, $\to g$

$f' = \frac{1}{2}(x^4+x^2)^{-\frac{1}{2}} \times (4x^3+2x)$

$= \frac{\cancel{2x}(2x^2+1)}{\cancel{2x}\sqrt{x^2+1}}$

$g' = 1$

$\Rightarrow \frac{dy}{dx} = \frac{f'g - g'f}{g^2}$

$= \frac{\frac{(2x^2+1)(x-1)}{\sqrt{x^2+1}} - x\sqrt{x^2+1}}{(x-1)^2}$

$= \frac{2x^3-2x^2+x-1 - x(x^2+1)}{(\sqrt{x^2+1})(x-1)^2}$

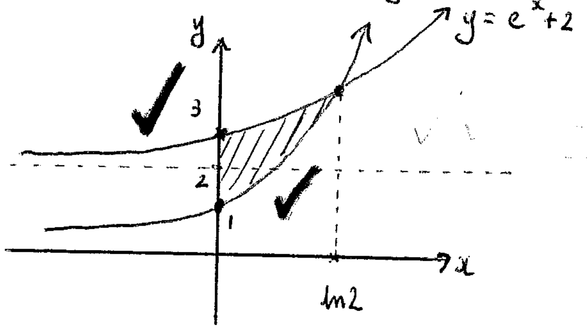
$= \frac{2x^3-2x^2+x-1-x^3-x}{(\sqrt{x^2+1})(x-1)^2}$

$= \frac{x^3-2x^2-1}{(\sqrt{x^2+1})(x-1)^2}$

QUESTION 2:

(a) $y = e^{2x}$ $\quad y = e^x + 2$

(i)



NOT TO SCALE

(ii) $y = e^{2x}$ & $y = e^x + 2$

$\therefore e^{2x} = e^x + 2$

$e^{2x} - e^x - 2 = 0$

$(e^x - 2)(e^x + 1) = 0$

$\therefore e^x = 2$ or $e^x = -1$ ×

$\therefore e^x = 2$

$\therefore x = \ln_e 2$ #

$y = 4$

$(\ln 2, 4)$

(iii) $A = \int_0^{\ln 2} e^x + 2 - (e^{2x})\, dx$

$= \left[e^x + 2x - \frac{1}{2}e^{2x}\right]_0^{\ln 2}$

$= \left[(2 + 2\ln 2 - 1) - (1 - \frac{1}{2})\right]$

$= 2\ln 2 - \frac{1}{2}$

$\doteqdot 0.89$ (2 dec) #

(b)

(i) Prove $8\cos^4 x = 3 + 4\cos 2x + \cos 4x$

L.H.S $= 8\cos^4 x$

$= 8\left[\cos^2 x\right]^2$

$= 8\left[\frac{1}{2}(\cos 2x + 1)\right]^2$

$= 2\left[\cos^2 2x + 2\cos 2x + 1\right]$

$= 2\left[\frac{1}{2}(\cos 4x + 1) + 2\cos 2x + 1\right]$

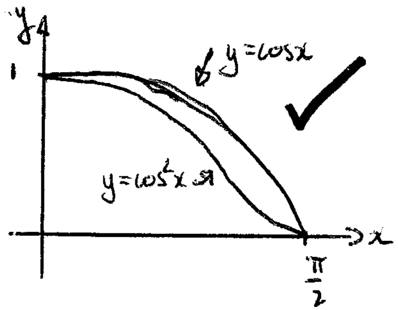
$= \cos 4x + 1 + 4\cos 2x + 2$

$= 3 + 4\cos 2x + \cos 4x$

$=$ R.H.S #

$\Rightarrow$ L.H.S = R.H.S hence proven

(ii) $y = \cos x$, $y = \cos^2 x$ $\quad 0 \le x \le \frac{\pi}{2}$



(iii) $A = \int_0^{\frac{\pi}{2}} \cos x - \cos^2 x\, dx$

$= \int_0^{\frac{\pi}{2}} \cos x - \frac{1}{2}(\cos 2x + 1)\, dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 2\cos x - \cos 2x - 1\, dx$

$= \frac{1}{2}\left[2\sin x - \frac{1}{2}\sin 2x - x\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[(2 - 0 - \frac{\pi}{2}) - 0\right]$

$= \left(1 - \frac{\pi}{4}\right) u^2$ #.

