SOUTH SYDNEY HIGH SCHOOL

SPECTEMUR AGENDO

Year 11

Preliminary Course

Assessment Task No. 1

20th November 1997

MATHEMATICS

3 UNIT

Time allowed - 2 PERIODS

Examiner : P. Ooi

DIRECTIONS TO CANDIDATES

- This assessment is worth **10%.**
- Board-approved calculators may be used.
- Write your Name on **EVERY** page .
- Attempt all **FIVE** questions.
- Start each question on a **NEW** page.

Question 1 (12 marks)	**Marks**
(a) (i) Expand $\sin(x-y)$	1
(ii) Hence show that $\sin 15^0 = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$	2
(b) If $\sin\alpha = \dfrac{4}{5}$ where α is acute, find the exact value of	4
(i) $\cos 2\alpha$ (ii) $\cot 2\alpha$	
(c) If $\sin(x-\alpha) = \cos(x-\beta)$, find an expression for $\tan x$ in terms of α and β.	2
(d) Show that $\dfrac{1}{\cos x + \sin x} + \dfrac{1}{\cos x - \sin x} = \dfrac{\tan 2x}{\sin x}$	3

Question 2 (12 marks)

(a) Solve the equations for $0^0 \le \theta \le 360^0$	5
(i) $\cos(\theta + 45^0) = \dfrac{1}{2}$ (ii) $\cos 2\theta + \cos\theta = 0$	
(b)(i) Express $3\sin\theta + 4\cos\theta$ in the form $R\sin(\theta+\alpha)$	2
(ii) Hence solve the equation $3\sin\theta + 4\cos\theta = 2$ for $0^0 \le \theta \le 360^0$ (Answer in degrees and minutes).	2
(c) Prove that $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = t$ where $t = \tan\dfrac{\theta}{2}$	3

Question 3 (11 marks)

(a) Find the coordinates of the point $P(x,y)$ which divides the line joining $A(-2,0)$ and $B(3,-7)$ externally in the ratio $AP:PB = 1:2$. **2**

(b) Find the acute angle (in degrees and minutes) between the lines $2x+y-7=0$ and $3x-y+2=0$. **2**

(c) Write down the cartesian equation of the curve given by : **4**

(i) $x = 2-t$ and $y = t^2+1$

(ii) $x = 2\cos t$ and $y = \cos 2t$

(d) Find the equation of (i) the tangent, (ii) the normal to the parabola $x^2 = 8y$ at the point $\left(2\sqrt{2}\,a, a^2\right)$. **3**

Question 4 (13 marks)

(a) Find the coordinates of the vertex, focus and directrix of the parabola **5**

(i) $x^2+20y=0$ (ii) $x^2+2x-8y-15=0$

(b) Prove by mathematical induction that :

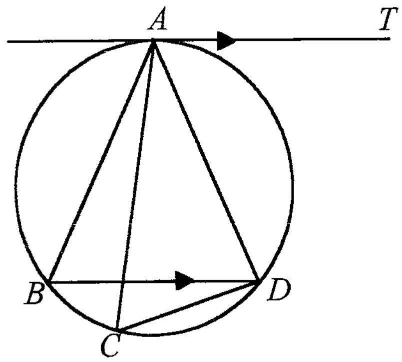
(i) $\sum_{r=1}^{n} \frac{1}{(4r-3)(4r+1)} = \frac{n}{4n+1}$ for all positive n **4**

(ii) $n(n+3)$ is always even for all positive n. **4**

Question 5 (12 marks) **Marks**

(a) In the figure AT is a tangent and $AT \| BD$. **2**

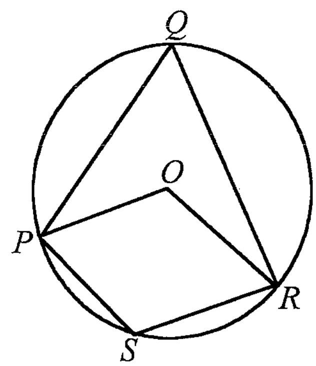
$\angle ADB = 54^0$, find $\angle ACD$ (giving reasons).



