SOUTH SYDNEY HIGH SCHOOL

2006

Higher School Certificate

Preliminary Examination

Mathematics

Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- All necessary working should be shown in every question
- Board approved calculators may be used
- A table of standard integrals is provided
- Write your student number and/or name at the top of every page

Total marks - 72

Attempt All Questions 1 – 6

All Questions are of equal value

Q1	Q2	Q3	Q4	Q5	Q6

This paper MUST NOT be removed from the examination room

Question 1 — Marks

(a) Differentiate $\dfrac{1+x^2}{\sqrt{1+2x}}$ — **2**

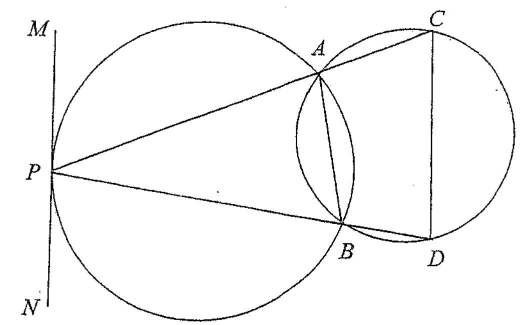
(b) A polynomial $P(x)$ is given by $P(x)=x^3+ax+b$ for some real numbers a and b. $(x-2)$ is a factor of $P(x)$, and when $P(x)$ is divided by x the remainder is 2, Find the values of a and b. — **2**

(c) The equation $x^3-6x^2+4x+2=0$ has three real roots α,β and γ.

(i) Write down the values of $\alpha+\beta+\gamma$, $\alpha\beta+\beta\gamma+\gamma\alpha$ and $\alpha\beta\gamma$. — **1**

(ii) Hence find the value of $(\alpha-2)(\beta-2)(\gamma-2)$. — **3**

(d)



In the diagram the two circles intersect at A and B. P is a point on one circle.
The lines PA produced and PB produced meet the other circle at C and D respectively.
MNP is the tangent to the first circle at P.

(i) Copy the diagram.

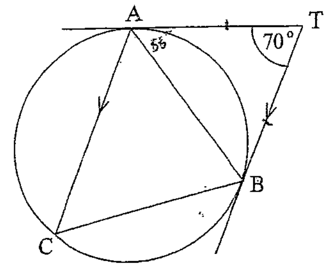
(ii) Give a reason why $\angle MPA=\angle PBA$. — **1**

(iii) Hence show that $CD \parallel MN$. — **3**

Question 2

Marks

(a) TA and TB are tangents to a circle such that $\angle ATB = 70^\circ$.
The parallel from A to TB meets the circle at C.



(i) Find the value of $\angle ACB$. Give reasons. **2**

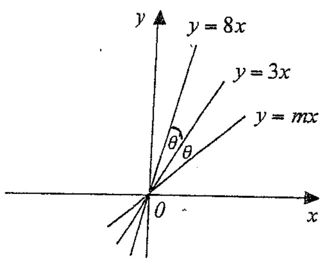
(ii) Find the value of $\angle ABC$. Give reasons **2**

(b) Sketch the graph of the function $f(x) = \dfrac{|x|}{x}$. **2**

(c) (i) Show that $\dfrac{1+\cos 2x}{\sin 2x} = \cot 2x$. **2**

(ii) Hence find the exact value of $\cot 15^\circ$. **1**

(d)



(i) The acute angle between the lines $y = 3x$ and $y = 8x$ is θ. Show that $\tan\theta = \dfrac{1}{5}$. **1**

(ii) The acute angle between the lines $y = mx$ and $y = 3x$ is also θ. Find the value of the real number m, where $m < 3$. **2**

Question 3

Marks

(a) $(-7,5)$ and $B(3,1)$ are two points. Find the coordinates of the point P which divides the interval AB externally in the ratio 3:1. **2**

(b) Find the coordinates of the points on the curve $y = x\sqrt{2-x^2}$ where the tangents to the curve are parallel to the x axis. **3**

(c) (i) Express $3\cos x - 4\sin x$ in the form $R\cos(x+\alpha)$ for some $R > 0$ and $0^\circ < \alpha < 90^\circ$, giving the value of α correct to the nearest minute. **2**

(ii) Hence solve the equation $3\cos x - 4\sin x = 5$ for $0^\circ \le x \le 360^\circ$, giving the answer correct to the nearest minute. **1**

(d) $P(a,b)$ is a point on the curve $y = \dfrac{1}{3}x^3 - x^2$.

