Student name/number: ________________

SOUTH SYDNEY HIGH SCHOOL

2002

AUGUST ASSESSMENT

Mathematics Extension 1

Total marks (63)

- Attempt Questions 1 – 5
- All questions are *NOT* of equal value
- Topics: Inequalities; Circle geometry; Further trigonometry; Differentiation.

General Instructions

- Working time – 2 periods
- Board-approved calculators may be used
- All necessary working should be shown in every question

Question 1 (13 marks)

(a) Solve for x: $2x^2+5x-3>0$ *(2m)*

(b) Complete these statements about the properties of a circle:
(i) Angles in the same segment are _______________.

(ii) Opposite angles of a cyclic quadrilateral are _____________. *(2m)*

(c) Solve for x: $\dfrac{6}{2x-1}\geq 3$ *(3m)*

(d) Find $\lim\limits_{x\to -5}\dfrac{x^2-25}{x+5}$ *(2m)*

(e) Differentiate with respect to x:

(i) $y=\sqrt{4x+3}$

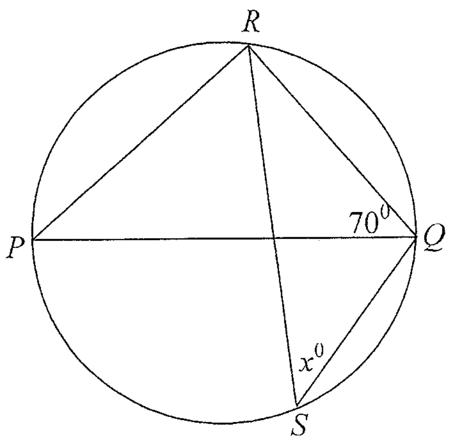
(ii) $y=\dfrac{1}{3x}-\dfrac{1}{2x^2}$ *(4m)*

Question 2 (10 marks)

(a) If $y=-\dfrac{1}{x}$, find $\dfrac{dy}{dx}$ from ***first principles***. *(4m)*

(b) If $\sin\alpha=\dfrac{4}{7}$ where $0<\alpha<90^0$, find the exact value of $\cot 2\alpha$. *(3m)*

(c)



PQ is a diameter, $\angle QSR = x^0$. Find the value of x, giving reasons for your answer. *(3m)*

Question 4 (15 marks)

(a) From a point P, due South of a flag pole, the angle of elevation of the top of the flag pole is 25^0. From another point Q, due East of the flag pole, the angle of elevation of the top of the flag pole is 40^0. Let the base of the flag pole be B, and the height of the flag pole h.

(i) If $PQ = 40$ metres, draw a clear diagram showing all the information.

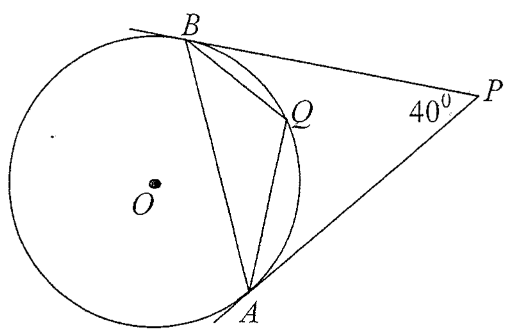
(ii) Write an expression for PB in terms of h.

(iii) Write an expression for QB in terms of h.

(iv) Using the answers in (ii) and (iii) find the height h, of the flag pole correct to *two* decimal places. *(6m)*

(b) Find the equation of the normal to the curve $y = \dfrac{1}{x+1}$ at $x = 1$. *(3m)*

(c) In the figure below PA and PB are tangents to the circle from P. $\angle APB = 40^0$



Find the size of the obtuse angle AQB giving reasons for your answer. *(3m)*

(d) Prove the trigonometric identity:

$$\frac{1}{\cos x + \sin x} + \frac{1}{\cos x - \sin x} = \frac{\tan 2x}{\sin x}$$ *(3m)*

Peggy Ngan

98%

Excellent work!

