SOUTH SYDNEY HIGH SCHOOL

Year 11 Preliminary Course
Half Yearly Exam 1996

MATHEMATICS

3 Unit

Instructions : **Time Allowed:** *2 periods*

1. All questions may be attempted.
2. Start each question on a new sheet of paper.
3. All necessary working should be shown.
4. Marks may be deducted for poorly arranged or missing working.
5. Approved calculators may be used.

Question 1 (Use a new page)

(a) On a number plane, indicate the region specified by $y \leq |x - 1|$ and $y \leq 1$.

(b) Factorise $2^{n+1} + 2^n$, and hence write $\dfrac{2^{1001} + 2^{1000}}{3}$ as a power of 2.

(c) The interval AB has endpoints $A(-2, 3)$ and $B(10, 11)$. Find the coordinates of the point P which divides the interval AB in the ratio 3:1.

(d) Find the equation of the straight line through $(3, -1)$ perpendicular to the line $3x - 2y - 7 = 0$.

Question 2 (Use a new page)

(a) Solve for $0^o < \theta < 360^o$

(i) $3\tan^2\theta - 1 = 0$ (Give answers in exact form)

(ii) $2\sin^2\theta = \sin 2\theta$

(b) Prove the following identity $\dfrac{2\tan A}{1+\tan^2 A} = \sin 2A$

(c) Express $\sqrt{\dfrac{1+\cos 2x}{1-\cos 2x}}$ as a single trigonometric ratio.

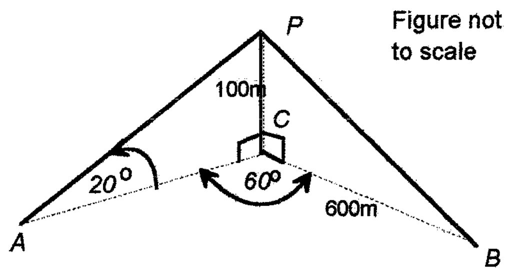
(d) Show that $\sin 195^\circ = \dfrac{1-\sqrt{3}}{2\sqrt{2}}$, without the aid of a calculator

(e) Find all angles θ, where $0^\circ \le \theta \le 360^\circ$, for which $\sqrt{3}\sin\theta - \cos\theta = 1$

Question 3 (Use a new page)

(a) Solve $|3+2x| = 4-x$

(b) Two yachts A and B subtend an angle of 60° at the base C of a cliff. From yacht A the angle of elevation of the point P, 100 metres vertically above C, is 20°. Yacht B is 600 metres from C.



(i) Calculate the length AC.

(ii) Calculate the distance between the two yachts.

c) Uluru is a large rock on flat ground in Central Australia. Three tourists A, B and C are observing Uluru from the ground. A is due north of Uluru, C is due east of Uluru, and B is on the line-of-sight from A to C and between them. The angles of elevation to the summit of Uluru from A, B and C are 26°, 28° and 30°, respectively.

Determine the bearing of B from Uluru.

Question 4 (Use a new page)

(a) Differentiate the following

(i) $\dfrac{x\sqrt{x}}{3}$

(ii) $\dfrac{x^4 - 4x^3 + 5}{x^3}$

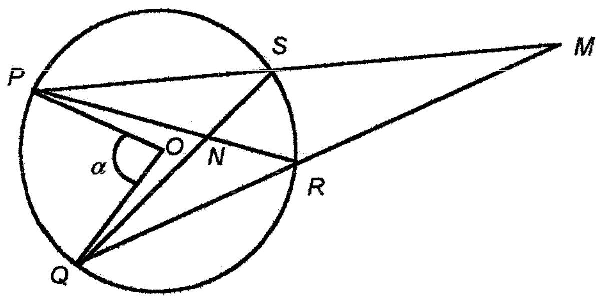
(iii) $3(x^2 + 5)^7$

(b) Differentiate $y = x^2 - 5x$ using first principles <u>and</u> hence find the gradient of the curve when x = 3