(b) O is the centre of the circle. **3**

If $\angle POR = \angle PSR$, prove that

they are both 120^0.



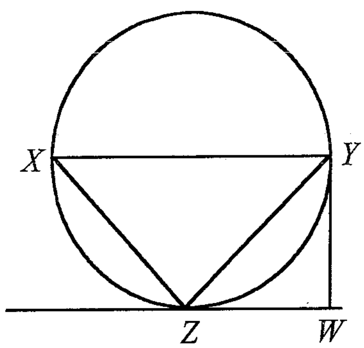
(c) XY is a diameter of a circle. **3**

Z is the point on the circle.

$YW \perp ZW$ which is a tangent at Z.

(i) Prove that YZ bisects $\angle XYW$.

(ii) Hence, or otherwise, prove that $YZ^2 = XY \times YW$



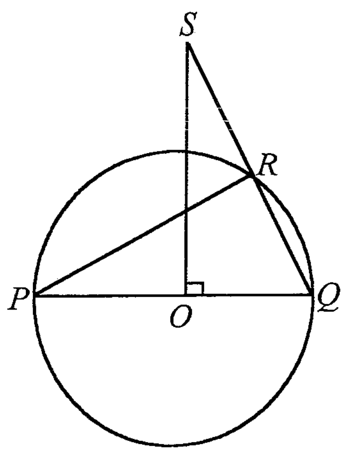
(d) PQ is a diameter of a circle with centre O. **4**

R is a point on its circumference.

The perpendicular to PQ at O, meets

QR produced at S.

Prove that $PORS$ is a cyclic quadrilateral.



End of paper

98%

Excellent effort! [redacted]

Q 1.

(a)(i) $\sin(\alpha - y) = \sin\alpha\cos y - \cos\alpha\sin y$ ✓

(ii) $\sin 15° = \sin(60 - 45)$

$= \sin 60 \cdot \cos 45 - \cos 60 \sin 45$ ✓

$= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$ ✓

$= \frac{\sqrt{3} - 1}{2\sqrt{2}}$

12

(b) $\sin\alpha = \frac{4}{5} \qquad 0 \le \alpha \le 90$

(i) $\cos 2\alpha = 1 - 2\sin^2\alpha = 1 - 2\left(\frac{4}{5}\right)^2 = 1 - 2 \cdot \frac{16}{25} = 1 - \frac{32}{25}$ ✓

$= -\frac{7}{25}$ ✓

(ii) $\cot 2\alpha = \frac{1}{\tan 2\alpha} = \frac{\cos 2\alpha}{\sin 2\alpha}$.

$\sin 2\alpha = 2\sin\alpha\cos\alpha$.

$\cos\alpha = \sqrt{1 - \sin^2\alpha} = \sqrt{1 - \frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$. ✓

$\therefore \sin 2\alpha = 2\sin\alpha \cdot \cos\alpha = 2 \cdot \frac{4}{5} \cdot \frac{3}{5} = \frac{24}{25}$

$\therefore \cot 2\alpha = \frac{\cos 2\alpha}{\sin 2\alpha} = \frac{-\frac{7}{\cancel{25}}}{\frac{24}{\cancel{25}}} = -\frac{7}{24}$. ✓

(c) $\sin(x-\alpha) = \cos(x-\beta)$

$\sin x \cos\alpha - \cos x \sin\alpha = \cos x \cdot \cos\beta + \sin x \sin\beta$

$\sin x(\cos\alpha - \sin\beta) = \cos x(\cos\beta + \sin\alpha)$ ✓

$\frac{\sin x}{\cos x} = \tan x = \frac{\cos\beta + \sin\alpha}{\cos\alpha - \sin\beta}$ ✓

(d) $\frac{1}{\cos x + \sin x} + \frac{1}{\cos x - \sin x}$

$= \frac{\cos x - \cancel{\sin x} + \cos x + \cancel{\sin x}}{(\cos x + \sin x)(\cos x - \sin x)}$ ✓

$= \frac{2\cos x}{\cos^2 x - \sin^2 x} = \frac{2\cos x}{\cos 2x} = \frac{1}{\cos 2x} \times \frac{2\cos x}{1}$ ✓

$= \frac{\sin 2x}{\cos 2x} \times \frac{2\cos x}{\sin 2x} = \tan 2x \times \frac{\cancel{2\cos x}}{\cancel{2\cos x} \cdot \sin x}$ ✓

$= \frac{\tan 2x}{\sin x}$

Q2.