1.

a) $2x^2 + 5x - 3 > 0$

$(2x-1)(x+3) > 0$

$\therefore \underline{x < -3 \text{ OR } x > \frac{1}{2}}$

b) i, Angles in the same segment are equal.

ii, Opposite angles of a cyclic quadrilateral are supplementary

c). $\frac{6}{2x-1} \geq 3 \quad (x \neq \frac{1}{2})$

$6(2x-1) \geq 3(2x-1)^2$

$12x - 6 \geq 3(4x^2 - 4x + 1)$

$12x - 6 \geq 12x^2 - 12x + 3$

$12x^2 - 24x + 9 \leq 0$

$4x^2 - 8x + 3 \leq 0$

$(2x-3)(2x-1) \leq 0 \qquad \therefore \underline{\frac{1}{2} < x \leq \frac{3}{2}}$

d) $\lim_{x \to -5} \frac{x^2-25}{x+5} = \lim_{x \to -5} \frac{(x+5)(x-5)}{(x+5)}$

$= \lim_{x \to -5} x - 5$

$= \underline{-10}$

e) i, $y = \sqrt{4x+3}$

$= (4x+3)^{\frac{1}{2}}$

$\therefore y' = \frac{1}{2}(4x+3)^{-\frac{1}{2}} \cdot 4$

$= \underline{\frac{2}{\sqrt{4x+3}}}$

ii, $y = \frac{1}{3x} - \frac{1}{2x^2}$

$= \frac{1}{3}x^{-1} - \frac{1}{2} \cdot x^{-2}$

$\therefore y' = \frac{-1}{3}x^{-2} + x^{-3}$

$= \underline{\frac{-1}{3x^2} + \frac{1}{x^3}}$

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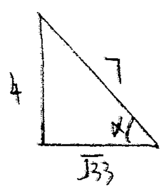
(2)

a)
$$y = -\frac{1}{x}$$
$$f(x) = -\frac{1}{x}$$
$$f(x+h) = \frac{-1}{x+h}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \checkmark$$
$$= \lim_{h\to 0} \frac{\frac{-1}{x+h} + \frac{1}{x}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{-x}{x(x+h)} + \frac{x+h}{x(x+h)}}{h} \checkmark$$
$$= \lim_{h\to 0} \frac{h}{x(x+h)} \times \frac{1}{h}$$
$$= \lim_{h\to 0} \frac{1}{x(x+h)} \checkmark$$
$$= \frac{1}{x^2} \checkmark$$

b)



$$\cot 2\alpha = \frac{1}{\tan 2\alpha} \checkmark$$
$$= \frac{1 - \tan^2\alpha}{2\tan\alpha}$$
$$= \frac{1 - \left(\frac{4}{\sqrt{33}}\right)^2}{2\left(\frac{4}{\sqrt{33}}\right)} \checkmark$$
$$= \frac{17}{33} \times \frac{\sqrt{33}}{8}$$
$$= \frac{17\sqrt{33}}{264} \checkmark$$

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c) $\angle PRQ = 90^\circ$ ($\angle$ in semi-circle) ✓

In $\triangle PRQ$,

$$90^\circ + 70^\circ + \angle RPQ = 180^\circ \quad (\angle \text{ sum of } \triangle)$$
$$\angle RPQ = 20^\circ \checkmark$$
$$\because \angle QSR = \angle QPR = x^\circ \quad (\angle \text{ in same segment})$$
$$\therefore x = 20 \checkmark$$

(7)

a) $\dfrac{2\tan 75^\circ}{1-\tan^2 75^\circ} = \tan(2\cdot 75^\circ)$

$= \tan 150^\circ$

$= -\tan 30^\circ$

$= -\dfrac{1}{\sqrt{3}} = \dfrac{-\sqrt{3}}{3}$

b) i, $\angle ACB = 90^\circ$ ($\angle$ in semi-circle)

$\because \angle ODB = 90^\circ$ (given)

$\therefore \angle ACB = \angle ODB = 90^\circ$

$\therefore$ $AC \parallel OE$ (corr. $\angle$s equal)

ii, $\angle BAC = x$ ($\angle$ in alternate segment)

$\because \angle BOD = \angle BAC$ (corr. $\angle$s, $AC \parallel OE$)

$\therefore \angle BOD = \angle BAC = x$

iii, In OBEC,

$\angle OCE = 90^\circ$ (line from centre $\perp$ tangent)

also, $\angle OBE = 90^\circ$ (line from centre $\perp$ tangent)

$\therefore \angle OCE + \angle OBE = 90^\circ + 90^\circ$

$= 180^\circ$

$\therefore \angle OCE$ & $\angle OBE$ are supplementary

$\therefore$ OBEC is a cyclic quad.