(c) Solve $\dfrac{2x - 5}{x + 3} \geq 1$

Question 5 (Use a new page)

(a)

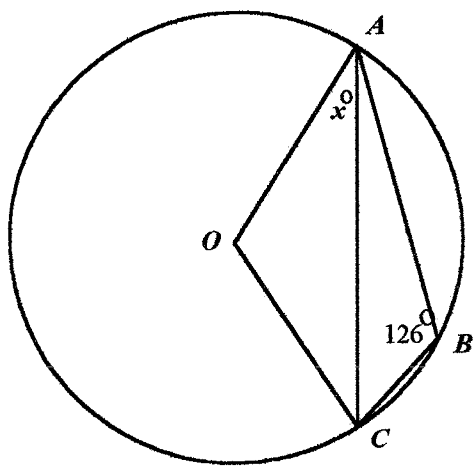


In the diagram *P, Q, R* and *S* are the points on a circle centre *O*, and $\angle POQ = \alpha$. The lines *PS* and *QR* intersect at *M* and the lines *QS* and *PR* intersect at *N*.

(i) Explain why $\angle PRM = \pi - \frac{1}{2}\alpha$

(ii) Show that $\angle PNQ + \angle PMQ = \alpha$

(b)



O is the centre of the circle
$\angle ABC = 126°$, $\angle OAC = x°$

Copy the diagram and find the value of *x*, giving reasons.

Question 6 (Use a new page)

(a) At the point $(2, -3)$ on the curve $y = ax^2 + bx + 7$, the tangent is inclined at $135°$ to the x-axis. Find the values of a and b.

(b) Find the distance between the parallel lines $7x - 15y - 3 = 0$ and $7x - 15y + 5 = 0$.

(c) The graphs of $y = x$ and $y = x^3$ intersect at $x = 1$.
Find the size of the acute angle between these curves at $x = 1$.

(d) Simplify $\dfrac{a^3(b^2)^4}{(a^{-1})^2 b^7}$ if $a = \dfrac{2}{3}$ and $b = \dfrac{4}{9}$

(e) Find $\lim\limits_{x \to 2} \dfrac{x - 2}{x^2 - 4}$

(f) A function is defined in the following way :

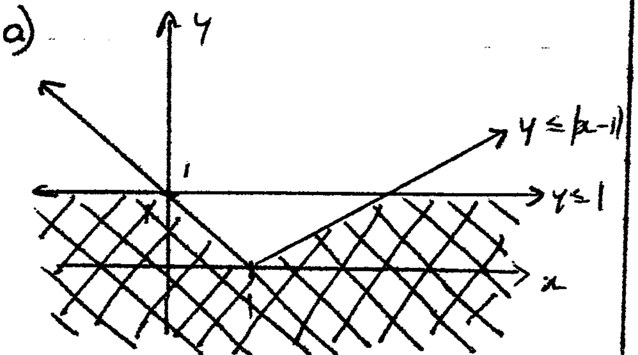
$$f(x) = \begin{cases} 1 - x & \text{for } x \le -1 \\ x^2 + 1 & \text{for } -1 < x \le 2 \\ 2x - 1 & \text{for } x > 2 \end{cases}$$

(i) Evaluate $2f(-1) + f(0) - 2f(3)$

(ii) Sketch the function for the domain $-3 \le x \le 3$.

Question 1.

a)



b) $2^{n+1} + 2^n = 2^n.2^1 + 2^n$

$= 2^n(2+1)$

$= 2^n.3$

$= 3.2^n$

$\therefore \dfrac{2^{1001}+2^{1000}}{3} = \dfrac{3.2^{1000}}{3}$

$= \underline{2^{1000}}$

c) $(X,Y) = \left(\dfrac{mx_2+nx_1}{m+n}, \dfrac{my_2+ny_1}{m+n}\right)$

$m:n = 3:1$

$A(-2,3)$, $B(10,11)$

$\therefore P(x,y) = \left[\dfrac{3(10)+1(-2)}{3+1}, \dfrac{3(11)+1(3)}{3+1}\right]$