(i) Find the gradient of the normal to the curve at P. **1**

(ii) If the normal at P is parallel to the line $y = x$, find the coordinates of P. **3**

Question 4

(a) Solve the inequality $\dfrac{x-2}{x} > 0$. **2**

(b) $A(a,b)$ and $B(c,d)$ are two points. $P(x,y)$ is a variable point which moves so that $\angle APB = 90^\circ$. Show that the locus of P has equation $(x-a)(x-c)+(y-b)(y-d) = 0$. **2**

(c) (i) Express $\sin x$ and $\cos x$ in terms of $t = \tan\dfrac{x}{2}$. **1**

(ii) Hence solve the equation $\cos x + \sin x = -1$ for $0^\circ \le x \le 360^\circ$. **3**

(d) A parabola has equation $4y = x^2 + 4x + 8$.

(i) Find its focal length. **1**

(ii) Find the coordinates of the vertex. **1**

(iii) State the equation of the axis of symmetry. **1**

(iv) Draw a neat sketch of the graph of the parabola **1**

Question 5 Marks

(a) Show that the line $y = mx - m^2$ is tangent to the parabola $x^2 = 4y$ for all real m. 2

(b) ABC is an acute angle angled triangle.

(i) Show that $\tan(A+B) = -\tan C$. 1

(ii) Hence show that $\tan A + \tan B + \tan C = \tan A . \tan B . \tan C$. 2

(c) Solve the equation $|x-2| > |x|$. 2

(d) Consider the equation $x^3 - 3x^2 + 2x + p = 0$ 2

Given that the sum of two roots is 3, find the value of p.

(e) (i) Sketch the graph of $x(x-1)^3(x+2)^2$ 2

(ii) Hence or otherwise, solve the inequation $x(x-1)^3(x+2)^2 \geq 0$. 1

Question 6

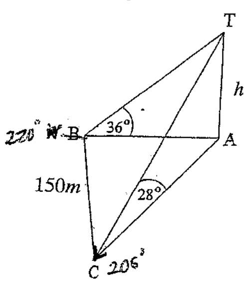
(a) Find the value of $\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h}$ by first rationalising the numerator. 2

(b) ~~Solve the inequation $\frac{1}{2x-3} \quad \frac{1}{x-4}$~~ 2

(c) Solve the equation $2\cos^3\theta - \sin 2\theta = 0$, $0^\circ \leq \theta \leq 360^\circ$. 3

(d) The diagram shows a vertical tower AT of height h metres stands with its base A on horizontal ground. B is a point on the ground due west of A, and C is a point on the ground 150m from B and has a true bearing of 205° from A.

From B and C the angle of elevation of the top T of the tower are 28° and 36° respectively.



(i) Show that the height h of the tower can be expressed as 4

$$h = \frac{150}{\sqrt{\tan^2 54^\circ + \tan^2 62^\circ - 2\tan 54^\circ \tan 62^\circ \cos 65^\circ}}$$

(ii) Hence, find the height of the tower, correct to the nearest minute. 1

Solutions to Year 11 Ext 1 Yearly Preliminary Exam 28/9/06

Question 1

(a) $y = \frac{1+x^2}{\sqrt{1+2x}}$

$y' = \frac{vu' - uv'}{v^2}$

$= \frac{\sqrt{1+2x}\cdot 2x - [1+x^2]\frac{1}{2}(1+2x)^{-\frac{1}{2}}\cdot 2}{1+2x}$

$= \frac{2x\sqrt{1+2x} - \left[\frac{1+x^2}{\sqrt{1+2x}}\right]}{1+2x}$

$= \frac{\frac{2x(1+2x) - 1 - x^2}{\sqrt{1+2x}}}{1+2x}$

$= \frac{2x + 4x^2 - 1 - x^2}{(1+2x)^{3/2}}$

$= \frac{3x^2 + 2x - 1}{(1+2x)^{3/2}}$

(b) $P(x) = x^3 + ax + b$

$P(2) = 0$ (Factor thm)

$8 + 2a + b = 0$

$2a + b = -8$ — (1)

$P(0) = 2$ (remainder thm)

$b = 2$ — (2)

Sub $b = 2$ into (1): $2a + 2 = -8$

$2a = -10$

$a = -5$

$\therefore a = -5,\ b = 2$

(c) (i) $x^3 - 6x^2 + 4x + 2 = 0$

$\alpha + \beta + \gamma = -\frac{b}{a}$

$= \frac{6}{1}$

$= 6$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

$= 4$

$\alpha\beta\gamma = -\frac{d}{a}$

$= -2$ (for 3 values)

