# South Sydney High School
# Year 11 Mathematics
# December Exam
# 1998

## 3/4 Unit Common Paper

**Instructions :** **Time Allowed:** 2 Periods

1. All questions may be attempted.
2. Start each question on a new sheet of paper.
3. All necessary working should be shown.
4. Marks may be deducted for poorly arranged or missing working.
5. Approved calculators may be used.

**Question 1**

a) Find the value of the following limits :

(i) $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3}$ (ii) $\lim\limits_{x \to \infty} \dfrac{3x^2 - 9}{2x^2 - x - 3}$

b) Find $\dfrac{dy}{dx}$ for the following

(i) $y = \dfrac{x^2}{x+3}$ (ii) $y = x\sqrt{x+1}$

c) For what values of $x$ is the curve $y = 5 - 9x + 6x^2 - x^3$ concave upwards ?

d) For a certain function $\dfrac{d^2y}{dx^2} = 6x - 12$ and $y = f(x)$ has a maximum turning point at $(1,-1)$.
Find $y$ in terms of $x$.

e) Find the equation of the normal to the curve $y = x^2 + 3x + 1$ at the point $(-1,-1)$.

**Question 2**

a) For what values of $k$ is $(k-1)x^2+(k+1)x+k-1$ positive definite ?

b) Find the values of $m$ for which the line $y = m(x+1)$ has no intersection with the parabola $y = 2x^2$

c) When the polynomial $P(x)$ is divided by $(x+1)(x-4)$, the quotient is $Q(x)$ and the remainder is $R(x)$.

(i) Why is the most general form of $R(x) = ax + b$ ?

(ii) Given that $P(4) = -5$, show that $R(4) = -5$.

(iii) Further, when $P(x)$ is divided by $(x+1)$, the remainder is 5. Find $R(x)$.

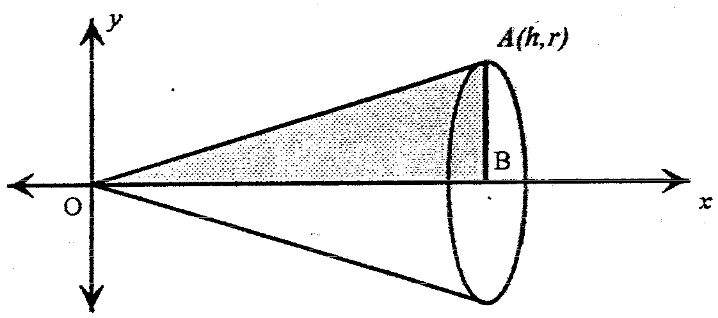
**Question 3**

a) Find $\int \frac{x^2+2}{x^2}\,dx$

b) Find the area enclosed between the curve $y = \sqrt{x+1}$ , the $y$-axis and the lines $y = 0$ and $y = 3$.

c) A football has a volume approximately the same as the volume generated by rotating the ellipse $9x^2 + 16y^2 = 144$ about the $x$-axis. Find its volume.

d) Prove that the volume of a right circular cone, height $h$ and radius of base $r$, is $\frac{1}{3}\pi r^2 h$ by rotating the area OAB about the $x$-axis.




e) If the slant edge of a right circular cone is 6cm in length, find the height of the cone when the volume is a maximum.

**Question 4**

a) Prove by Mathematical Induction that $n^2 + 2n$ is divisible by 3, for all positive integers $n$.

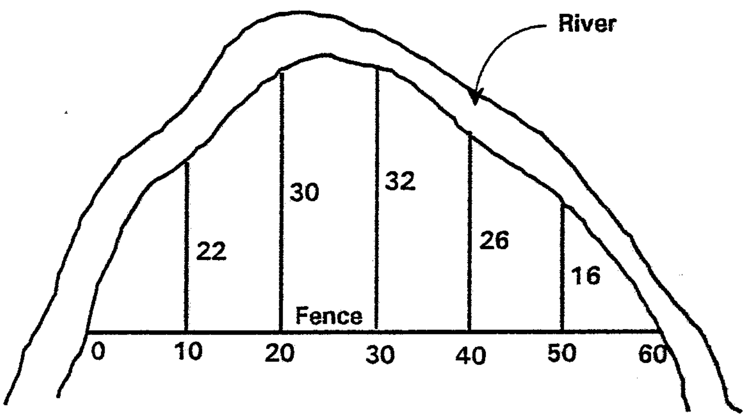
b) Sketch $y = \frac{x-2}{x^2}$ showing all important features.