$= \left(\dfrac{30-2}{4}, \dfrac{33+3}{4}\right)$

$= \left(\dfrac{28}{4}, \dfrac{36}{4}\right)$

$\therefore P = \underline{(7,9)}$

d) $3x-2y-7=0$

$m = \dfrac{-a}{b}$

$= \dfrac{3}{2}$

$\therefore$ perp. $m = \dfrac{-2}{3}$ point $(3,-1)$

$y-y_1 = m(x-x_1)$

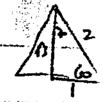
$y+1 = \frac{-2}{3}(x-3)$

$3y+3 = -2x+6$

$\therefore \underline{2x+3y-3=0}$ is eqn. of line

Question 2

a) (i) $3\tan^2\theta - 1 = 0$



$\tan^2\theta = \frac{1}{3}$

$\tan\theta = \pm\frac{1}{\sqrt{3}}$

$\therefore \underline{\theta = 30°, 150°, 210°, 330°}$

(ii) $2\sin^2\theta = \sin 2\theta$

$2\sin^2\theta = 2\sin\theta\cos\theta$

$\sin^2\theta - \sin\theta\cos\theta = 0$

$\sin\theta(\sin\theta - \cos\theta) = 0$

$\therefore \sin\theta = 0$ or $\sin\theta - \cos\theta = 0$

$\theta = 0°, 180°, 360°$ $\qquad \sin\theta = \cos\theta$

$\dfrac{\sin\theta}{\cos\theta} = 1$

$\tan\theta = 1$

$(\cos\theta \neq 0)$

$\theta = 45°, 22$

$\therefore \underline{\theta = (0°)\ 45°, 180°, 225°, (360°)}$

b) LHS $= \dfrac{2\tan A}{1+\tan^2 A}$

$= \dfrac{2\tan A}{\sec^2 A}$

$= \dfrac{2\sin A}{\cos A} \times \dfrac{\cos^2 A}{1}$

$= 2\sin A\cos A$

$= \sin 2A$

$=$ RHS.

$\therefore \underline{\dfrac{2\tan A}{1+\tan^2 A} = \sin 2A}$

c) $\sqrt{\dfrac{1+\cos 2x}{1-\cos 2x}} = \sqrt{\dfrac{1+(2\cos^2 x-1)}{1-(1-2\sin^2 x)}}$

$= \sqrt{\dfrac{2\cos^2 x}{2\sin^2 x}}$

$= \sqrt{\cot^2 x}$

$= \underline{|\cot x|}$

d) $\sin 195° = +\sin(180°+15°)$

$= -\sin 15°$ 3rd Qua

$= -\sin(45°-30°)$

$= -(\sin 45\cos 30 - \cos 45\sin$

$= -\left(\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2}\right)$

$= \dfrac{-\sqrt{3}}{2\sqrt{2}} + \dfrac{1}{2\sqrt{2}}$

$= \dfrac{1-\sqrt{3}}{}$

Question 2 (Cont.)

e) $\sqrt{3}\sin\theta - \cos\theta = 1$

Method 1.

$a\sin\theta + b\cos\theta = r\sin(\theta+\alpha)$

$r = \sqrt{a^2+b^2}$ $\qquad \tan\alpha = \frac{b}{a}$

$r = \sqrt{3+1}$ $\qquad = \frac{-1}{\sqrt{3}}$

$= 2$ $\qquad \alpha = -30^\circ$

$\therefore\ 2\sin(\theta-30)^\circ = 1$

$\sin(\theta-30) = \frac{1}{2}$

$\theta - 30 = 30,\ 150^\circ$

$\theta = 60^\circ,\ 180^\circ$

Method 2 (t formula)

$\sin\theta = \frac{2t}{1+t^2}$ $\qquad \cos\theta = \frac{1-t^2}{1+t^2}$ where $t = \tan\frac{\theta}{2}$

