SOUTH SYDNEY HIGH SCHOOL

Year 11 Preliminary Course
June Exam 1998

MATHEMATICS

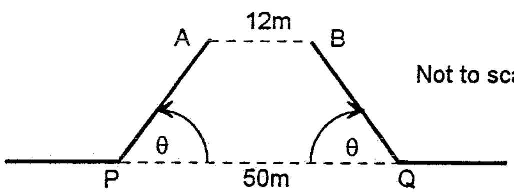
3 Unit

Instructions : **Time Allowed:** *2 periods*

1. All questions may be attempted.
2. Start each question on a new sheet of paper.
3. All necessary working should be shown.
4. Marks may be deducted for poorly arranged or missing working.
5. Approved calculators may be used.

Question 1 (Use a new page)

(a)



The figure shows the side view of a bridge opened to let boats pass underneath. When the equal arms of the bridge *PA* and *QB* are lowered, they meet exactly to form the straight roadway *PQ*, which is 50m long. When the arms *PA* and *QB* are raised through an angle θ as shown, the "corridor" *AB* is 12m wide

(b) Find the equation of the straight line through (3,-1) perpendicular to the line 3x - 2y -7 = 0

(c) The interval *AB* has endpoints *A*(-2,3) and *B*(10,11).
Find the coordinates of the point *P* which divides the interval *AB* in the ratio 3:1.

(d) For the function $y = \frac{-1}{\sqrt{x^2 - 9}}$ state the domain and range.

Question 2 (Use a new page)

(a) Solve for $0^o < \theta < 360^o$

(i) $\cot^2\theta - 3 = 0$ (Give answers in exact form)

(ii) $2\sin^2\theta = \sin 2\theta$

(b) Prove the following identity $\dfrac{2\tan A}{1+\tan^2 A} = \sin 2A$

(c) Express $\sqrt{\dfrac{1+\cos 2x}{1-\cos 2x}}$ as a single trigonometric ratio.

(d) Show that $\sin 195° = \dfrac{1-\sqrt{3}}{2\sqrt{2}}$, without the aid of a calculator

(e) Find all angles θ, where $0° \le \theta \le 360°$, for which $\sqrt{3}\sin\theta - \cos\theta = 1$

Question 3 (Use a new page)

(a) Differentiate the following

(i) $\dfrac{x\sqrt{x}}{3}$

(ii) $\dfrac{x^4 - 4x^3 + 5}{x^3}$

(iii) $3(x^2 + 5)^7$

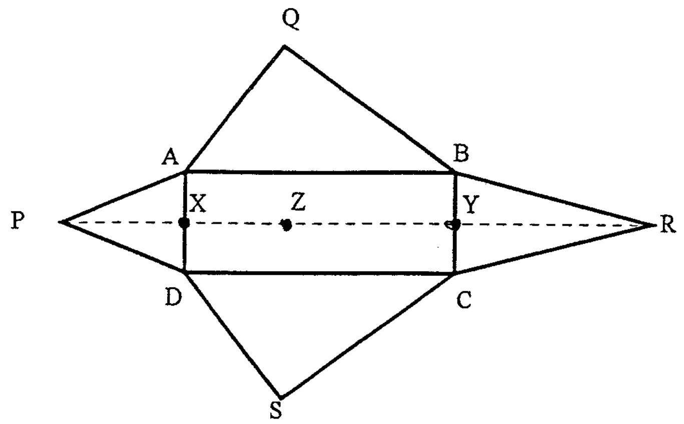
(iv) $(x+2)^3(3x^2 - 4x + 2)$

(b) Differentiate $y = x^2 - 5x$ using first principles and hence find the gradient of the curve when $x = 3$

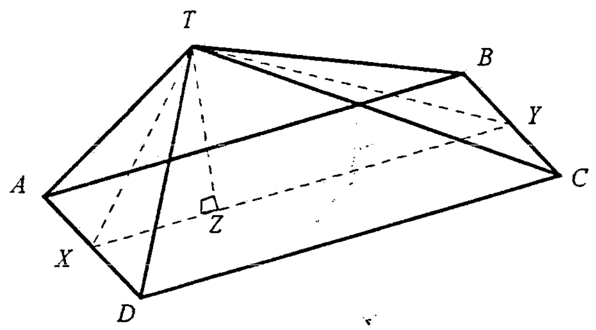
(c) Solve $\dfrac{2x-5}{x+3} \ge 1$

Question 4 (Use a new page)