iv, $\angle OCE = \angle OCD + x$

$\therefore \angle OCD = 90^\circ - x$ (line from centre $\perp$ tangent)

$\because \triangle OCB$ is isosceles $\triangle$ (same radii)

$\therefore \angle OBD = \angle OCD$ (base $\angle$s, isos. $\triangle$)

$= 90^\circ - x$

In $\triangle CBP$,

$x + \angle P = \angle OBD$ (ext. $\angle$ of $\triangle$)

$\therefore \angle P = 90^\circ - 2x$

c) $\sin 15^\circ = \sin(45^\circ - 30^\circ)$

$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$

$= \left[\dfrac{\sqrt{2}}{2}\cdot\dfrac{\sqrt{3}}{2}\right] - \left[\dfrac{\sqrt{2}}{2}\cdot\dfrac{1}{2}\right]$

$= \dfrac{\sqrt{6}-\sqrt{2}}{4}$

d)

$$f(x) = (x-1)\sqrt{x+1}$$
$$= (x-1)(x+1)^{\frac{1}{2}}$$

$$f'(x) = (x-1)\tfrac{1}{2}(x+1)^{-\frac{1}{2}} \cdot 1 + (x+1)^{\frac{1}{2}} \cdot 1$$
$$= \frac{x-1}{2\sqrt{x+1}} + \sqrt{x+1}$$

$$\therefore f'(2) = \frac{2-1}{2\sqrt{2+1}} + \sqrt{2+1}$$
$$= \frac{1}{2\sqrt{3}} + \sqrt{3}$$
$$= \frac{1}{2\sqrt{3}} + \frac{6}{2\sqrt{3}}$$
$$= \frac{7}{2\sqrt{3}} = \underline{\underline{\frac{7\sqrt{3}}{6}}}$$

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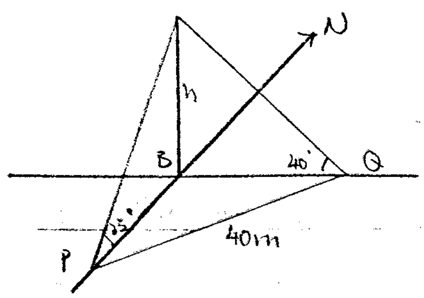
(4)

a) i)



ii)

$$\cancel{\tan 25^\circ =} \frac{h}{PB}$$

$$\therefore PB = \frac{h}{\tan 25^\circ}$$

iii)

$$\tan 40^\circ = \frac{h}{QB}$$

$$\therefore QB = \frac{h}{\tan 40^\circ}$$

iv)

$$40^2 = PB^2 + QB^2 - 2(PB)(QB)\cos 90^\circ \quad \text{(Pyth. Theorem?)}$$

$$1600 = \frac{h^2}{\tan^2 25^\circ} + \frac{h^2}{\tan^2 40^\circ} - 2 \cdot \frac{h^2}{\tan 25 \tan 40} \cdot 0$$

$$1600 = \frac{h^2(\tan^2 40^\circ + \tan^2 25^\circ)}{\tan^2 25^\circ \tan^2 40^\circ}$$

$$\therefore h^2 = 1600 \times \frac{\tan^2 25^\circ \cdot \tan^2 40^\circ}{\tan^2 25^\circ + \tan^2 40^\circ}$$

$$\therefore h = 16.30\text{ m}$$

b)

$$y = \frac{1}{x+1}$$

$$= (x+1)^{-1} \qquad \therefore y' = -1(x+1)^{-2} \cdot 1$$

$$= -(x+1)^{-2}$$

when $x = 1$, $y' = -(1+1)^{-2}$

$$= -\frac{1}{4}$$

$$\because m \times -\frac{1}{4} = -1$$

$$m = 4$$

when $x = 1$, $y = \frac{1}{2}$

$\therefore$ Eqn of normal: $\dfrac{y - \frac{1}{2}}{x - 1} = 4$ $\quad 8x - 2y - 7 = 0$

$$y - \frac{1}{2} = 4x - 4$$

$$2y - 1 = 8x - 8$$

$PB = PA$ (tangents from external pt have equal length)

$\therefore \angle PBA = \angle PAB = x$ (base $\angle$s, isos. $\triangle$)

$\because 2x + 40° = 180°$ ($\angle$ sum of $\triangle$)

$\therefore x = 70°$ ✓

$\angle AQB = 180° - \angle PAB$ ($\angle$ in alt segment)