## Question 5

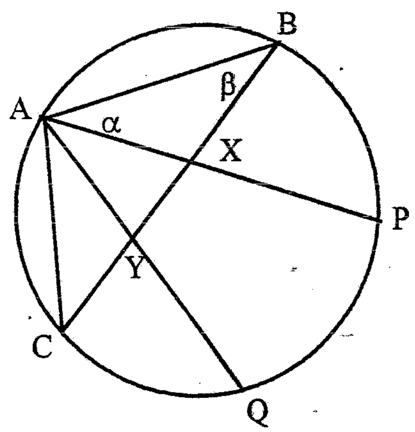
a) Evaluate $\int_0^1 \frac{t}{\sqrt{(1+t)}}\,dt$ by using the substitution $u = 1 + t$.

b) A field bounded by a river and a straight fence is surveyed. Offset measurements were taken at 10m intervals along the fence, as shown in the diagram.

Use Simpson's Rule to find the area of the field.




c)




Let $ABPQC$ be a circle such that $AB = AC$, $AP$ meets $BC$ at $X$, and $AQ$ meets $BC$ at $Y$, as in the diagram.
Let $\angle BAP = \alpha$ and $\angle ABC = b$.

(i) Copy the diagram and state why $\angle AXC = \alpha + b$.

(ii) Prove that $\angle BQP = \alpha$.

(iii) Prove that $\angle BQA = \beta$.

(iv) Prove that $PQYX$ is a cyclic quadrilateral.

## Question 1

a) (i) $\lim_{x\to 3} \frac{(x+3)\cancel{(x-3)}}{\cancel{x-3}} = 6$

(ii) $\lim_{x\to\infty} \frac{3-\frac{9}{x^2}}{2-\frac{1}{x}-\frac{3}{x^2}} = \frac{3}{2}$

