Student Name / Number ____________________

SOUTH SYDNEY HIGH SCHOOL

SPECTEMUR AGENDO

Preliminary Examination

2000

MATHEMATICS

Extension 1

Instructions :

Time Allowed: 2 *hours*
(plus 5 mins reading time)

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown.
- Marks may be deducted for poorly arranged or missing working.
- Approved calculators may be used.

Question 1. **Marks**

(a) Solve the equation for x in the simplest surd form, if **3**

$$x^2 - 2x - 17 = 0$$

(b) Solve for x : $\dfrac{4}{5-x} \geq 1$ **2**

(c) Find the values of a, b and c if **3**

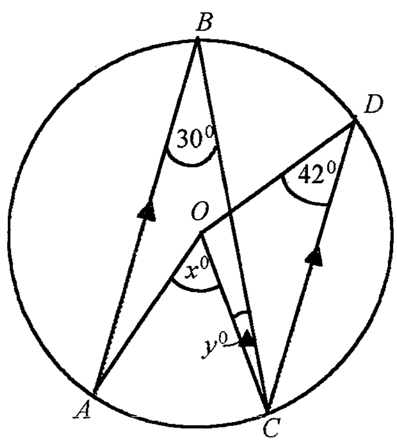
$$7x^2 + 3x - 6 \equiv ax(x+1) + bx^2 + c(x+3)$$

(d) Prove that $4 + 4\cot^2\theta = \dfrac{4}{1-\cos^2\theta}$ **2**

(e) Solve for $0 \leq \theta \leq 360^0$, if $\sin 2\theta = -\dfrac{\sqrt{3}}{2}$ **2**

Question 2.

(a) **4**



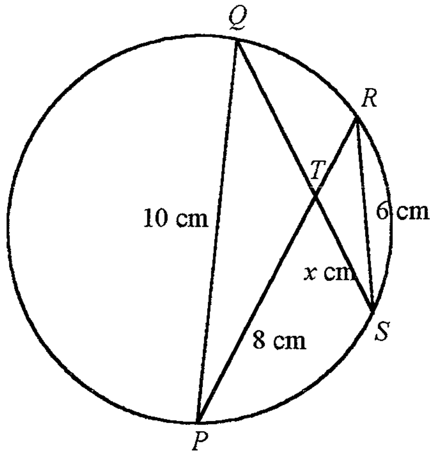
In the diagram above, O is the centre of a circle with chords $AB \| CD$.
$\angle ABC = 30^0$ and $\angle ODC = 42^0$.

Find the values of x (i.e. $\angle AOC$) and y (i.e. $\angle OCB$), giving reasons for your answers.

Continue ....

Question 2 (continued) **Marks**

(b) **4**

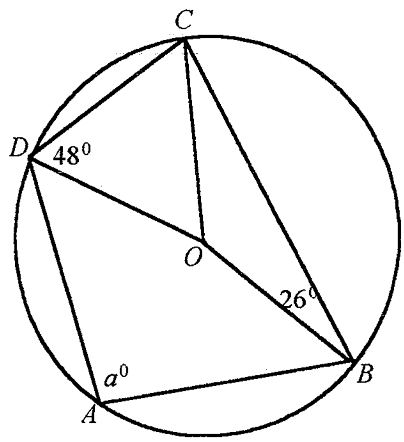


In the diagram above, PR intersects QS at T, $PQ = 10$ cm, $PT = 8$ cm and $SR = 6$ cm.

(i) Prove that ΔPQT and ΔSRT are similar.

(ii) Hence, find the length of ST.

(c) **4**



In the diagram above, O is the centre of the circle with $\angle CDO = 48^0$ and $\angle CBO = 26^0$.

Find $\angle DAB$ giving reasons for your answer.

Continue ....

Question 3. **Marks**

(a) Find the following limits : **4**

(i) $\lim\limits_{x\to 3}\dfrac{x^2-2x-3}{x-3}$

(ii) $\lim\limits_{x\to\infty}\dfrac{3x^2+4x-5}{4x^2-1}$

(b) If $\dfrac{dy}{dx}=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$, find the derivative of $f(x)=2x^2-1$ **3**

from first principles.

(c) Differentiate the following with respect to x. **5**

(i) $y=\dfrac{x^5}{5}-4x+8$

(ii) $y=\dfrac{2x^2+3x-1}{x}$

(iii) $y=(4x+1)^2$

Question 4.