$\sqrt{3}\cos\theta - \cos\theta = 1$

$\frac{2\sqrt{3}t}{1+t^2} - \frac{1-t^2}{1+t^2} = 1$

$2\sqrt{3}t - 1 + t^2 = 1 + t^2$

$2\sqrt{3}t = 2$

$t = \frac{1}{\sqrt{3}}$

$\therefore\ \tan\frac{\theta}{2} = \frac{1}{\sqrt{3}}$ $\qquad 0 \le \frac{\theta}{2} \le 180^\circ$

$\frac{\theta}{2} = 30^\circ$

$\theta = 60^\circ$

Note that the "t method" fails when $\frac{\theta}{2} = \frac{180}{2}$, i.e. when $\theta = 180$ as $\tan\frac{180}{2}$ is undefined. We need to check $\theta = 180$ seperately

check $\theta = 180$

LHS $= \sqrt{3}\sin 180^\circ - \cos 180^\circ$

$= 0 + 1$

$= 1$

$=$ RHS $\qquad \therefore \theta = 180^\circ$ is also a soln.

$\therefore\ \theta = 60^\circ, 180^\circ$

Question 3

a) $|3+2x| = 4 - x$

$3+2x = 4-x$ or $3+2x = -(4-x)$

$3x = 1$ $\qquad 3+2x = -4+x$

$x = \frac{1}{3}$ $\qquad x = -7$

check $x = \frac{1}{3}$ RHS $= 4 - \frac{1}{3} = 3\frac{2}{3}$

$x = -7$ RHS $= 4+7 = 11$

$\therefore$ soln. $x = \frac{1}{3}, -7$.

b) (i) In ΔPAC

$\tan 20^\circ = \frac{100}{AC}$

$AC = \frac{100}{\tan 20}$

$\doteqdot 274.7$

(ii) In ΔABC

$AB^2 = AC^2 + BC^2 - 2.AC.BC.\cos 60^\circ$

$= 274.7^2 + 600^2 - 2\times 274.7 \times 600 \cos 6$

$\doteqdot 270640$

$AB = 520$

$\therefore$ dist between yachts $\doteqdot 520$ m.

c)

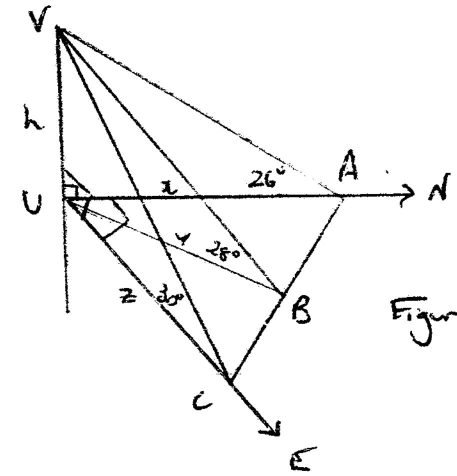


Figure 1

Let $UV = h$, $UA = x$, $UB = y$, $UC = z$

$\angle VUA = \angle VUB = \angle VUC = \angle AUC = 90^\circ$

$\therefore\ \angle UVA = 64^\circ$

$\angle UVB = 62^\circ$

$\angle UVC = 60^\circ$

In ΔUAV, $\tan 64° = \frac{x}{h}$

$\therefore x = h\tan 64°$

In ΔUBV, $\tan 62° = \frac{y}{h}$

$\therefore y = h\tan 62°$

In ΔUCV, $\tan 60° = \frac{z}{h}$

$z = h\tan 60°$

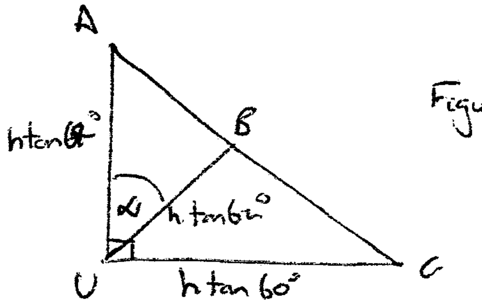


Figure 2

we need to find α

in ΔAUC, $\tan\angle UAC = \frac{\tan 60}{\tan 64}$

$\therefore \angle UAC = 40°11'$ (to nearest minute)