(ii) $(\alpha-2)(\beta-2)(\gamma-2)$

$= (\alpha-2)(\beta\gamma - 2\beta - 2\gamma + 4)$

$= \alpha\beta\gamma - 2\alpha\beta - 2\alpha\gamma + 4\alpha - 2\beta\gamma + 4\beta + 4\gamma - 8$

$= \alpha\beta\gamma - 2(\alpha\beta + \alpha\gamma + \beta\gamma) + 4(\alpha + \beta + \gamma) - 8$

$= -2 - 2(4) + 4(6) - 8$

$= 6$

(d) (ii) Angle between tangent and chord is equal to angle in alternate segment

(iii) $\angle MPA = \angle PBA$ (from (ii))

$\angle PBA = \angle ACD$ (exterior angle of cyclic quad is equal to opp int angle)

$\therefore \angle MPA = \angle ACD$

$\therefore MN \parallel CD$ (alternate angles equal)

Question 2

(i) $AT = BT$ (tangents from external point equal)

$\therefore \angle TAB = 55$ (Base angles of isosceles $\triangle$)

$\therefore \angle ACB = 55$ (Angle between tangent and chord equal to angle in alternate seg)

(ii) $\angle TBC = 125$ (Co-interior; AC // TB) ✓

$\angle TBA = 55$ (Base angles of isosceles △) ✓

$\angle ABC = 125 - 55$

$= \underline{70}$

b) $f(x) = \frac{|x|}{x}$

$= \frac{x}{x}$ if $x > 0$

$f(x) = 1$ if $x > 0$ ✓ ½

$f(x) = \frac{-x}{x}$ if $x < 0$

$f(x) = -1$ if $x < 0$ ✓ ½

✓ 1

c) i) $\frac{1+\cos 2x}{\sin 2x} = \cot 2x$

LHS $= \frac{1+\cos 2x}{\sin 2x}$

$= \frac{1+\cos^2 x - \sin^2 x}{2\sin x \cos x}$ ✓

$= \frac{1-\sin^2 x + \cos^2 x}{2\sin x\cos x}$

$= \frac{\cos^2 x + \cos^2 x}{2\sin x \cos x}$

$= \frac{2\cos^2 x}{2\sin x \cos x}$ ✓

$= \frac{\cos x}{\sin x}$

$= \cot x$

$= \underline{RHS}$

ii) $\cot 15 = \frac{1+\cos 30}{\ldots} = \frac{1+\frac{\sqrt{3}}{2}}{\ldots}$

(d) (i) $y = 8x \Longrightarrow m_1 = 8$

$y = 3x \Longrightarrow m_2 = 3$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{8-3}{1+24}\right|$

$= \frac{5}{25}$

$= \frac{1}{5}$ ✓

(ii) $\left|\frac{3-m}{1+3m}\right| = \frac{1}{5}$ ✓

$5(3-m) = 1+3m$ (since $m<3$)

$15 - 5m = 1 + 3m$

$14 = 8m$

$m = \frac{14}{8}$

$= \frac{7}{4}$ ✓ 2

Question 3

(a) $A(-7, 5)$, $B(3, 1)$ $m : n = 3 : 1$

$\left[\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right]$

$\left[\frac{3.3 + (-1)(-7)}{3-1}, \frac{3.1 + (-1)(5)}{3-1}\right]$ ✓

$\left[\frac{9+7}{2}, \frac{3-5}{2}\right]$

$\left[8, -1\right]$ ✓ 2

b) $y = x\sqrt{2-x^2}$

$y' = uv' + vu'$

$= x \cdot \frac{1}{2}(2-x^2)^{-1/2} \cdot (-2x)$

$+ \sqrt{2-x^2} \cdot 1$ ✓

$= \frac{-x^2}{\sqrt{2-x^2}} + \sqrt{2-x^2}$

$= \frac{-x^2 + 2 - x^2}{\sqrt{2-x^2}}$

$= \frac{2-2x^2}{\sqrt{2-x^2}}$

Now if parallel to x axis

then $m = 0$

ie $\frac{2-2x^2}{\sqrt{2-x^2}} = 0$ ✓

$2 - 2x^2 = 0$

$2(1-x)(1+x) = 0$

$x = \pm 1$

$\therefore \underline{(1, 1) \text{ and } (-1, -1)}$ ✓ 3

c) i) $3\cos x - 4\sin x = R\cos(x+\alpha)$

$R = \sqrt{3^2+4^2}$

$= 5$ ✓

$3\cos x - 4\sin x = 5\cos(x+\alpha)$

$\frac{3}{5}\cos x - \frac{4}{5}\sin x = \cos x\cos\alpha - \sin x\sin\alpha$

$\therefore \cos\alpha = \frac{3}{5}$, $\sin\alpha = \frac{4}{5}$ $\therefore \tan\alpha = \frac{4}{3}$