(a) Find the coordinates of the points where the tangents to the curve **4**

$$y=\frac{x^3}{3}+\frac{x^2}{2}-2x+5$$

are parallel to the x–axis.

(b) Find the equation of the normal to the curve $y=(x-1)\sqrt{x}$ **3**
at the point where $x=4$.

(c) If $y=2x^3+\dfrac{7x^2}{2}-3x+7$, solve the equation $x.f'(x)=0$ **3**

Continue ....

Question 3 (continued) **Marks**

(d) Find the value of a and b, if **2**

$(x+a)^2 + b = x^2 + 4x - 5$

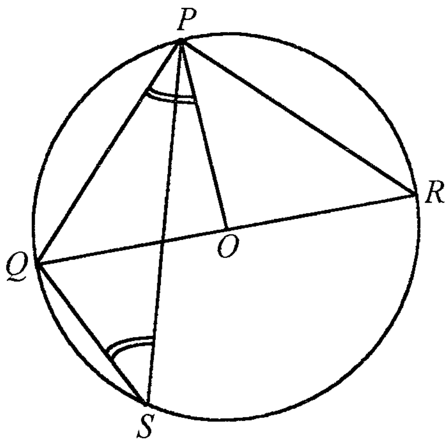
Question 4.

(a) In the diagram, O is the centre of the circle with radius a units. Given that $\angle OPQ = \angle PSQ$. **3**

Show that :

(i) OP is perpendicular to QR.

(ii) $PR = \sqrt{2}\,a$ units.



(b) If α and β are the roots of the quadratic equation **6**

$2x^2 - x + 4 = 0.$

Find :

(i) $\alpha + \beta$ (ii) $\alpha\beta$

(iii) $\alpha^2 + \beta^2$ (iv) $\alpha\beta^2 + \alpha^2\beta$

(v) $\dfrac{\beta}{\alpha} + \dfrac{\alpha}{\beta}$

(c) Find the quadratic equation with roots $2+\sqrt{5}$ and $2-\sqrt{5}$. **3**

Continue ....

Question 5. **Marks**

(a) (i) Show that $\frac{x+1}{x+3} = 1 - \frac{2}{x+3}$. **5**

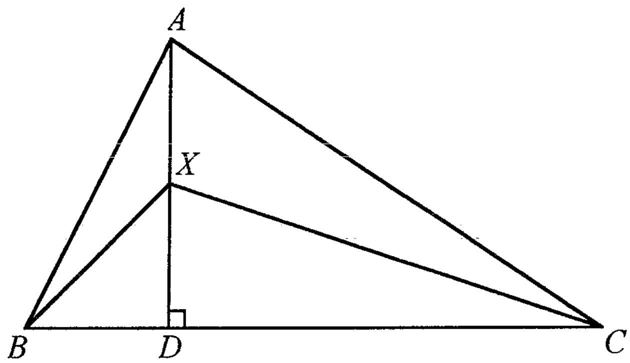
(ii) Given that $y = \frac{x+1}{x+3}$

(α) find the equation of the vertical asymptote

(β) find $\lim_{x\to\infty} \frac{x+1}{x+3}$

(γ) sketch the graph of $y = \frac{x+1}{x+3}$ showing all the main features. (Do not use **Calculus**).

(b) Copy the diagram below into your examination booklet. **3**



Prove that $AB^2 - AC^2 = BD^2 - DC^2 = XB^2 - XC^2$

(c) Solve : **4**

(i) $9^x - 6.3^x - 27 = 0$

(ii) $x + \frac{6}{x} > 5$

Continue ...

Question 6. **Marks**

(a) The points A, B and C have coordinates $(3,5)$, $(5,0)$ and $(-4,3)$ respectively. **9**

(i) Sketch triangle ABC on the number plane.

(ii) Find the length of BC.

(iii) Find the gradient of BC.

(iv) Show that the equation of BC is $x+3y-5=0$.

(v) Find the perpendicular distance from A to BC.

(vi) Find the area of triangle ABC.