in ΔAUB (using the Sine Rule)

$$\frac{h\tan 62°}{\sin 40°11'} = \frac{h\tan 64°}{\sin\angle ABU}$$

$$\sin\angle ABC = \frac{\tan 64° \sin 40°11'}{\tan 62°}$$

$= 44°43'$ or $135°17'$ (to nearest minute)

hence $\alpha = (180° - 40°11' - 44°43')$ or $(180° - 40°11' - 135°17')$

$= 95°6'$ or $4°32'$

$\alpha \neq 95°6'$ as B lies between A and C

$\therefore \alpha = 4°32'$

ie the bearing of B from Uluru is $004°32'$

Question 4.

a) (i) $\frac{d}{dx}\left(\frac{x\sqrt{x}}{3}\right) = \frac{d}{dx}\left(\frac{x^{\frac{3}{2}}}{3}\right)$

$= \frac{\frac{3}{2}x^{\frac{1}{2}}}{3}$

$= \underline{\frac{\sqrt{x}}{2}}$

(ii) $\frac{d}{dx}\left(\frac{x^4 - 4x^3 + 5}{x^3}\right) = \frac{d}{dx}\left(x - 4 + \frac{5}{x^3}\right)$

$= \frac{d}{dx}\left(x - 4 + 5x^{-3}\right)$

$= 1 - 15x^{-4}$

$= \underline{1 - \frac{15}{x^4}}$

(iii) $\frac{d}{dx}\ 3(x^2+5)^7 = 3.7(x^2+5)^6$

$= \underline{42x(x^2+5)^6}$

b) $y = x^2 - 5x$

$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{(x+h)^2 - 5(x+h) - (x^2 - 5x)}{h}$

$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{5x} - 5h - \cancel{x^2} + \cancel{5x}}{h}$

$= \lim_{h\to 0} \frac{2xh + h^2 - 5h}{h}$

$= \lim_{h\to 0} 2x + h - 5$

$\therefore \frac{dy}{dx} = \underline{2x - 5}$

When $x = 3$ $\frac{dy}{dx} = 2(3) - 5$

$= 1$

hence when $x=3$ gradient $\underline{= 1}$

c) $\dfrac{2x-5}{x+3} \geq 1$ $\qquad \times (x+3)^2$

$x \neq -3$

$(x+3)(2x-5) \geq (x+3)^2$

$2x^2 - 5x + 6x - 15 \geq x^2 + 6x + 9$

$x^2 - 5x - 24 \geq 0$

$(x+3)(x-8) \geq 0$

$-3 \qquad 8$

$\therefore x \leq -3$ or $x \geq 8$

Question 5

a) (i) $\angle PRQ = \frac{1}{2} \angle POQ$

$= \frac{1}{2}\alpha$

(angle at centre is twice the angle at the circumference standing on the same arc.)

$\angle PRM = 180 - \frac{1}{2}\alpha$

($\angle QRM$ is a straight angle)

$\therefore \angle PRM = 180 - \frac{1}{2}\alpha$

(ii) Similarly $\angle QSM = 180 - \frac{1}{2}\alpha$

$\angle SNR + \angle NSM + \angle NRM + \angle RMS = 360^\circ$

(quadrilateral MRNS)

$\angle SNR + (180 - \frac{1}{2}\alpha) + (180 - \frac{1}{2}\alpha) + \angle RMS = 360$

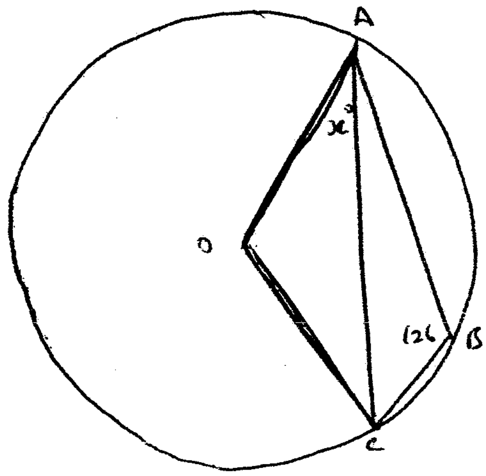
$\angle SNR + \angle RMS + 360 - \alpha = 360^\circ$

$\angle SNR + \angle RMS = \alpha$

but $\angle PNQ = \angle SNR$ (Vertically Opp.)