(a)



The figure shows the net of an oblique pyramid with a rectangular base.
In this figure, $PXZYR$ is a straight line, $PX = 15$ cm, $RY = 20$ cm, $AB = 25$ cm, and $BC = 10$ cm.
Further, $AP = PD$ and $BR = RC$.
When the net is folded, points P, Q, R, and S all meet at the apex T, which lies vertically above the point Z in the horizontal base, as shown below.



(i) Show that triangle TXY is right-angled

(ii) Hence show that T is 12 cm above the base.

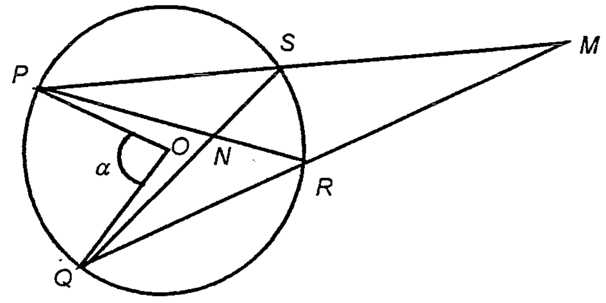
(iii) Hence find the angle that the face DCT makes with the base.

(b) The line $y = mx + b$ is a tangent to the curve $y = x^3 - 3x + 1$ at the point (-2, -1). Find m and b.

(c) Seventy-five tagged fish were released into a dam known to contain fish. Later a sample of forty-two fish was netted from this dam and then released. Of these forty-two fish it was noted that five were tagged.
Estimate the total number of fish in the dam.

Question 5 (Use a new page)

(a)

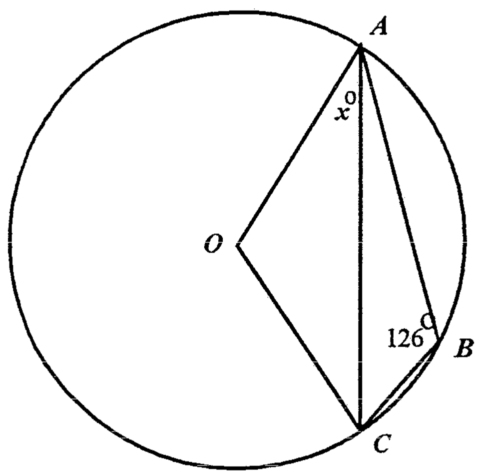


In the diagram P, Q, R and S are the points on a circle centre O, and $\angle PQR = \alpha$. The lines PS and QR intersect at M and the lines QS and PR intersect at N.

(i) Explain why $\angle PRM = 180 - \frac{1}{2}\alpha$

(ii) Show that $\angle PNQ + \angle PMQ = \alpha$

(b)



O is the centre of the circle
$\angle ABC = 126°$, $\angle OAC = x°$.

Copy the diagram and find the value of x, giving reasons.

Question 6 (Use a new page)

(a) At the point (2,-3) on the curve $y = ax^2 + bx + 7$, the tangent is inclined at 135° to the x-axis. Find the values of a and b.

(b) Find the distance between the parallel lines $7x - 15y - 3 = 0$ and $7x - 15y + 5 = 0$.

(c) The graphs of $y = x$ and $y = x^3$ intersect at $x = 1$.
Find the size of the acute angle between these curves at $x = 1$.