(b) Sketch the region given by $y \leq \sqrt{4-x^2}$ and $x-y<2$. **3**

End of paper

Preliminary Examination

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Mathematics Extension 1

Question 1

$x^2-2x-17=0$

$x_{1,2} = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \dfrac{-(-2) \pm \sqrt{4-4(1)(-17)}}{2}$

$= \dfrac{2 \pm \sqrt{4+68}}{2}$

$= \dfrac{2 \pm \sqrt{72}}{2}$

$x_1 = \dfrac{2+\sqrt{72}}{2}$ OR $x_2 = \dfrac{2-\sqrt{72}}{2}$

$= \dfrac{2+6\sqrt{2}}{2}$ $\quad = \dfrac{2-6\sqrt{2}}{2}$

$= \dfrac{2(1+3\sqrt{2})}{2}$ $\quad = \dfrac{2(1-3\sqrt{2})}{2}$

$= \underline{\underline{1+3\sqrt{2}}}$ $\quad = \underline{\underline{1-3\sqrt{2}}}$ ✓

$\dfrac{4}{5-x} \geq 1$

$\dfrac{4}{5-x} - 1 \geq 0$ $\qquad 5-x \neq 0$

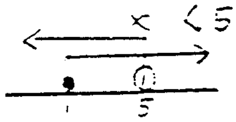
$\dfrac{4-1(5-x)}{5-x} \geq 0$ $\qquad x \neq 5$

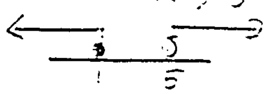
$\dfrac{4-5+x}{5-x} \geq 0$

$-1+x \geq 0 \cap 5-x \geq 0$, $-1+x \leq 0 \cap 5-x \leq 0$

$x \geq 1 \quad -x \geq -5$ $\qquad x \leq 1 \quad -x \leq -5$

$x < 5$ $\qquad x > 5$





$\underline{\underline{1 \leq x < 5}}$ ✓

$4+4\cot^2\theta = \dfrac{4}{1-\cos^2\theta}$

LHS $= 4(1+\cot^2\theta) = 4(\operatorname{cosec}^2\theta) = \dfrac{4}{\sin^2\theta}$ ✓

$= \dfrac{4}{1-\cos^2\theta}$

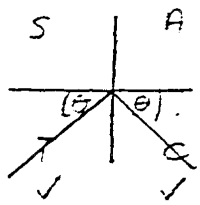
$= \underline{\underline{RHS}}$

$\sin 2\theta = -\dfrac{\sqrt{3}}{2} \quad 0° \leq 2\theta \leq 720°$

(Acute $2\theta = 60°$) ✓

$2\theta = 240°, 300°, 600°, 660°$

$\underline{\underline{\theta = 120°, 150°, 300°, 330°}}$ ✓



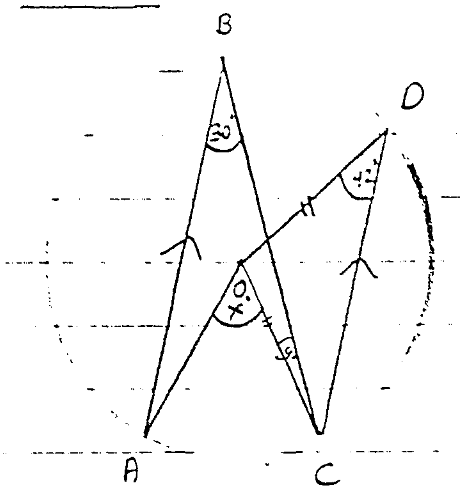
c. $7x^2+3x-6 \equiv ax^2+ax+bx^2+cx+3c$

$\therefore \quad a+b=7 \; ; \; a+c=3, \quad 3c=-6$

$\therefore \quad \underline{c=-2} \qquad \therefore \underline{a=5}$

$\underline{b=2}$

Question 2



$x\ (\angle AOC) = 2 \cdot \angle ABC$ ($\angle$ at the centre is twice $\angle$ at the circumference)

$\therefore x = 2 \cdot 30°$

$\therefore x = 60°$

$OD = OC$ (equal radii).