$\therefore \angle PNQ + \angle PMQ = \alpha$

b)



Reflex $\angle AOC = 2 \times \angle ABC$ (angle at centre is twice the angle at circumference standing on the same arc)

$= 252^\circ$

$\angle AOC = 360^\circ - 252$ (Revolution)

$= 108^\circ$

ΔAOC is isosceles ($OA = OC$ equal radii)

$\therefore \angle OAC = \angle OCA$ (angles opposite equal sides)

$x + x + 108^\circ = 180$ (Angle sum ΔAOC)

$x = 36^\circ$

Question 6

a) $m = \tan\alpha$

$m = \tan 135^\circ$ and $m = \frac{-a}{b}$

$= -1$

hence $\frac{-a}{b} = -1$

$\therefore a = b$ ——— ①

also $(2, -3)$ satisfies $y = ax^2 + bx + 7$

$\therefore -3 = 4a + 2b + 7$ ——— ②

using ① $-3 = 4a + 2a + 7$

$-3 = 6a + 7$

$-10 = 6a$

$a = \frac{-5}{3}$

$\therefore a = \frac{-5}{3} \quad b = \frac{-5}{3}$

b) $7x - 15y - 3 = 0$ ——— ①

$7x - 15y + 5 = 0$ ——— ②

Find a point on ② let $x = 10$

$70 - 15y + 5 = 0$

$75 = 15y$

$5 = y$

$\therefore (10, 5)$ lies on ②

find perp dist of $(10, 5)$ to ①

$$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}\right|$$

$$= \left|\frac{7(10) - 15(5) - 3}{\sqrt{7^2 + 15^2}}\right|$$

$$= \left|\frac{70 - 75 - 3}{\sqrt{49 + 225}}\right|$$

$$= \left|\frac{-[illegible]}{\sqrt{274}}\right|$$

$$= \frac{[illegible]}{\sqrt{274}} \text{ units}$$

([illegible])

c) $y = x$ —— ①

$y = x^3$ —— ②

for ① $\frac{dy}{dx} = 1 \quad \therefore m_1 = 1$

for ② $\frac{dy}{dx} = 3x^2$

at $x = 1 \quad m_2 = 3$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{1-3}{1+1\times 3}\right|$$

$$= \left|\frac{-2}{4}\right|$$

$$= \frac{1}{2}$$

$\therefore \alpha = 26^\circ 34'$ (nearest minute)

(26.565°)

d) $$\frac{a^3 (b^2)^4}{(a^{-1})^2 b^7} = \frac{a^3 b^8}{a^{-2} b^7}$$

$$= a^5 b$$

$$= \left(\frac{2}{3}\right)^5 \cdot \frac{4}{9}$$

$$= \frac{2^5}{3^5} \cdot \frac{4}{9}$$

$$= \frac{32}{243} \cdot \frac{4}{9}$$

$$= \frac{128}{2187}$$

e) $$\lim_{x\to 2} \frac{x-2}{x^2-4}$$

$$= \lim_{x\to 2} \frac{\cancel{x-2}\ ^1}{(x+2)\cancel{(x-2)}}$$

$$= \frac{1}{2+2}$$

$$= \frac{1}{4}$$

f) (i) $2f(-1) + f(0) - 2f(3)$

$= 2(1+1) + (0^2+1) - 2[2(3) - 1]$

$= 4 + 1 - 10$

$= -5$

(ii) $f(-3) = 1 + 3 = 4$

$f(3) = 2(3) - 1 = 5$