(d) Simplify $$\frac{a^3\left(b^2\right)^4}{\left(a^{-1}\right)^2 b^7}$$ if $a = \frac{2}{3}$ and $b = \frac{4}{9}$

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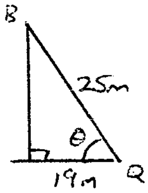
Question 1

a)



$\cos\theta = \frac{19}{25}$

$\theta = 40°32'$

b) $3x - 2y - 7 = 0$

$m = \frac{-a}{b} = \frac{3}{2}$ $\quad (3, -1)$

$y - y_1 = m(x - x_1)$ $\quad m_p = \frac{-2}{3}$

$y + 1 = \frac{-2}{3}(x - 3)$

$3y + 3 = -2x + 6$

$2x + 3y - 3 = 0$

c) $P = \left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n}\right)$

$= \left(\frac{1(-2) + 3(10)}{3+1}, \frac{1(3) + 3(11)}{3+1}\right)$

$= \left(\frac{-2+30}{4}, \frac{3+33}{4}\right)$

$\therefore P = \left(\frac{28}{4}, \frac{36}{4}\right) = (7, 9)$

d) Domain : $x < -3$ or $x > 3$

Range : $y < 0$

Question 2

(i) $\cot^2\theta - 3 = 0$

$\cot^2\theta = 3$

$\tan^2\theta = \frac{1}{3}$

$\tan\theta = \pm\frac{1}{\sqrt{3}}$

$\therefore \theta = 30°, 150°, 210°, 330°$

(ii) $2\sin^2\theta = \sin 2\theta$

$2\sin^2\theta - 2\sin\theta\cos\theta = 0$

$2\sin^2\theta(\sin\theta - \cos\theta) = 0$

$\therefore 2\sin^2\theta = 0$ or $\sin\theta = \cos\theta$

$\sin\theta = 0$ $\quad \tan\theta = 1$

$\theta = 180°$ $\quad \theta = 45°, 225°$

$\therefore \theta = 45°, 180°, 225°$

b) $\frac{2\tan A}{1+\tan^2 A} = \sin 2A$

$LHS = \frac{2\tan A}{1+\tan^2 A}$

$= \frac{2\tan A}{\sec^2 A}$

$= 2\frac{\sin A}{\cos A} \div \frac{1}{\cos^2 A}$

$= \frac{2\sin A}{\cos A} \times \frac{\cos^2 A}{1}$

$= 2\sin A\cos A$

$= \sin 2A$

$= RHS.$

c) $\sqrt{\frac{1+\cos 2x}{1-\cos 2x}} = \sqrt{\frac{1+2\cos^2 x - 1}{1 - 1 + 2\sin^2 x}}$