$\therefore \angle ODC = \angle OCD = 42°$ (base $\angle$ of isosceles Δ)

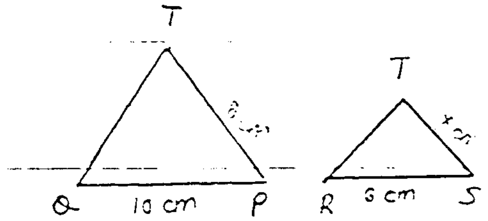
$\angle ABC = \angle BCD = 30°$ (alternate $\angle$ of $AB \parallel CD$)

$\therefore \angle OCB = 42° - 30°$

$= 12°$

$\therefore y = 12°$

(i)



ΔTQP, ΔTRS

$\angle QTP = \angle RTS$ (vertically opposite $\angle$s)

$\angle QPT = \angle RST$ ($\angle$ in the same segment)

$\angle TQP = \angle TRS$ ($\angle$ in the same segment).

$\therefore \Delta TQP \parallel\parallel \Delta TRS$ (equiangular)

(ii).

$$\frac{QP}{RS} = \frac{TP}{TS}$$

$$\frac{10\,cm}{6\,cm} = \frac{8\,cm}{x\,cm}$$

$$x = \frac{8\,cm}{\frac{10\,cm}{6\,cm}}$$

$$x = 4.8\,cm$$

$OD = OC$ (equal radii)

$\therefore \angle ODC = \angle OCD = 48°$ (base $\angle$ of isosceles Δ)

$OB = OC$ (equal radii)

$\therefore \angle OBC = \angle OCB = 26°$ (base $\angle$ of isosceles Δ)

$\therefore \angle DCB = \angle OCD + \angle OCB$

$= 48° + 26° = 74°$

$\angle DCB + \angle DAB = 180°$ (opposite $\angle$ of a cyclic quadrilateral are supplementary)

$\therefore \angle DAB = 180° - 74°$

$= 106°$

$\therefore a = 106°$

Question 3

a. (i) $\lim_{x \to 3} \frac{x^2-2x-3}{x-3}$

$= \frac{(x-3)(x+1)}{(x-3)}$

$= (x+1)$

$= 4$

(ii) $\lim_{x \to \infty} \frac{3x^2+4x-5}{4x^2-1}$

$= \frac{\cancel{x^2}(3+\frac{4}{x}-\frac{5}{x^2})}{x^2(4-\frac{1}{x^2})}$

$= \frac{3+\frac{4}{x}-\frac{5}{x^2}}{4-\frac{1}{x^2}}$

$= \frac{3}{4}$

b. $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$f(x) = 2x^2-1$

$f(x+h) = 2(x+h)^2-1 = 2(x^2+2xh+h^2)-1$

$= 2x^2+4xh+2h^2-1$

$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h \to 0} \frac{2x^2+4x+2h^2-1-(2x^2-1)}{h}$

$= \lim_{h \to 0} \frac{\cancel{2x^2}+4x+2h^2 \cancel{-1} \cancel{-2x^2} \cancel{+1}}{h}$

$= \lim_{h \to 0} \frac{4x+2h^2}{h}$

$= 4x$

(i) $y = \frac{x^5}{5} - 4x + 8$

$= (x^5)(5^{-1}) - 4x + 8$

$= 5x - 4x + 8$

$\frac{dy}{dx} = \cancel{20x^3-4}$ $\quad \frac{5x^4}{5} - 4$

$= x^4 - 4$

(iii) $y = (4x+1)^2$

$\frac{dy}{dx} = 2(4x+1)^1(4)$

$= 8(4x+1)$

$= 32x+8$

(ii). $y = \frac{2x^2+3x-1}{x}$

$\frac{dy}{dx} = (2x^2+3x^1-1)(x^{-1})$

$= 2x^1 + 3x^0 - x^{-1}$

$y = 2x + 3.1 - x^{-}$

$= 2x + 3 - x^{-1}$

$= 2x + 3 - \frac{1}{x}$

$y' = 2 + x^{-2}$

$= 2 + \frac{1}{x^2}$

Question 4

$y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + 5$

$y' = 3 \cdot \frac{1}{3}x^2 + 2 \cdot \frac{1}{2}x - 2$

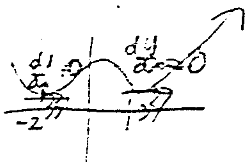
$= x^2 + x - 2$

Parallel to x-axis, gradient $= 0$

$0 = x^2 + x - 2$

$0 = (x+2)(x-1)$

$x = -2 \quad x = 1$



For $x = -2$

$y = \frac{-8}{3} + 2 + 4 + 5$

$= \cancel{12\frac{1}{3} = 12.3}$

$= (-2, 8\frac{1}{3})$ $\cancel{(-2, 12\frac{1}{3})}$

For $x = 1$

$y = \frac{1}{3} + \frac{1}{2} - 2 + 5 = 3\frac{5}{6} = 3.8$

$(1, 3\frac{5}{6})$

$y = (x-1)\sqrt{x}$

$u = (x-1) \qquad v = \sqrt{x} = x^{\frac{1}{2}}$

$u' = 1 \qquad v' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$y' = uv' + vu$