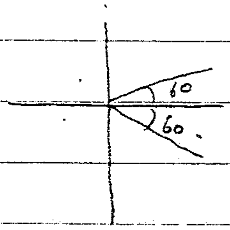
12

(a) $0 \leq \theta \leq 360$

(i) $\cos(\theta + 45) = \frac{1}{2}$ $\quad 45^\circ \leq \theta + 45^\circ \leq 360^\circ + 45^\circ$

$\therefore \theta + 45 = 60, 300$ ✓

$\therefore \theta = 15^\circ, 255^\circ$ ✓



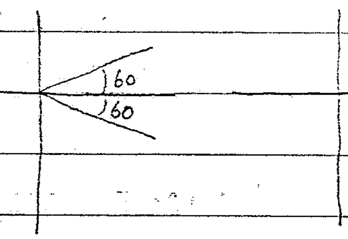
(ii) $\cos 2\theta + \cos\theta = 0$.

$2\cos^2\theta - 1 + \cos\theta = 2\cos^2\theta + \cos\theta - 1 = 0$.

$(2\cos\theta - 1)(\cos\theta + 1) = 0$. ✓

$\therefore \cos\theta = \frac{1}{2}, -1$. ✓

$\therefore \theta = 60^\circ, 300^\circ, 180^\circ$ ✓



(b) $3\sin\theta + 4\cos\theta = \sqrt{3^2+4^2}\left(\frac{3}{\sqrt{3^2+4^2}}\sin\theta + \frac{4}{\sqrt{3^2+4^2}}\cos\theta\right)$

$= 5\sin(\theta + \alpha)$. ✓

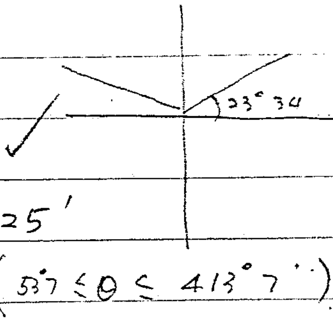
$\tan\alpha = \frac{b}{a} = \frac{4}{3}$ $\quad \therefore \alpha = 53^\circ 7'$

$\therefore 5\sin(\theta + 53^\circ 7')$ ✓

(ii) $3\sin\theta + 4\cos\theta = 2 \qquad 0 \leq \theta \leq 360$

$5\sin(\theta + 53^\circ 7') = 2$

$\therefore \sin(\theta + 53^\circ 7') = \frac{2}{5}$

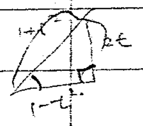


$\therefore \theta + 53^\circ 7' = \cancel{23^\circ 34}, \; 156^\circ 25'$

$383^\circ 34' \qquad (53^\circ 7 \leq \theta \leq 413^\circ 7')$

$\therefore \theta = 156^\circ 25' - 53^\circ 7', \; 383^\circ 34' - 53^\circ 7'$

$= 103^\circ 18', \; 330^\circ 27'$



(C) $\frac{1+\sin\theta - \cos\theta}{1+\sin\theta+\cos\theta} = \frac{\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\cos\theta}}{\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\cos\theta}} = \frac{\cot\theta + \cancel{\tan\theta} - 1}{\cot\theta + \tan\theta + 1}$

$\tan\theta = \frac{2t}{1-t^2}$

$\therefore = \frac{\frac{1-t^2}{2t} + \frac{2t}{1-t^2} - 1}{\frac{1-t^2}{2t} + \frac{2t}{1-t^2} + 1}$