b) (i) $y = \frac{x^2}{x+3}$

$y' = \frac{vu' - uv'}{v^2}$

$= \frac{(x+3)2x - x^2 \cdot 1}{(x+3)^2}$

$= \frac{x^2+6x}{(x+3)^2}$

(ii) $y = x(x+1)^{\frac{1}{2}}$

$y' = vu' + uv'$

$= (x+1)^{\frac{1}{2}} \cdot 1 + x \cdot \frac{1}{2}(x+1)^{-\frac{1}{2}}$

$= \sqrt{x+1} + \frac{x}{2\sqrt{x+1}}$

$= \frac{2(x+1) + x}{2\sqrt{x+1}}$

$= \frac{3x+2}{2\sqrt{x+1}}$

c) $y = 5 - 9x + 6x^2 - x^3$

$y' = -9 + 12x - 3x^2$

$y'' = 12 - 6x$

Concave up → $y'' > 0$

ie $12 - 6x > 0$

$12 > 6x$

$2 > x$

∴ concave up when $x < 2$

d) $y'' = 6x - 12$

$y' = 3x^2 - 12x + c$

but $y' = 0$ when $x = 1$

$\therefore 0 = 3 - 12 + c$

$c = 9$

$\therefore y' = 3x^2 - 12x + 9$

$y = x^3 - 6x^2 + 9x + k$

but $y = -1$ when $x = 1$

$\therefore -1 = 1 - 6 + 9 + k$

$-1 = 4 + k$

$k = -5$

$\therefore y = x^3 - 6x^2 + 9x - 5$

e) $y = x^2 + 3x + 1$

$y' = 2x + 3$

at $x = -1$ $m = -2 + 3 = 1$

grad. normal $= -1$

normal $y - y_1 = m(x - x_1)$

$y + 1 = -1(x+1)$

$y + 1 = -x - 1$

$x + y + 2 = 0$

## Question 2

a) pos. def. $a > 0$ $b^2 - 4ac < 0$

$b^2 - 4ac = (k+1)^2 - 4(k-1)(k-1)$

$= k^2 + 2k + 1 - 4k^2 + 8k - 4$

$= -3k^2 + 10k - 3$

$\therefore -3k^2 + 10k - 3 < 0$

$-(\underset{\frac{1}{3}}{3k-1})(\underset{3}{k-3}) < 0$

$\therefore x < \frac{1}{3}$ or $x > 3$

but $a > 0$ ∴ $k - 1 > 0$

$k > 1$

∴ solution $k > 3$

b) $y = m(x+1)$ —①

$y = 2x^2$ —②

no intersection when

$2x^2 = m(x+1)$ has no roots

$2x^2 = mx + m$

$2x^2 - mx - m = 0$

$\therefore b^2 - 4ac < 0$

$m^2 - 4(2)(-m) < 0$

$m^2 + 8m < 0$

$\underset{0}{m}(\underset{-8}{m+8}) < 0$

$\therefore -8 < m < 0$

no intersection when $-8 < m < 0$

c) (i) the divisor is of degree 2 ∴ the remainder must be of degree (2−1) or less.

∴ general representation of a degree 1 expression is $ax + b$

$\therefore R(x) = ax + b$

(ii) $P(x) = (x+1)(x-4)Q(x) + R(x)$ is the general expression for $P(x)$

∴ if $P(4) = -5$ then

$-5 = (x+1)(0)Q(4) + R(4)$

$\therefore -5 = R(4)$

$\therefore R(4) = (-5)$

(iii) if $(x+1)$ is divided into $P(x)$ the remainder is $P(-1)$

$\therefore P(-1) = 5$ also $R(4) = -5$

$\therefore 4a + b = -5$ —①

$P(-1) = R(-1)$

$\therefore -a + b = 5$ —②

①−② $5a = -10$

$a = -2$

$\therefore \quad 2+b=5$ from ②

$b=3$

$\therefore R(x) = -2x+3$

## Question 3

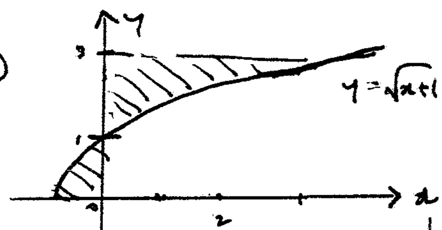
a) $\int \frac{x^2+2}{x^2}\,dx$

$= \int (1+2x^{-2})\,dx$

$= x - 2x^{-1} + c$

$= x - \frac{2}{x} + c$

b)




$y = \sqrt{x+1}$

$y^2 = x+1$

$x = y^2 - 1$

$A = \left|\int_0^1 (y^2-1)\,dx\right| + \int_1^3 (y^2-1)\,dx$

$= \left|\left[\frac{y^3}{3} - y\right]_0^1\right| + \left|\frac{y^3}{3} - y\right|$

$= \left|\frac{-2}{3}\right| + 6 + \frac{2}{3}$

$= 7\frac{1}{3}$ units$^2$

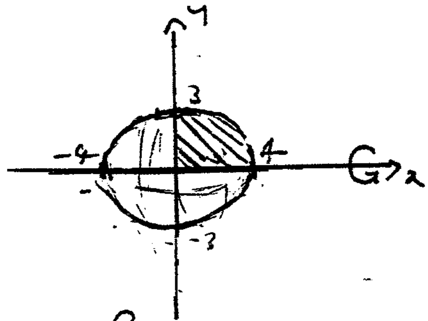
c) $9x^2 + 16y^2 = 144$

$\therefore y^2 = \frac{144-9x^2}{16}$

when $y=0$ $\quad 0 = 144 - 9x^2$

$x^2 = 16$

$x = \pm 4$




Even fn.