$= (x-1)\left(\frac{1}{2\sqrt{x}}\right) + (\sqrt{x})(1)$

$= \frac{x-1}{2\sqrt{x}} + \frac{\sqrt{x}}{1}$

$y = (x-1)x^{\frac{1}{2}}$

$= x^{\frac{3}{2}} - x^{\frac{1}{2}}$

$y' = \frac{3}{2}x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{3\sqrt{x}}{2} - \frac{1}{2\sqrt{x}}$

Try again

When $x = 4$, $y' = 3 - \frac{1}{4} = 2\frac{3}{4}$

when $x = 4$, $y' = \frac{4-1}{2\sqrt{4}} + \frac{\sqrt{4}}{1} = \frac{3}{4} + 2 = 2\frac{3}{4}$

$\therefore$ Gradient of tangent, $m = 2\frac{3}{4}$

When $x = 4$, $y = (4-1)\sqrt{4} = (3)(2) = 6$ ✓

So the tangent has gradient $2\frac{3}{4}$, and passes through $(4,6)$

Equation of tangent : $y - y_1 = m(x - x_1)$

$y - 6 = 2\frac{3}{4}(x-4)$

$y - 6 = \frac{11}{4}x - 11$ ✓

$4y - 64 = 11x - 44$

$0 = 11x - 4y + 20$ ✓

c. $y = 2x^3 + \frac{7x^2}{2} - 3x + 7$

$y = 2x^3 + \frac{7}{2}x^2 - 3x + 7$

$\frac{dy}{dx}$ $y' = 6x^2 + 2.\frac{7}{2}x - 3$

$= 6x^2 + 7x - 3$ /

$x \cdot f'(x) = 0$

$x(6x^2 + 7x - 3) = 0$

$x = 0$ / $\quad 6x^2 + 7x - 3 = 0$

$6x^2 + 9x - 2x - 3 = 0$

$6x^2 - 2x + 9x - 3 = 0$

$2x(3x-1) + 3(3x-1) = 0$

$(2x+3)(3x-1) = 0$

$2x + 3 = 0 \qquad 3x = 1$

$2x = -3 \qquad x = \frac{1}{3}$ ✓

$x = -\frac{3}{2}$ /

$(x+a)^2 + b = x^2 + 4x - 5$

$x^2 + 4x + 4 - 5 - 4$

$(x+2)(x+2) - 5 - 4$

$(x+2)^2 - 5 - 4$

$\therefore a = 2 \qquad b = \cancel{5}\ -9$

Question 5

(i) Let $\angle QPO = \angle PSQ = x$ (given)

$\angle PRO = \angle PSQ = x$ ($\angle$ in the same segment)

$OR = OP$ (equal radii)

$\therefore \angle PRO = \angle RPO = x$ (base $\angle$s of isosceles $\triangle$ are equal)

$\therefore \angle POR = 180° - 2x$ (straight line) /

Similarly, $OP = OQ$ (equal radii)

$\therefore \angle PQO = \angle QPO = x$ (base $\angle$s of isosceles $\triangle$ are equal)

$\therefore \angle POQ = 180° - 2x$ (straight line)

$\therefore \angle POQ + \angle POR = 180°$ (straight line)

$180 - 2x + 180 - 2x = 180$ /

$360 - 180 = 4x$

$180 = 4x$

$x = 45°$

$\therefore \angle POQ = 180° - 2($

$= 180° - 90$

$= 90°$ /

Similarly, $\angle POR =$

$= 180° - 2(45°)$

$= 90°$ /

$\therefore PO \perp QR$

(ii). FROM (i),

in ΔPOR

$$\frac{PR}{\sin\angle P\hat{O}R} = \frac{OR}{\sin\angle OPR}$$

$$\frac{PR}{\sin 90^\circ} = \frac{a}{\sin 45^\circ}$$

$$\frac{PR}{1} = \frac{a}{\frac{\sqrt{2}}{2}}$$

$$PR = \frac{a}{\frac{\sqrt{2}}{2}}$$

$$PR = a\cdot\frac{2}{\sqrt{2}} = \frac{2a}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{2}a = \sqrt{2}\ a \text{ units}$$