$= \frac{\frac{(1-t^2)^2 + (2t)^2 - 2t(1-t^2)}{2t \cdot \cancel{(1-t^2)}}}{\frac{(1-t^2)^2 + (2t)^2 + 2t(1-t^2)}{2t \cdot \cancel{(1-t^2)}}}$

$= \frac{(1-t^2)^2 + (2t)^2 - 2t(1-t^2)}{(1-t^2)^2 + (2t)^2 + 2t(1-t^2)}$

[redacted]

$$= \frac{1 + t^4 - 2t^2 + 4t^2 - 2t + 2t^3}{1 + t^4 - 2t^2 + 4t^2 + 2t - 2t^3}$$

$$= \frac{t^4 + 2t^3 + 2t^2 - 2t + 1}{t^4 - 2t^3 + 2t^2 + 2}$$

(c) $\dfrac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta} = t$

$$\sin\theta = \frac{2t}{1+t^2}, \quad \cos\theta = \frac{1-t^2}{1+t^2}.$$

$$\text{L.H.S} = \frac{1 + \frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} = \frac{\frac{2t - 1 + t^2 + 1 + t^2}{\cancel{1+t^2}}}{\frac{2t + 1 - \cancel{t^2} + 1 + \cancel{t^2}}{\cancel{1+t^2}}}$$

$$= \frac{2t^2 + 2t}{2t + 2} = \frac{t(\cancel{2t+2})}{(\cancel{2t+2})} = t = \text{R.H.S}$$

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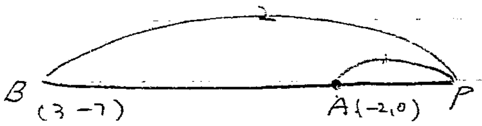
Q3.

(a)



$$\therefore P\left(\frac{\overset{-4\ -3}{2\times -2 - (1\times 3)}}{2-1}, \frac{2\times 0 - (1\times(-7))}{2-1}\right)$$

$$\therefore P(-7, 7)$$

(b) $m_1 = -2$, $m_2 = 3$.

$$\therefore \tan\theta = \left|\frac{-2-3}{1+-2\cdot 3}\right| = \left|\frac{-5}{-5}\right| = +1$$

$$\therefore \tan\theta = \cancel{a}\ 1 \qquad \therefore \theta = 45^\circ$$

(c) (i) $t = 2-x$

$$\therefore y = (2-x)^2 + 1 = 4 + x^2 - 4x + 1$$

$$\therefore y = x^2 - 4x + 5$$

(ii) $x = 2\cos t \qquad y = 2\cos^2 t - 1$

$$\therefore \cos t = \frac{x}{2}$$

$$\therefore y = 2\cdot\left(\frac{x}{2}\right)^2 - 1 = 2\cdot\frac{x^2}{4} - 1 = \frac{x^2}{2} - 1$$

(d) $y = \frac{x^2}{8}$ $\quad \frac{\Delta y}{\Delta x} = \frac{2x}{8} = \frac{x}{4}$. $\quad (2\sqrt{2}a, a^2)$

$\therefore m_1 = \frac{2\sqrt{2}a}{4}$

$\frac{2\sqrt{2}a}{4} \times m_2 = -1 \quad \therefore m_2 = \frac{-4}{2\sqrt{2}a}$ ✓

$\therefore$ tangent : $(y - a^2) = \frac{2\sqrt{2}a}{4}(x - 2\sqrt{2}a)$

$y = \frac{2\sqrt{2}}{4}a \cdot x - \frac{8a^2}{4} + a^2$

$y = \frac{\sqrt{2}}{2}a x - a^2$. ✓

$\therefore$ normal : $(y - a^2) = \frac{-4}{2\sqrt{2}a}(x - 2\sqrt{2}a)$

$\therefore y = \frac{-4}{2\sqrt{2}a}x + 4 + a^2$.

