South Sydney High School

YR 12

2006

MATHEMATICS
EXTENSION 1 ASSESSMENT

Time allowed – 1.5hours

Name: ......Mohammad Nasiry..........

General Instructions

- Write using blue or black pen
- All necessary working should be shown for every question
- Approved calculators may be used
- Begin each question on a new page clearly marked "Question 1", "Question 2", etc.

QUESTION 1. (17 marks)

Marks

Consider the function $y = \dfrac{2x-1}{(x-2)^2} = \dfrac{2x-1}{x^2-4x+4}$.

(a) What is the domain of the function? **1**

(b) Determine the coordinates of the points where the graph crosses the x- and y- axes. **2**

(c) Determine if the function is odd or even. Justify your answer. **2**

(d) What happens to y as x approaches positive infinity? **1**

(e) What happens to y as x approaches negative infinity? **1**

(f) Find the coordinates of any turning points and determine their nature. **6**

(g) Sketch the curve showing important features including asymptote(s). **2**

(h) From the graph, determine the values of x for which the function is decreasing. **2**

QUESTION 2. (12 marks)

Marks

(a) Solve $2 \times \binom{n}{4} = 5 \times \binom{n}{2}$. **3**

(b) Expand $(3+2x)^6$ in ascending powers of x as far as the term in x^2. **3**

(c) Find the coefficient of x^2 in the expansion of $(3+x)(1-2x)^5$. **2**

(d) Write down the expression for the $(r+1)^{\text{th}}$ term in the expansion of $\left(x^2+\frac{3}{x}\right)^{10}$. **4**

Hence find the coefficient of x^{11} in the expansion of $\left(x^2+\frac{3}{x}\right)^{10}$.

QUESTION 3. (~~7~~ 10 marks)

(a) For what values of r is the coefficient of the $(r+1)^{\text{th}}$ term greater than the coefficient of the r^{th} term in the expansion of $(5+2x)^{15}$, in ascending powers of x? **4**

(b) By considering the expansion of $(1+x)^n$, find the value of $\sum_{r=1}^{n} \binom{n}{r} 3^r$. **3**

(c) Write down the general term in the expansion of $\left(3x-\frac{2}{x^2}\right)^9$ and use it to determine the value of the term that is independent of x. **3**

QUESTION 4

(a) Considering the expansion of $(a+b)^n$:

(i) By letting $a = b = 1$, show that $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+....+\binom{n}{n}=2^n$ **1**

(ii) By letting $a = 1$, and $b = -1$, show that
$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+....+(-1)^n\binom{n}{n}=0$ **1**

(iii) Hence show that $\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+....=2^{n-1}$ **2**

(b) (i) State the binomial theorem for $(1+x)^n$ where n is a positive integer. **1**

(ii) If k is a positive integer, show that $\left(1+\frac{1}{n}\right)^k$ approaches 1 as $n \to \infty$ **1**

(iii) Show that $\frac{1}{n!} < \frac{1}{2^{n-1}}$ for all positive integral $n \geq 3$. **3**

QUESTION 1

$$y=\frac{2x-1}{(x-2)^2}$$

(a) Domain: all x except $x=2$ 1

(b) When $x=0$, $y=-\frac{1}{4}$. Cuts y-axis at $-\frac{1}{4}$.

When $y=0$, $x=\frac{1}{2}$. Cuts x-axis at $\frac{1}{2}$. 2

(c) $f(-x)=\frac{2(-x)-1}{(-x-2)^2}$

$=\frac{-2x-1}{(x+2)^2}$

$f(-x)\neq f(x)$ $\therefore$ $f(x)$ is not even.

$f(-x)\neq -f(x)$ $\therefore$ $f(x)$ is not odd. 2

(d) $y=\frac{2x-1}{x^2-4x+4}$

$=\frac{\frac{2}{x}-\frac{1}{x^2}}{1-\frac{4}{x}+\frac{4}{x^2}}$

As $x\to\infty$, $y\to 0$ $\frac{0-0}{1-0+0}=0$ from above (positive values) 1

(e) As $x\to-\infty$, $y\to 0$ $\frac{0-0}{1-0+0}=0$ from below (negative values) 1

(f) By the quotient rule,

$$\frac{dy}{dx}=\frac{(x-2)^2\times 2-(2x-1)\times 2(x-2)}{(x-2)^4}$$

$$=\frac{2(x-2)[(x-2)-(2x-1)]}{(x-2)^4}$$

$$=\frac{2(-x-1)}{(x-2)^3}$$

$$=\frac{-2(x+1)}{(x-2)^3}$$

When $\frac{dy}{dx}=0$, $x=-1$

Stationary point at $\left(-1,-\frac{1}{3}\right)$

When $x=-1-\varepsilon$, $\frac{dy}{dx}=\frac{-2(-1-\varepsilon+1)}{(-1-\varepsilon-2)^3}=\frac{(-)\times(-)}{(-)}<0$