Quicker to use Pythagoras Thm in Δ OPR

$$PR^2 = a^2 + a^2$$
$$= 2a^2$$
$$PR = \sqrt{2a^2}$$
$$= \sqrt{2}\,a \quad \text{as req}^d$$

Question 5

(i). $\frac{x+1}{x+3} = 1 - \frac{2}{x+3}$

RHS $= 1 - \frac{2}{x+3}$

$= \frac{x+3-2}{x+3}$

$= \frac{x+1}{x+3}$

$=$ LHS

(ii). $y = \frac{x+1}{x+3}$

(α). vertical asymptote $x \neq -3$

(β) $\lim_{x\to\infty} \frac{x+1}{x+3}$

$= \frac{\frac{x}{x} + \frac{1}{x}}{\frac{x}{x} + \frac{3}{x}}$

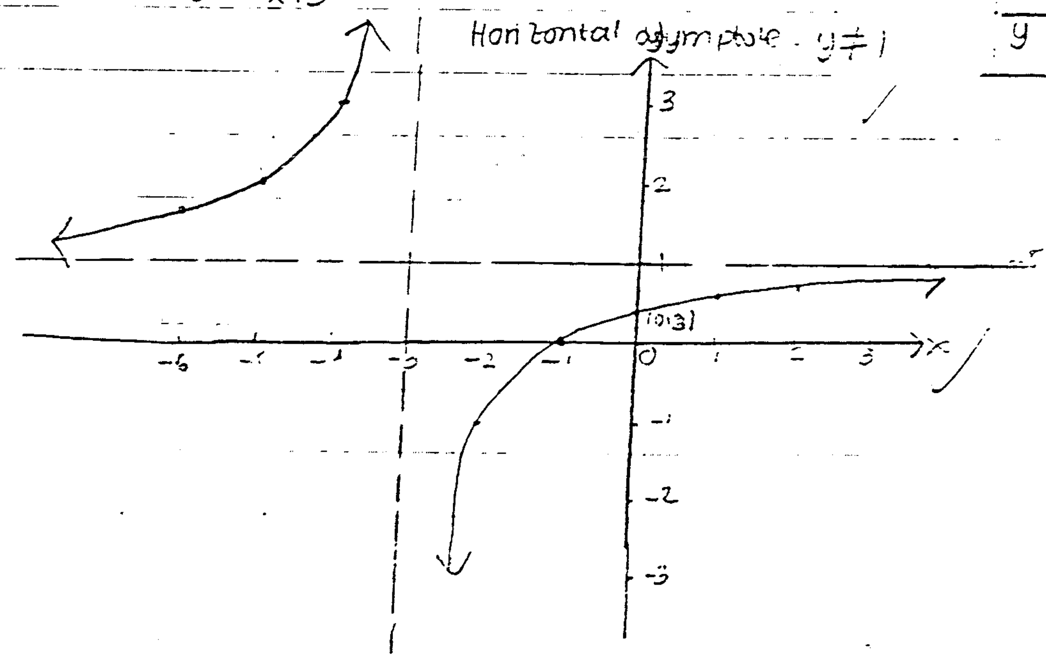
$= \frac{1}{1}$

$= 1$

(γ) $y = \frac{x+1}{x+3}$ vertical asymptote $x \neq -3$

Horizontal asymptote $y \neq 1$

x	-3	-2	-1	0	1	2
y	∞	-1	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{5}$



(4) (b) (i) $\alpha + \beta = -\frac{b}{a} = \frac{1}{2}$

(ii) $\alpha\beta = \frac{c}{a} = 2$

(iii) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= (\frac{1}{2})^2 - 2(2)$