$\therefore y = \frac{-4\sqrt{2}}{4a}x + 4 + a^2 = -\frac{\sqrt{2}}{a}x + a^2 + 4$ ✓

11 1/2

Q4

a) $x^2 + 20y = 0$

$x^2 = -4 \cdot 5y$.

vertex $(0,0)$, focus $(0,-5)$, directrix $(0,5)$ $\quad y=5$

b) $(x+1)^2 = 8y + 15 + 1$

$(x+1)^2 = 4 \cdot 2(y+2)$ $\quad y = -4$

vertex $(-1,-2)$ focus $(-1, 0)$ ✓, directrix $(-1,-4)$

(b)

(i)

① Prove true for ~~n~~=1.

$$\frac{1}{(4-3)(4+1)} = \frac{1}{5} = \frac{1}{4+1}$$

∴ proven

② Assume that true for ~~n~~=k.

$$\sum_{r=1}^{k} \frac{1}{(4k-3)(4k+1)} = \frac{k}{4k+1}$$

③ Prove true for n=k+1

$$\underbrace{\sum_{r=1}^{k} \frac{1}{(4k-3)(4k+1)}}_{\text{Assumption} = \frac{k}{4k+1}} + \frac{1}{[4(k+1)-3][4(k+1)+1]} = \frac{k+1}{4(k+1)+1}$$

$$L.H.S = \frac{k}{4k+1} + \frac{1}{(4k+1)(4k+5)} = \frac{k(4k+5)+1}{(4k+1)(4k+5)}$$

$$= \frac{4k^2+5k+1}{(4k+1)(4k+5)} = \frac{(4k+1)(k+1)}{(4k+1)(4k+5)}$$

$$= \frac{k+1}{4k+5}$$

$$R.H.S = \frac{k+1}{4k+4+1} = \frac{1}{4k+5}$$

∴ Proven true for n=k, n=k+1. So, the statement is true for all positive n.

[redacted]

(ii) $n(n+3) = 2m$ (m is integer)

① Prove true for $n=1$

$1 \cdot (1+3) = 4 = 2 \cdot 2$

true

② Assume true for $n=k$

$k(k+3) = 2m$

③ prove true for $n=k+1$

$(k+1)(k+1+3) = (k+1)(k+4) = k^2+5k+4$

$k^2+3k.$

$= k(k+3) + 2k+4$

$= 2m + 2k+4$

$= 2(m+k+2)$ ∴ true

proven for $n=k$, $n=k+1$ ∴ the statement is true for all positive n

Q5.

12

(a) $\angle ADB = \angle DAT = 54°$ (alterate angle , $AT \parallel BD$)

$\angle DAT = \angle DBA = 54°$ ($\angle$'s in altermate segment)

$\angle DBA = \angle DCA = 54°$ ($\angle$'s in same segment)

(b) Let $\angle POR = \angle PSR = \alpha$

reflex $\angle POR = 360 - \alpha$ ($\angle$'s in revolution)

reflex $\angle POR = 2 \times \angle PSR$ ($\angle$ at centre is double $\angle$ at circumference)

$\therefore 360 - \alpha = 2\alpha$. $3\alpha = 360$

$\therefore \alpha = 120$.

(c) (i) $\angle ZWY = 90°$ (given)

$\angle XZY = 90°$ ($\angle$'s in semi-circle)

$\angle YZW = \angle ZXY$ ($\angle$'s in altermate segnent)

$\therefore \triangle XYZ$ and $\triangle ZYW$ are simillar (two correspondi angles are equal)

$\therefore \angle XYZ = \angle ZYX$.

$\therefore YZ$ bisects $\angle XYW$

(ii) $\triangle XYZ \ ||| \ \triangle ZYW$

$\therefore \frac{YZ}{YW} = \frac{XY}{YZ}$

$\therefore YZ^2 = XY \times YW$

In $\triangle SOQ$, $\triangle PRQ$

(d) $\angle SOQ = 90^\circ$, $\angle QRP = 90^\circ$ ($\angle$'s in semi-circle)

~~$\angle PQR = \angle SQO$ (common)~~

$\therefore \triangle PRQ \ ||| \ \triangle SOQ$ (two corresponding angles are equal)

$\therefore \angle RPQ = \angle OSQ$. (~~Reason~~)

$\therefore$ PORS is a cyclic quadrilateral

($\because \angle RPQ = \angle OSQ$ which are in same segment)