$= \sqrt{\frac{2\cos^2 x}{2\sin^2 x}}$

$= \cot x$

d) $\sin 195° = \sin(150° + 45°)$

$= \sin 150\cos 45 + \cos 150 \sin 45°$

$= \sin 30\cos 45 - \cos 30\sin 45°$

$= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}$

$= \frac{1-\sqrt{3}}{2\sqrt{2}}$

e) $\sqrt{3}\sin\theta - \cos\theta = 1$

if $a\sin\theta + b\cos\theta = r\sin(\theta + \alpha)$

$a = \sqrt{3}$, $b = -1$

$r = \sqrt{3+1} = 2$

$\tan\alpha = \frac{b}{a} = \frac{-1}{\sqrt{3}}$

sin -ve, cos +ve $\therefore$ 4th Quad

$\alpha = (360 - 30)°$

$\therefore \alpha = 300°$ or $-30°$

$\therefore \sqrt{3}\sin\theta - \cos\theta = 2\sin(\theta - 30°)$

$2\sin(\theta - 30)° = 1$

$\sin(\theta - 30)° = \frac{1}{2}$

$\theta - 30° = 30°, 150°$

$\theta = 60°, 180°$

Question 3

(a)(i) $y = \frac{x\sqrt{x}}{3} = \frac{x^{\frac{3}{2}}}{3}$

$y' = \frac{3}{2} \cdot \frac{x^{\frac{1}{2}}}{3} = \frac{\sqrt{x}}{2}$

(ii) $y = \frac{x^4 - 4x^3 + 5}{x^3}$

$y = x - 4 + 5x^{-3}$

$y' = 1 - 15x^{-4}$

$y' = 1 - \frac{15}{x^4}$ (or $\frac{x^4 - 15}{x^4}$

(iii) $y = 3(x^2+5)^7$

$y' = 21(x^2+5)^6 \cdot 2x$

$y' = 42x(x^2+5)^6$

(iv) $y = (x+2)^3(3x^2 - 4x + 2)$

$y' = (x+2)^3(6x - 4) + (3x^2 - 4x + 2) \cdot 3(x+2)^2 \cdot 1$

$= (x+2)^2[(x+2)(6x-4) + 3(3x^2 - 4x$

$= (x+2)^2[6x^2 + 8x - 8 + 9x^2 - 12x + 6$

$= (x+2)^2(15x^2 - 4x - 2)$

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b) $y = x^2 - 5x$

$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{(x+h)^2 - 5(x+h) - (x^2 - 5x)}{h}$

$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 5x - 5h - x^2 + 5x}{h}$

$= \lim_{h \to 0} 2x + h - 5$

$\frac{dy}{dx} = 2x - 5$

$x = 3 \quad m = 2(3) - 5$

$\therefore m = 1$

c) $\frac{2x-5}{x+3} \geq 1$

$(2x-5)(x+3) \geq (x+3)^2$

$2x^2 + x - 15 \geq x^2 + 6x + 9$

$x^2 - 5x - 24 \geq 0$

$(x-8)(x+3) \geq 0$

$\therefore x \leq -3$ or $x \geq 8$

Question 4

a) (i) When folded $TX = PX = 15$ cm

$TY = RY = 20$ cm

$XY = AB = 25$ cm

In ΔTXY, $TX^2 + TY^2 = 15^2 + 20^2$

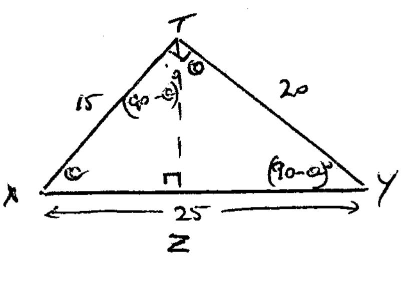
$= 625$

$= 25^2$

$= XY^2$

∴ Pythagoras' thm is satisfied hence ΔTXY is right angled at T

(ii)



Note $\Delta TXZ \;|||\; \Delta YTZ \;|||\; \Delta YXT$ (equiangular)

$\therefore \frac{TZ}{TX} = \frac{YT}{YX}$ (corresponding sides in same ratio)

$\frac{TZ}{15} = \frac{20}{25}$

$TZ = 12$

∴ T is 12 cm above the base

OR Area $\Delta TXY = \frac{1}{2} \times 15 \times 20$

and Area $\Delta TXY = \frac{1}{2} \times 25 \times h$

$\therefore 25h = 15 \times 20$

$h = \frac{15 \times 20}{25}$

$= 12$ cm.

OR Let $\angle TXY = \theta$

then $\sin\theta = \frac{TY}{XY} = \frac{20}{25} = \frac{4}{5}$

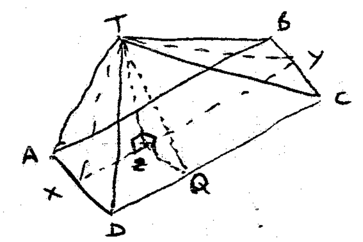
In ΔTZX, $\sin\theta = \frac{TZ}{XT} = \frac{TZ}{15}$

$\therefore \frac{TZ}{15} = \frac{4}{5}$

$TZ = \frac{4}{5} \times 15$

$= 12$ cm

(iii)



X, Y are midpoints of AD, BC

$\therefore XD = YC = ZQ = 5$ cm

Angle between face DCT and base is given by $\angle T$

$\tan \angle TQZ = \frac{TZ}{ZQ} = \frac{12}{5}$

$\therefore \angle TQZ = 67°23'$

b) $y = x^3 - 3x + 1$

$y' = 3x^2 - 3$

at $(-2, -1)$ $m = 3(-2)^2 - 3$

$= 9$

eqn. tangent $y - y_1 = m(x - x_1)$

$y + 1 = 9(x + 2)$

$y + 1 = 9x + 18$

$y = 9x + 17$

$\therefore m = 9, \; b = 17$

c) sample $5 : 42$

dam $75 : x$

if sample is representative then $\frac{5}{42} = \frac{75}{x}$

$x = 75 \times \frac{42}{5}$

$= 630$

∴ 630 fish in dam

Question 5

a)(i) $\angle PRQ = \frac{1}{2}\angle POQ$ (angle at the circumference is ½ the angle at the centre standing on the same arc)