When $x=-1+\varepsilon$, $\frac{dy}{dx}=\frac{-2(-1+\varepsilon+1)}{(-1+\varepsilon-2)^3}=\frac{(-)\times(+)}{(-)}>0$

$\therefore$ Relative minimum turning point at $\left(-1,-\frac{1}{3}\right)$. 6

Alternative Solution :

Find second derivative.

$$\frac{d^2y}{dx^2}=\frac{(x-2)^3(-2)- -2(x+1)\times 3(x-2)^2}{(x-2)^6}$$

$$=\frac{-2(x-2)^2[(x-2)-3(x+1)]}{(x-2)^6}$$

$$=\frac{-2(-2x-5)}{(x-2)^4}$$

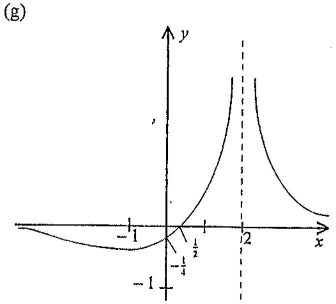
$$=\frac{2(2x+5)}{(x-2)^4}$$

When $x=-1$, $\frac{d^2y}{dx^2}=\frac{6}{81}>0$

Concave up at x = –1.

$\therefore$ Relative minimum at $(-1, \frac{1}{-3})$.

(g)



2

(h) Function is decreasing for

$-\infty<x<-1$, $x>2$ 2

QUESTION 2

(a) $2\binom{n}{4}=5\binom{n}{2}$

$\frac{2\times n(n-1)(n-2)(n-3)}{4\times 3\times 2\times 1}=\frac{5\times n(n-1)}{2\times 1}$ 1

$\frac{(n-2)(n-3)}{12}=\frac{5}{2}$

$(n-2)(n-3)=30$

$n^2-5n+6=30$

$n^2-5n-24=0$

$(n-8)(n+3)=0$

$n=8$ or $n=-3$ 1

But $n>0$, $\therefore$ $n=8$ 1

(b) $(3+2x)^6$

$=3^6+\binom{6}{1}3^5(2x)+\binom{6}{2}3^4(2x)^2+\ldots$

$=729+2916x+4860x^2+\ldots\ldots$ 1,1,1

(c) $(3+x)(1-2x)^5$

Term in $x^2=3\times\binom{5}{2}(-2x)^2+x\times\binom{5}{1}(-2x)$ 1

$=120x^2-10x^2$

$\therefore$ Coefficient of x^2 is 110. 1

(d) For $\left(x^2+\frac{3}{x}\right)^{10}$

$T_{r+1}=\binom{10}{r}(x^2)^{10-r}\left(\frac{3}{x}\right)^r$ 1

$T_{r+1}=\binom{10}{r}x^{20-2r}\times\frac{3^r}{x^r}$

$T_{r+1}=\binom{10}{r}3^r x^{20-3r}$ 1

For the term in x^{11}, $20-3r=11$

$r=3$ 1

$\therefore$ coefficient of $x^{11}=\binom{10}{3}3^3$

$=3240$ 1

QUESTION 3

(a) For $(5+2x)^{15}$

$T_{r+1}=\binom{15}{r}5^{15-r}(2x)^r$

$T_r=\binom{15}{r-1}5^{15-(r-1)}(2x)^{r-1}$ 1

For coefficient $T_{r+1}>$ coefficient T_r

$\frac{15!}{r!(15-r)!}\times 5^{15-r}\times 2^r>\frac{15!}{(r-1)!(16-r)!}\times 5^{16-r}\times 2^{r-1}$ 1

Divide by 15!, multiply by $r!$ and $(16-r)!$

Divide by 5^{15-r}, and 2^{r-1}

$\frac{(16-r)!}{(15-r)!}\times 2>\frac{r!}{(r-1)!}5$

$(16-r)\times 2>r\times 5$

$32-2r>5r$

$r<4\frac{4}{7}$ 1

$\therefore$ $r=1, 2, 3, 4$ 1

(b) $(1+x)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\ldots+\binom{n}{n}x^n$ 1

Let $x=3$:

$4^n=\binom{n}{0}+\binom{n}{1}\times 3+\binom{n}{2}\times 3^2+\ldots+\binom{n}{n}3^n$ 1

Now

$\sum_{r=1}^{n}\binom{n}{r}3^r=\binom{n}{1}\times 3+\binom{n}{2}\times 3^2+\ldots+\binom{n}{n}3^n$

$=4^n-\binom{n}{0}$

$=4^n-1$ 1

(c) The general term of the expansion of $\left(3x-\frac{2}{x^2}\right)^9$

$={}^9C_r(3x)^{9-r}\left(\frac{-2}{x^2}\right)^r$

For independence from x: $9-r-2r=0$, i.e. $r=3$.