$V = \pi \int_a^b y^2\,dx$

$= \frac{2\pi}{16}\int_0^4 (144 - 9x^2)\,dx$

$= \frac{\pi}{8}\left[144x - \frac{3x^3}{}\right]_0^4$

$= \frac{\pi}{8}\left[(576-192) - (0)\right]$

$= \frac{\pi}{8} \times 384$

$= 48\pi$ units$^3$

d) Equation OA $\quad y = mx + b$

$m = \frac{\text{rise}}{\text{run}} = \frac{r}{h} \qquad b = 0$

$\therefore \quad y = \frac{r}{h}x$

Area bounded by $y = \frac{r}{h}x$, the $x$-axis and the ordinates $x=0$ and $x=h$ is rotated around the $x$ axis

$V = \pi \int_0^h \frac{r^2}{h^2} x^2\,dx$

$= \pi \left[\frac{r^2}{h^2}\frac{x^3}{3}\right]_0^h$

$= \pi \left[\frac{r^2}{h^2}\frac{h^3}{3} - 0\right]$

$= \pi r^2 \frac{h}{3}$

$\therefore V = \frac{1}{3}\pi r^2 h$

e)


$r = \sqrt{6^2 - h^2}$

$\therefore V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(6^2 - h^2)h$

$= \frac{1}{3}\pi(36h - h^3)$

$\frac{dV}{dh} = \frac{1}{3}\pi \cdot (36 - 3h^2)$

Max/Min when $\frac{dV}{dh} = 0$

ie $36 - 3h^2 = 0$

$36 = 3h^2$

$12 = h^2$

$h = \pm\sqrt{12}$

$\therefore \quad h = 2\sqrt{3}$

check Max $\quad \frac{d^2V}{dh^2} = \frac{1}{3}\pi(-6h)$

at $h = 2\sqrt{3}$ $\quad \frac{d^2V}{dh^2} = \frac{-12\sqrt{3}\pi}{3}$

$\therefore$ Max when $h = 2\sqrt{3}$

## Question 4

a) $n^2 + 2n$ is divisible by 3

Step 1. let $n=1$

$n^2 + 2n = 1+2 = 3$

which is divisible by 3

$\therefore$ true for $n=1$

Step 2 Assume true for $n=k$

ie. $k^2 + 2k = 3M$

where $M$ is a +ve integer

**Step 3** prove true for $n=k+1$

i.e. prove $(k+1)^2+2(k+1)=3N$

where $N$ is a positive integer

$$LHS=(k+1)^2+2(k+1)$$
$$=k^2+2k+1+2k+1$$
$$=3M+2k+2$$
(Not divisible by 3)

Hence the expression is divisible by 3 for all $n\geq 1$ since if it is true for some $n=k$ then it is also true for $n=k+1$ and since it is true for $n=1$ then it must be true for $n=1+1=2$ and true then for $n=1+2=3$ and so on.

b) $y=\dfrac{x-2}{x^2}$

- Vertical asymptote at $x=0$

as $x\to 0^{\pm}$ $y\to -\infty$

- Horizontal asymptote

$$\lim_{x\to\infty}\frac{x-2}{x^2}=\lim_{x\to\infty}\frac{\frac{1}{x}-\frac{2}{x^2}}{1}=0^+$$

$$\lim_{x\to-\infty}\frac{x-2}{x^2}=0^-$$

$$\frac{dy}{dx}=\frac{x^2\cdot 1-(x-2)2x}{x^4}=\frac{x^2-2x^2+4x}{x^4}=\frac{-x+4}{x^3}$$