(iv) $V = \pi\int_0^{\frac{\pi}{2}} \cos^2 x\, dx - \pi\int_0^{\frac{\pi}{2}} \cos^4 x\, dx$ from (i)

$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \cos 2x + 1\, dx - \frac{\pi}{8}\int 3 + 4\cos 2x + \cos 4x\, dx$

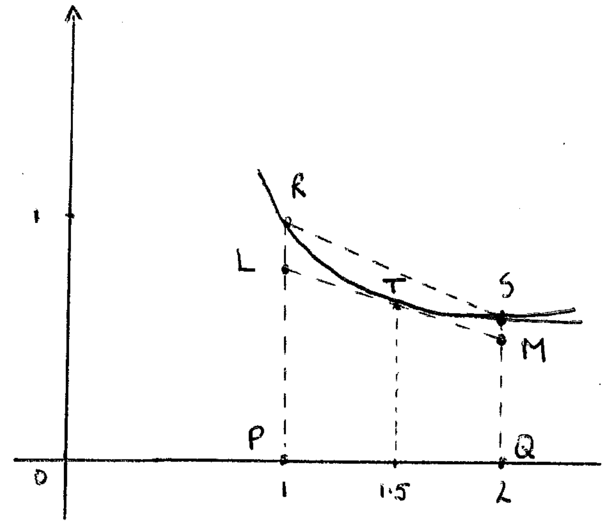
$= \frac{\pi}{8}\int 4\cos 2x + 4 - 3 - 4\cos 2x - \cos 4x\, dx$

$= \frac{\pi}{8}\int_0^{\frac{\pi}{2}} 1 - \cos 4x\, dx$

$= \frac{\pi}{8}\left[x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{8}\left[\frac{\pi}{2} - 0 - 0\right] = \frac{\pi^2}{16}\ u^3$ #.

(c) $P(1,0)$ $Q(2,0)$ $y = \frac{1}{x}$



(i) $A = \int_1^2 \frac{1}{x}\, dx$

$= [\ln x]_1^2$

$= \ln 2_{\#}$

(ii)

x	1	2
y	1	$\frac{1}{2}$
	y_0	y_n

$A \doteqdot \frac{1}{2}\left[1 + \frac{1}{2}\right] \doteqdot \frac{3}{4}\ u^2_{\#}$

2

(iii) $T = \left(\frac{3}{2}, \frac{2}{3}\right)$

$y' = -\frac{1}{x^2}$

$\therefore M_{TAN} = -\frac{4}{9}$

ie eqn of tangent at $\left(\frac{3}{2}, \frac{2}{3}\right)$

$y - \frac{2}{3} = -\frac{4}{9}\left(x - \frac{3}{2}\right)$

Now when $x = 1$

$y - \frac{2}{3} = -\frac{4}{9}\left(1 - \frac{3}{2}\right)$

ie $y = 8/9$

$\therefore L\left(1, \frac{8}{9}\right)$

when $x = 2$

$y - 2/3 = -\frac{4}{9}\left(2 - 3/2\right)$

ie $y = \frac{4}{9}$

$\therefore M\left(2, \frac{4}{9}\right)$

$\therefore$ AREA TRAP PQML

$= \frac{1}{2}\left(\frac{8}{9} + \frac{4}{9}\right) \times 1$

$= \frac{2}{3}\ u^2_{\#}$

QUESTION 2

#4

(-6)

(iv) Now

AREA TRAP PQML < AREA UNDER CURVE $\frac{1}{x}$ between $x = 1$ & 2 < AREA TRAP PQSR

$\frac{2}{3} < \ln 2 < \frac{3}{4}_{\#}$

QUESTION 3:

(a) HOCKEYROO

$\therefore$ ARRANGEMENTS $= \frac{9!}{3!} = 60\,480_{\#}$

2

(b)