$\alpha = 53°8'$ ✓

$\therefore \underline{3\cos x - 4\sin x = 5\cos(x + 53°8')}$ 2

(ii) $3\cos x - 4\sin x = 5$

$5\cos(x+53°8') = 5$

$\cos(x + 53°8') = 1$

$x + 53°8' = 360$

$\underline{x = 306°52'}$ ✓

d) (i) $y = \frac{1}{3}x^3 - x^2$

$y' = x^2 - 2x$

when $x = a$: $y' = a^2 - 2a$

$\therefore m_T = a^2 - 2a$

$m_N = \frac{-1}{a^2-2a}$ ✓ 1

(ii) If parallel then:

$\frac{-1}{a^2-2a} = 1$ (Since $m_1 = m_2$) ✓

$-1 = a^2 - 2a$

$a^2 - 2a + 1 = 0$

$(a-1)^2 = 1$

$a = 1$ ✓

when $a=1$: $y = \frac{1}{3}(1)^3 - 1^2$

$= \frac{1}{3} - 1$

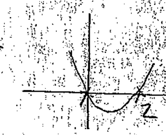
$= -\frac{2}{3}$

$\therefore P\left(1, -\frac{2}{3}\right)$ ✓ 3

Question 4

(a) $\frac{x-2}{x} > 0$

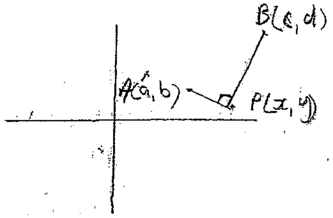
$x^2 \times \frac{x-2}{x} > 0 \times x^2$



$x(x-2) > 0$ ✓

$\therefore \underline{x > 2 \text{ or } x < 0}$ 2

(b)



$PA \perp PB \therefore m_1 m_2 = -1$

$m_1 = \frac{y-b}{x-a}, \quad m_2 = \frac{y-d}{x-c}$

2 $\therefore \frac{y-b}{x-a} \times \frac{y-d}{x-c} = -1$ ✓

$(y-b)(y-d) = -(x-a)(x-c)$

$\underline{(y-b)(y-d) + (x-a)(x-c) = 0}$ ✓

1 (c)(i) $\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$ ✓

(ii) $\cos x + \sin x = -1$

$\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = -1$

$2t + 1 - t^2 = -1 - t^2$

3 $2t = -2$

$t = -1$ ✓

$\therefore \tan\frac{x}{2} = -1 \quad 0 \le \frac{x}{2} \le 180$

$\frac{x}{2} = 135$ ✓

$\underline{x = 270 \text{ or } x = 180}$ ✓

d) $4y = x^2 + 4x + 8$

$x^2 + 4x + 8 = 4y$

$x^2 + 4x = 4y - 8$

$x^2 + 4x + 4 = 4y - 8 + 4$

$(x+2)^2 = 4y - 4$

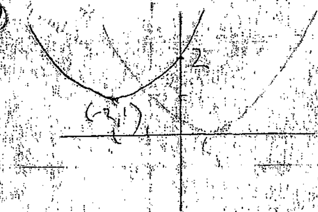
$(x+2)^2 = 4(y-1)$

(i) focal length = 1 ✓

(ii) vertex = (-2, 1) ✓

(iii) axis of symmetry: $x = -2$ ✓

(iv)



✓

Question 5

(a) $y = mx - m^2$ — (1)

$x^2 = 4y$ — (2)

$x^2 = 4(mx - m^2)$

$x^2 = 4mx - 4m^2$

$x^2 - 4mx + 4m^2 = 0$ ✓

For one root

$\Delta = 0$

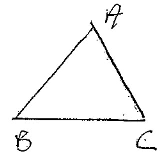
$b^2 - 4ac = 0$ 2

$16m^2 - 4(1)(4m^2) = 0$

$16m^2 - 16m^2 = 0$

$\underline{\text{Hence } \Delta = 0 \text{ for all } m}$ ✓

(b)