$\therefore \angle PRQ = \frac{1}{2}\alpha$

$\therefore \angle PRM = 180 - \frac{1}{2}\alpha$ ($\angle QRM$ is a straight angle)

(ii) Similarly from (i)

$\angle QSM = 180 - \frac{1}{2}\alpha$

In Quad NSMR

$\angle NSM + \angle SMR + \angle MRN + \angle RNS = 360°$

$\therefore \angle QSM + \angle PMQ + \angle PRM + \angle RNS = 360°$

$\therefore (180 - \frac{1}{2}\alpha) + \angle PMQ + (180 - \frac{1}{2}\alpha) + \angle RNS = 36$

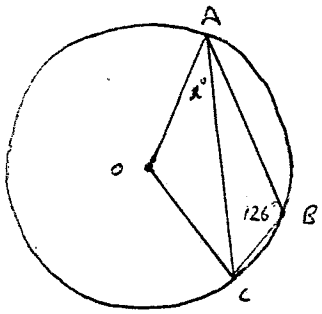
$\angle PMQ + \angle RNS + 360 - \alpha = 360$

$\angle PMQ + \angle RNS = \alpha$

but $\angle PNQ = \angle RNS$ (vertically opp)

so $\angle PMQ + \angle PNQ = \alpha$.

b)



Reflex $\angle AOC = 2\angle ABC$

(angle at centre twice angle at circumference on same arc)

$= 252°$

$\therefore$ obtuse $\angle AOC = 360 - 252°$ (angles at point)

$= 108°$

ΔAOC is isosceles as $AO = OC$ (equal radii)

$\therefore \angle OAC = \angle OCA = x$ (base angles ΔAOC)

$\therefore 2x + 108° = 180°$ (angle sum ΔAOC)

$2x = 72°$

$\therefore x = 36°$

Question 6

a) $m = \tan\theta$

$= \tan 135°$

$= -1$

$y' = 2ax + b$

but $f'(2) = -1 \quad \therefore 4a + b = -1$ ①

and at $(2,-3)$ $\quad -3 = a(2)^2 + b(2) + 7$

$-3 = 4a + 2b + 7$

$4a + 2b = -10$ ②

① − ② $\quad -b = 9$

$b = -9$

$\therefore 4a - 9 = -1$

$4a = 8$

$\therefore a = 2, \; b = -9$

b) for $7x - 15y + 5 = 0$

if $y = -2 \quad 7x + 30 + 5 = 0$

$7x = -35$

$x = -5$

$\therefore (-5,-2)$ lies on $7x - 15y + 5 = 0$

find perp. dist of $(-5,-2)$ from

$7x - 15y - 3 = 0$

$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}\right|$

$= \left|\frac{7(-5) + 15(-2) - 3}{\sqrt{7^2 + 15^2}}\right|$

$= \left|\frac{-35 + 30 - 3}{\sqrt{45 + 225}}\right|$

$= \left|\frac{-8}{\sqrt{270}}\right|$

$= \frac{8}{3\sqrt{30}}$ units

c) $y = x \qquad y = x^3$

$y' = 1 \qquad y' = 3x^2$

$f'(1) = 3$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \frac{3-1}{1+(3)(1)}$

$= \frac{2}{4}$

$= \frac{1}{2}$

$\theta = 26°34'$

d) $\frac{a^3(b^2)^4}{(a^{-1})^2 b^7} = \frac{a^3 b^8}{a^{-2} b^7}$

$= a^5 b$

$= \left(\frac{2}{3}\right)^5 \cdot \frac{4}{9}$

$= \frac{2^5}{3^5} \cdot \frac{2^2}{3^2}$

$= \frac{2^7}{3^7}$

$= \frac{128}{2187}$