$= -3\frac{3}{4}$

(iv) $\alpha\beta^2 + \alpha^2\beta = \alpha\beta(\beta+\alpha)$

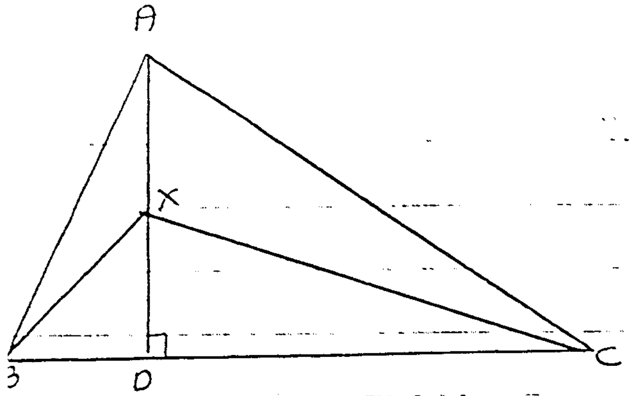
$= 2(\frac{1}{2}) = 1.$

(v) $\frac{\beta}{\alpha} + \frac{\alpha}{\beta} = \frac{\beta^2+\alpha^2}{\alpha\beta}$

$= \frac{-3\frac{3}{4}}{2}$

$= -\frac{15}{8} = -1\frac{7}{8}.$

(c) $x^2 - 4x - 1 = 0.$



Using Phytagoras Theorem,

In ΔABD, $AB^2 = AD^2 + BD^2$

In ΔACD, $AC^2 = AD^2 + DC^2$ $(-)$

$\therefore AB^2 - AC^2 = BD^2 - DC^2$

Similarly, using phytagoras Theorem,

In ΔBXD $BX^2 = BD^2 + XD^2$

In ΔCXD, $CX^2 = CD^2 + XD^2$ $(-)$

$\therefore BX^2 - CX^2 = BD^2 - CD^2$

$\therefore AB^2 - AC^2 = BD^2 - CD^2 = BX^2 - CX^2$

(i). $9^x - 6 \cdot 3^x - 27 = 0$

$(3^2)^x - 6 \cdot 3^x - 27 = 0$

Let $3^x = u$

$u^2 - 6u - 27 = 0$

$(u-9)(u+3) = 0$

$u - 9 = 0$ $\quad$ $u + 3 = 0$

$u = 9$ $\quad$ $u = -3$

$3^x = u$ $\quad$ $3^x = u$

$3^x = 9$ $\quad$ $3^x = -3$

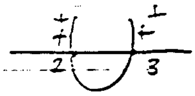
$\underline{\underline{x = 2}}$ $\quad$ No solution required

(ii). $x + \frac{6}{x} - 5 > 0$ $\quad \times x$

$x^2 + 6 - 5x > 0$

$x^2 - 5x + 6 > 0$

$(x-3)(x-2) > 0$



$\underline{\underline{x < 2, x > 3}}$

Test:

$4 \Rightarrow 4 + \frac{6}{4} - 5 > 0$

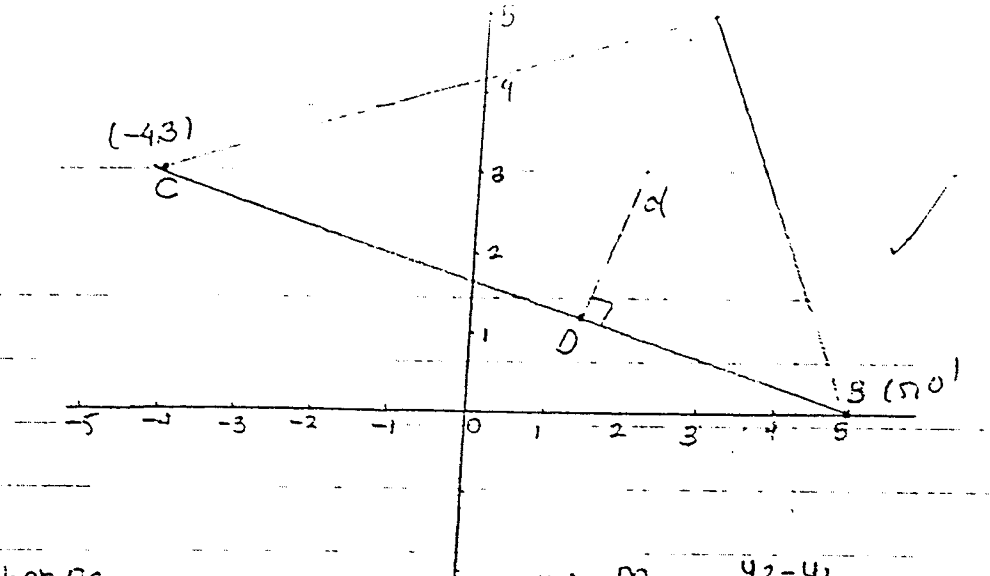
$\frac{1}{2} > 0$ (TRUE)

$1 \Rightarrow 1 + \frac{6}{1} - 5 > 0$

$2 > 0$ (TRUE)

Question 6

(i).