(i) $11! = 39\,916\,800_{\#}$

(ii) $6! \times 5! = 86\,400_{\#}$

2

(c)

(i) No restrictions

$= 5 \times 5 \times 5 \times 5 \times 5$

$= 3125_{\#}$

(ii) ${}^5P_5 = 120_{\#}$

5	4	3	2	1

(iii) Group H/W together

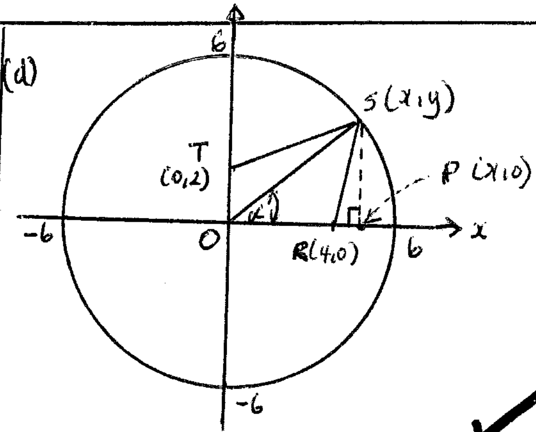
$\therefore$ arrangements (No rest)

$= 5 \cdot 4 \cdot 4 \cdot 4$

$= 320_{\#}$

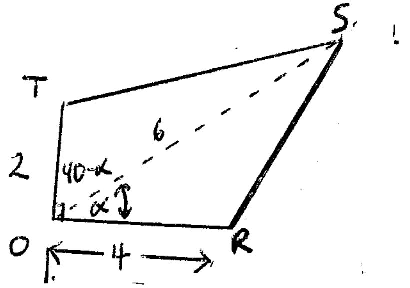
2

(d)



(i) $A = \frac{1}{2}ab\sin C = \frac{1}{2} \times 4 \times 6\sin\alpha = 12\sin\alpha_{\#}$

(ii)



AREA QUAD ORST := A_{Δ} OTS + A_{Δ} ORS

$= \frac{1}{2} \times 2 \times 6 \times \sin(90 - \alpha) + 12\sin\alpha$

$= 6\cos\alpha + 12\sin\alpha$

$= 6\cos\alpha(1 + 2\tan\alpha)$

2

(iii) $\therefore\ A = 6\cos\alpha\,(2\tan\alpha + 1)$

$A = 12\sin\alpha + 6\cos\alpha$

$\frac{dA}{d\alpha} = 12\cos\alpha - 6\sin\alpha$ ✓

MAX/MIN OCCUR WHERE $\frac{dA}{d\alpha} = 0$

ie $6\sin\alpha = 12\cos\alpha$

$\tan\alpha = 2$

$\therefore\ \alpha = \tan^{-1}(2)$ (1ST QUAD ONLY)

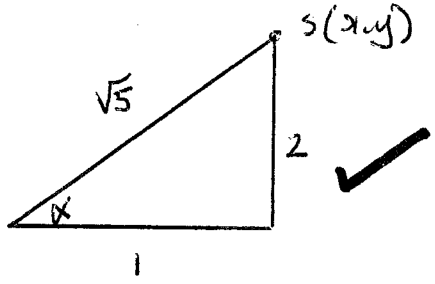
$\therefore\ \alpha \doteqdot 1.11^c$ ✓

α	1^c	1.11^c	1.2^c
$\frac{dA}{d\alpha}$	+	0	−
	/	—	\

$\therefore$ MAX AREA ✓

OCCURS WHEN $\alpha = 1.11^c$

(iv)



✓

Now $x = 6\cos\alpha$ $\quad y = 6\sin\alpha$

$x = 6 \times \frac{1}{\sqrt{5}}$ $\quad y = 6 \times \frac{2}{\sqrt{5}}$

$= \frac{6}{\sqrt{5}}$ ✓ $\quad = \frac{12\sqrt{5}}{5}$ ✓

$= \frac{6\sqrt{5}}{5}$

$\therefore$ For MAX AREA

$S = \left(\frac{6\sqrt{5}}{5}, \frac{12\sqrt{5}}{5}\right)$ #