$= 180° - 70°$

$= 110°$ ✓

$\therefore$ ~~$\angle AQ$~~ obtuse $\angle AQB = 110°$

d) L.H.S. $= \dfrac{1}{\cos x + \sin x} + \dfrac{1}{\cos x - \sin x}$

$= \dfrac{\cos x - \sin x + \cos x + \sin x}{\cos^2 x - \sin^2 x}$ ✓

$= \dfrac{2\cos x}{\cos 2x}$

$= \dfrac{2 \cdot \sin x \cdot \cos x}{\cos 2x} \times \dfrac{1}{\sin x}$ ✓

$= \dfrac{\sin 2x}{\cos 2x} \times \dfrac{1}{\sin x}$

$= \dfrac{\tan 2x}{\sin x}$ ✓

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(5)

a) i, $\frac{d}{dx}\left[\frac{2x-1}{3x+5}\right]$

$= \frac{(3x+5)2 - (2x-1)3}{(3x+5)^2}$ ✓

$= \frac{6x+10-6x+3}{(3x+5)^2}$

$= \frac{13}{(3x+5)^2}$ ✓

ii, $g(x) = x^2 - x - 12$

$g'(x) = 2x-1$ ✓

$h(x) = \frac{1}{x^2-x-12}$

$= (x^2-x-12)^{-1}$

$h'(x) = -(x^2-x-12)^{-2}\cdot(2x-1)$

$= \frac{-(2x-1)}{(x^2-x-12)^2}$ ✓

Put $g'(x) = 0$

$2x-1=0$

$x = \frac{1}{2}$

Put $h'(x) = 0$

$\frac{-(2x-1)}{(x^2-x-12)^2} = 0$

$1-2x = 0$

$x = \frac{1}{2}$

$\therefore$ $g'(x) = h'(x) = 0$ for some value of x

& that x-value is $\frac{1}{2}$ ✓

b) $\cos 79^\circ \cos 311^\circ + \sin 101^\circ \sin 49^\circ$

$= \cos 30^\circ$ ✓

$= \frac{\sqrt{3}}{2}$

missing step.

$= \cos 79^\circ \cos(360-311)^\circ + \sin(180-101)^\circ \sin 49^\circ$

$= \cos 79^\circ \cos 49^\circ + \sin 79^\circ \sin 49^\circ$

$= \cos(79^\circ - 49^\circ)$

$= \cos 30^\circ$

$= \frac{\sqrt{3}}{2}$

c) $\dfrac{x^2-3x-4}{3-x} \leq 0 \qquad (x \neq 3)$

$\therefore \quad x^2-3x-4 \geq 0 \quad$ and $\quad 3-x \leq 0$

$(x-4)(x+1) \geq 0 \qquad\qquad x \geq 3$

$\therefore \quad x \geq 4$

$\therefore \quad x^2-3x-4 \leq 0 \quad$ and $\quad 3-x \geq 0$

$(x-4)(x+1) \leq 0 \qquad\qquad x \leq 3$

$\therefore \quad -1 \leq x \leq 3$

$\therefore \quad \underline{\underline{-1 \leq x < 3 \quad \text{OR} \quad x \geq 4}}$

10

Q 8 (b).

(i) $f(\theta) = \dfrac{2-\cos\theta}{\sin\theta}$; $0 < \theta < \pi/2$

$$f'(\theta) = \frac{\sin\theta(\sin\theta) - (2-\cos\theta).\cos\theta}{\sin^2\theta}$$

$$= \frac{\sin^2\theta - 2\cos\theta + \cos^2\theta}{\sin^2\theta}$$

$$= \frac{1-2\cos\theta}{\sin^2\theta} \qquad f''\theta = \frac{2\sin\theta(1-\cos\theta)}{\sin^4\theta}$$

For Max or Min values $f'(\theta) = 0$

$\therefore\ 1 - 2\cos\theta = 0$

$\cos\theta = 1/2$.