(i) $A + B + C = 180$

$A + B = 180 - C$

$\therefore \tan(A+B) = \tan(180 - C)$

$= -\tan C$ ✓

(ii) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ ✓

$-\tan C = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$-\tan C(1 - \tan A \tan B) = \tan A + \tan B$

2 $-\tan C + \tan A \tan B \tan C = \tan A + \tan B$

$\therefore \underline{\tan A + \tan B + \tan C = \tan A \tan B \tan C}$ ✓

(c) $|x-2| > |x|$

$(x-2)^2 > x^2$ ✓

2 $x^2 - 4x + 4 > x^2$

$-4x + 4 > 0$

$4x < 4$

$\underline{x < 1}$ ✓

(d) $x^3 - 3x^2 + 2x + p = 0$

$\alpha + \beta + \gamma = -\frac{b}{a}$

$3 + \gamma = 3$

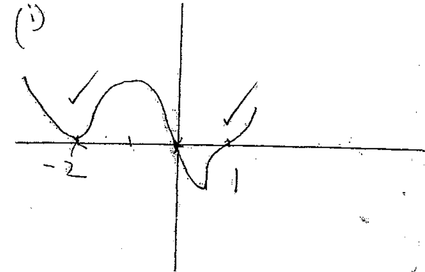
$\gamma = 0$ ✓

$\alpha\beta\gamma = -\frac{d}{a}$

$0 = -p$

$\underline{p = 0}$ ✓

e) (i)



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(ii) $\underline{x \ge 1 \text{ or } x \le 0}$ ✓

Question 6

(a) $\lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} = \lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} \times \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1}$

$= \lim_{h \to 0} \frac{1 + h - 1}{h(\sqrt{1+h} + 1)}$

$= \lim_{h \to 0} \frac{h}{h(\sqrt{1+h} + 1)}$

$= \lim_{h \to 0} \frac{1}{\sqrt{1+h} + 1}$ ✓ 2

$= \underline{\frac{1}{2}}$ ✓

(b) $2\cos^3\theta - \sin 2\theta = 0$

$2\cos^3\theta - 2\sin\theta\cos\theta = 0$ ✓

$2\cos\theta(\cos^2\theta - \sin\theta) = 0$

$\cos\theta(1 - \sin^2\theta - \sin\theta) = 0$

$\cos\theta(-\sin^2\theta - \sin\theta + 1) = 0$ 3

$\cos\theta = 0$ or $-\sin^2\theta - \sin\theta + 1 = 0$

$\underline{\theta = 90, 270}$ ✓ $\quad \sin^2\theta + \sin\theta - 1 = 0$

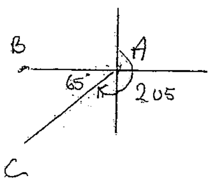
$\sin\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

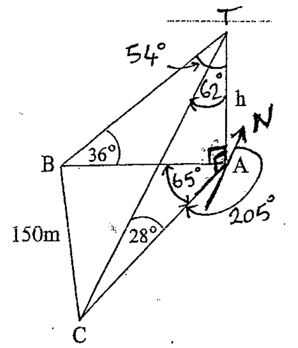
$= \frac{-1 \pm \sqrt{1+4}}{2}$

$= \frac{-1 \pm \sqrt{5}}{2}$ ✓

$\theta = 38°10', 141°51'$

(d) (i)





In $\triangle ATB$: $\tan 54° = \frac{AB}{h}$

$AB = h\tan 54$ ✓

In $\triangle ATC$: $\tan 62° = \frac{AC}{h}$ ✓

$AC = h\tan 62$

Using Cosine Rule:

$a^2 = b^2 + c^2 - 2bc\cos A$

i.e. $BC^2 = AC^2 + AB^2 - 2(AC)(AB)\cos A$

$150^2 = h^2\tan^2 62 + h^2\tan^2 54 - 2h^2\tan 62\tan 54\cos 62$ ✓

$150^2 = h^2\left[\tan^2 62 + \tan^2 54 - 2\tan 54\tan 62\cos 65\right]$

$h^2 = \frac{150^2}{\tan^2 62 + \tan^2 54 - 2\tan 54\tan 62\cos 65}$

$h = \frac{150}{\sqrt{\tan^2 62 + \tan^2 54 - 2\tan 54\tan 62\cos 65}}$ ✓

(ii) $h = 83$ metres ✓