(ii). Length of BC.

$d_{BC} = \sqrt{(y_2-y_1)^2+(x_2-x_1)^2}$

$= \sqrt{(3-0)^2+(-4-5)^2}$

$= \sqrt{9+81}$

$= \sqrt{90}$

$= 3\sqrt{10}$ units

(iii). $m_{BC} = \dfrac{y_2-y_1}{x_2-x_1}$

$= \dfrac{3-0}{-4-5}$

$= \dfrac{3}{-9}$

$= -\dfrac{1}{3}$

(iv). $y-y_1 = m(x-x_1)$

$y-0 = -\frac{1}{3}(x-5)$ $\times 3$

$3y = -1(x-5)$

$3y = -x+5$

$x+3y-5=0$

(v). $d = \dfrac{|ax_1+by_1+c_1|}{\sqrt{a^2+b^2}}$

$= \dfrac{|x+3y-5|}{\sqrt{1+9}}$

$= \dfrac{|3+15-5|}{\sqrt{10}}$

$= \dfrac{|18-5|}{\sqrt{10}}$

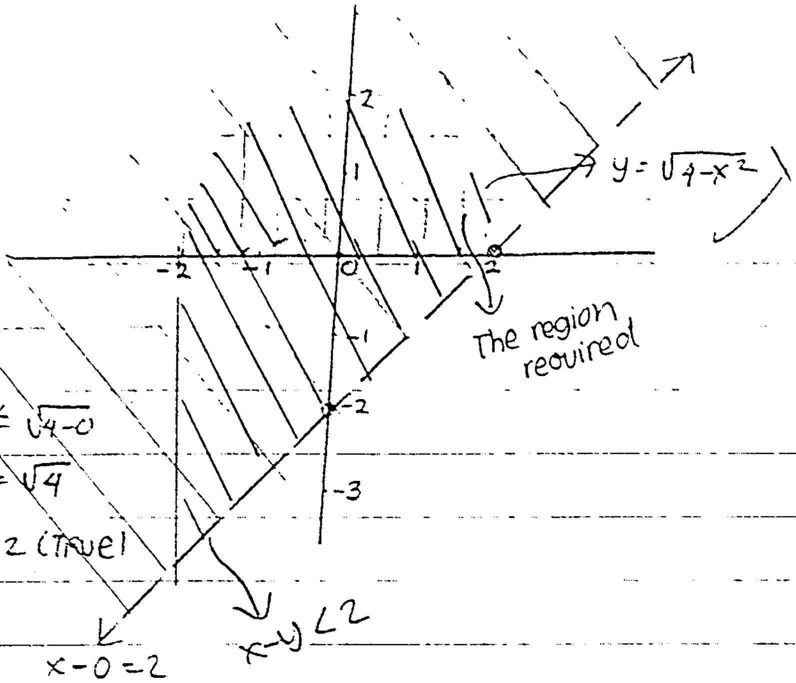
$= \dfrac{13}{\sqrt{10}} \cdot \dfrac{\sqrt{10}}{\sqrt{10}}$

$= \dfrac{13\sqrt{10}}{10}$ units

(vi). Area $= \frac{1}{2}(AD)(BC)$

$= \frac{1}{2}\left(\frac{13\sqrt{10}}{10}\right)(3\sqrt{10})$

$= 19\frac{1}{2}$ units^2



$y \leq \sqrt{4-x^2}$

Test $(0,0) \rightarrow 0 \leq \sqrt{4-0}$

$0 \leq \sqrt{4}$

$0 \leq 2$ (True)

$x-y<2$

x-intercept $y=0$ $\quad x-0=2$

$x=2$

y-intercept $x=0$ $\quad 0-y=2$

$y=-2$

Test $(0,0) \rightarrow 0-0<2$

$0<2$ (True)