Stat pts $y'=0$

$0=-x+4$

$x=4$

$$\frac{d^2y}{dx^2}=\frac{x^3(-1)-(-x+4)3x^2}{x^6}=\frac{-x^3+3x^3-12x^2}{x^6}=\frac{2x-12}{x^4}$$

at $x=4$ $\dfrac{d^2y}{dx^2}=\dfrac{2(4)-12}{4^4}<0$

$y=\dfrac{4-2}{16}=\dfrac{1}{8}$

$\therefore$ max $t_p$ at $(4,\frac{1}{8})$

- pt of inflexion; $y''=0$

$\therefore 0=2x-12$

$x=6$

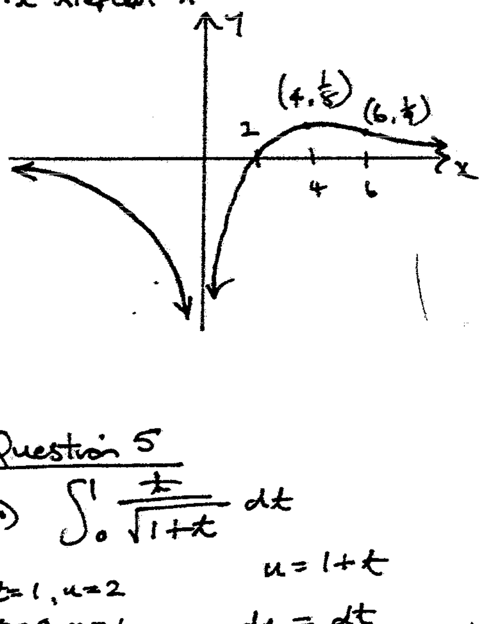
test

| $x$ | $6^-$ | $6$ | $6^+$ |
|---|---|---|---|
| $y''$ | $-ve$ | $0$ | $+ve$ |

$y=\dfrac{6-2}{36}=\dfrac{4}{36}=\dfrac{1}{9}$

$\therefore$ pt of inflexion at $(6,\frac{1}{9})$

- $x$ intercept $x=2$




**Question 5**

a) $\displaystyle\int_0^1\frac{t}{\sqrt{1+t}}\,dt$

$u=1+t$

$t=1,u=2$

$t=0,u=1$

$du=dt$

$t=u-1$

$$=\int_1^2\frac{u-1}{\sqrt{u}}\,du$$
$$=\int_1^2\left(u^{\frac{1}{2}}-u^{-\frac{1}{2}}\right)du$$
$$=\left[\frac{2}{3}u^{\frac{3}{2}}-2u^{\frac{1}{2}}\right]_1^2$$
$$=\left(\frac{2}{3}(2^{\frac{3}{2}})-2(2)^{\frac{1}{2}}\right)-\left(\frac{2}{3}-2\right)$$
$$=\left(\frac{2}{3}-1\right)2^{\frac{3}{2}}+\frac{4}{3}$$
$$=-\frac{1}{3}\sqrt{2^3}+\frac{4}{3}$$
$$=\frac{4-2\sqrt{2}}{3}$$

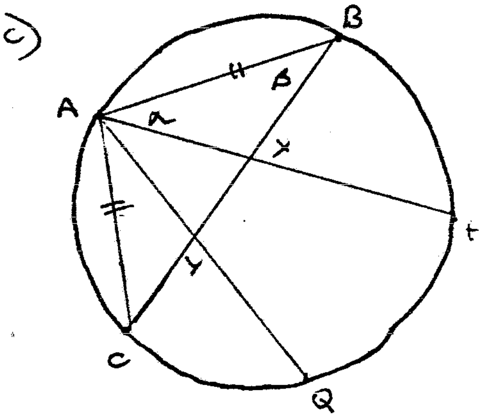
b) $A=\dfrac{h}{3}\left\{\Sigma f(\text{ends})+2\Sigma f(\text{odds})+4\Sigma f(\text{evens})\right\}$

$h=\dfrac{b-a}{n}=\dfrac{60-0}{6}=10$

$\therefore A=\dfrac{10}{3}\left\{(0+0)+2(30+26)+4(22+32+16)\right\}$

$=1306\frac{2}{3}\ m^2$

c)




(i) $\angle AXC=\alpha+\beta$ because $\angle AXC$ is the exterior angle of $\Delta AXB$ and is therefore equal to the sum of the two opposite interior angles $(\alpha+\beta)$

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(ii) $\angle BQP=\angle BAP$ (angles in the same segment are equal)

$\therefore \angle BQP=\alpha$