When $\cos\theta = 1/2$ $f(\theta) = \dfrac{2-1/2}{\sqrt{3}/2}$

$= \dfrac{3}{2} \times \dfrac{2}{\sqrt{3}}$

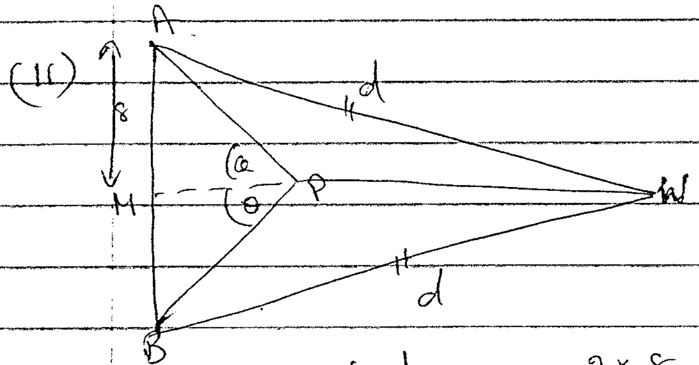
$= \dfrac{3}{\sqrt{3}}$

$= \sqrt{3}$

For $\cos\theta = 1/2$ $f'' > 0$ $\therefore$ Minimum value at $\cos\theta = 1/2$

$\therefore$ Minimum value of $f(\theta) = \sqrt{3}$

(ii)



Let $PA = x$.

$\therefore BP = x$, and Let $MP = y$

$x = \dfrac{8}{\sin\theta}$. $y = \dfrac{8}{\tan\theta}$.

From ΔMWA $MW = d^2 - 8^2$

$\therefore L = \dfrac{2\times 8}{\sin\theta} + PW$

$= 8 \times \dfrac{2}{\sin\theta} + \sqrt{d^2-8^2} - \dfrac{8}{\tan\theta}$.

$= 8\left[\dfrac{2}{\sin\theta} - \dfrac{\cos\theta}{\sin\theta}\right] + \sqrt{d^2-8^2}$

$= 8\left[\dfrac{2-\cos\theta}{\sin\theta}\right] + \sqrt{d^2-64}$

$= 8f(\theta) - \sqrt{d^2-64}$ $\left(\text{where } \dfrac{8}{d} \le \sin\theta \le 1\right)$

(iii) If $d = 20$, Min L occures when $\dfrac{dL}{d\theta} = 0$.

i.e. $f'(\theta) = 0$

$\therefore \cos\theta = 1/2$.

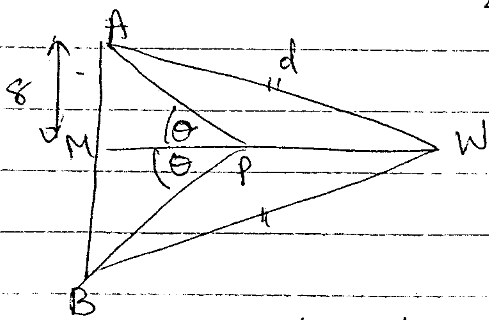
Min value of $L = 8 \times \sqrt{3} - \sqrt{20^2 - 64}$

$= 8\sqrt{3} - \sqrt{400-64}$

$= 8\sqrt{3} - \sqrt{336}$

$= 8\sqrt{3} - \sqrt{6\times 8\times 7}$ $= 4\sqrt{3}(2-\sqrt{7})$

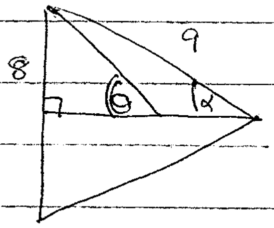
(iv) For min $f(\theta)$ $\cos\theta = \frac{1}{2}$.



Where $\frac{8}{d} \leq \sin\theta \leq 1$

Let $\angle AWP = \alpha$

When $d = 9$



$\sin\alpha = \frac{8}{9}$. $\therefore \alpha \doteqdot 62^\circ 44'$

But for min L, $\sin\theta = \frac{\sqrt{3}}{2}$

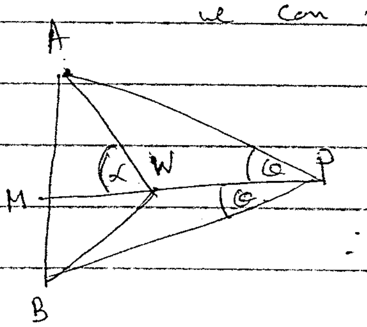
$\theta = 60^\circ$

But in this method $\theta > \alpha$.

But $62^\circ 44' \not> 60^\circ$

$\therefore$ $\theta < \alpha$ and ~~thi~~ using this method

we can not find the min. value of L.



~~Let~~ $AW = d$.

$\therefore$ Min value of the length of pipes

$= AW + BW$ only //