The term independent of x

$={}^9C_3(3x)^6\left(\frac{-2}{x^2}\right)^3={}^9C_3\times 3^6\times(-2)^3$

$=84\times 3^6\times(-8)$ OR $(-2^3\times 3^6)\,{}^9C_3$

$=-489\,888$

Q4

a) i) $(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + ... + \binom{n}{n}a^0 b^n$

When $a=1$ and $b=1$, this becomes

$(1+1)^n = \binom{n}{0}1^n 1^0 + \binom{n}{1}1^{n-1}1^1 + \binom{n}{2}1^{n-2}1^2 + ... + \binom{n}{n}1^0 1^n$

$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n}$

a) ii)

$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + ... + \binom{n}{n}a^0 b^n$

When $a=1$ and $b=-1$, this becomes

$(1+(-1))^n = \binom{n}{0}1^n(-1)^0 + \binom{n}{1}1^{n-1}(-1)^1 + \binom{n}{2}1^{n-2}(-1)^2 + ... + \binom{n}{n}1^0(-1)^n$

$(0)^n = \binom{n}{0}1 + \binom{n}{1}(-1) + \binom{n}{2}1 + ... + \binom{n}{n}(-1)^n$

$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} .... + (-1)^n\binom{n}{n}$

a) iii) From ii) above

$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + ... + (-1)^n\binom{n}{n} = 0$

$\therefore \binom{n}{0} + \binom{n}{2} + ... = \binom{n}{1} + \binom{n}{3} + ...$

Now since $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n} = 2^n$

$\therefore \binom{n}{0} + \binom{n}{2} + ... = \binom{n}{1} + \binom{n}{3} + ... = \frac{1}{2}\left[\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n}\right]$

$\binom{n}{0} + \binom{n}{2} + ... = \binom{n}{1} + \binom{n}{3} + ... = \frac{1}{2}(2^n)$

$\binom{n}{0} + \binom{n}{2} + ... = \binom{n}{1} + \binom{n}{3} + ... = 2^{n-1}$

(b) (i)

$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_r x^r + \cdots + x^n$

$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!}x^r + \cdots + x^n$

(ii) $\left(1+\frac{1}{n}\right)^k = 1 + k\cdot\frac{1}{n} + \frac{k(k-1)}{2!}\left(\frac{1}{n}\right)^2 + \cdots + \frac{1}{n^k}$

If k is fixed and as $n \to \infty$, then $\frac{1}{n}, \frac{1}{n^2}, \frac{1}{n^3}, \cdots \frac{1}{n^k} \to 0$ and so $\left(1+\frac{1}{n}\right)^k \to 1$

(iii) $\left(1+\frac{1}{n}\right)^n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!}\cdot\frac{1}{n^r} + \cdots + \frac{1}{n^n}$

$= 1 + 1 + \frac{1}{2!}\cdot\frac{n}{n}\cdot\frac{(n-1)}{n} + \cdots + \frac{1}{r!}\cdot\frac{n}{n}\cdot\frac{n-1}{n}\cdots\frac{n-r+1}{n} + \cdots + \frac{1}{n^n}$

$= 1 + 1 + \frac{1}{2!}\cdot 1\cdot\left(1-\frac{1}{n}\right) + \cdots\cdots + \frac{1}{r!}\cdot 1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right) + \cdots + \frac{1}{n^n}$

As $n \to \infty$, $\left(1+\frac{1}{n}\right)^n \to 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots\cdots$

$\therefore$ As $n \to \infty$, $\left(1+\frac{1}{n}\right)^n$ approaches the sum of an infinite series and the sum is clearly greater than 2.

Now $n! = 1 \cdot 2 \cdot 3 \cdot \cdots\cdots (n-1) \cdot n$ there are n terms and in

$2^{n-1} = 2 \bullet 2 \bullet 2 \bullet 2 \cdots\cdots\cdots \bullet 2$ there are $(n-1)$ terms

$\therefore n! > 2^{n-1}$ for all except $n \le 2$ when $n! = 2^{n-1}$

Hence $\frac{1}{n!} < \frac{1}{2^{n-1}}$ for all integral $n \ge 3$