SOUTH SYDNEY HIGH SCHOOL

SPECTEMUR AGENDO

Year 12

Half-Yearly Examination 1998

MATHEMATICS

3 UNIT

Time allowed - 2 hours

DIRECTIONS TO CANDIDATES

- Board-approved calculators may be used.
- Write your Name on **EVERY** page .
- Attempt all **SIX** questions.
- Standard Integrals are provided.
- Start each question on a **NEW** page.

Question 1 (13 marks) **Marks**

(a) Differentiate and simplify with respect to x, the function **2**

$$x \tan x$$

(b) Evaluate $e^{2x} - e^{x} - 20 = 0$ **3**

(c) Solve the following equations for $0^0 \leq \theta \leq 360$ **2**

$$5\cos 2\theta - 4\sin 30^0 = 0$$

(d) Evaluate $\int_1^2 \frac{x}{x^2+4}\,dx$ **2**

(e) Differentiate the following with respect to x : **4**

(i) $\cos^2 3x$

(ii) $\ln\left(\frac{2x-1}{x+5}\right)$

Continue next page .../2

Question 2 (14 marks) **Marks**

(a) Solve for x : **2**

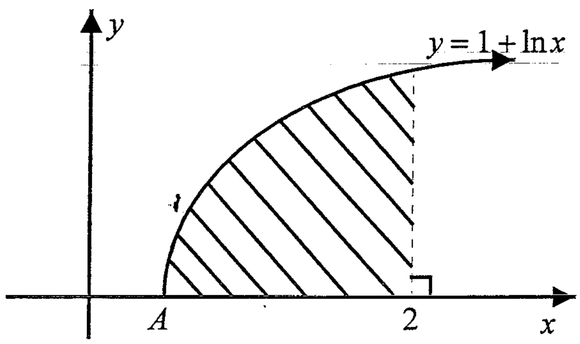
Given that $\log_e x - \log_e(x^2+3) + 2\log_e 2 = 0$

(b) Evaluate $\int_1^9 \frac{1}{\sqrt{1+7x}}\,dx$ **3**

(c) Using the identity $\sin 3x = 3\sin x - 4\sin^3 x$, **4**

evaluate $\int_0^{\pi} \sin^3 x\,dx$

(d) The diagram shows part of the curve $y = 1 + \ln x$.



(i) Find the x–co-ordinates of A. **1**

(ii) Show that $\frac{d}{dx}(x\ln x) = 1 + \ln x$. **1**

(iii) Hence, find the shaded area under the curve to two decimal places. **3**

Continue next page .../3

Question 3 (12 marks) **Marks**

(a) A particle moves in a straight line so that, t seconds after passing through O, its velocity v cm/sec is given by $v = t^2 - 6t + 5$. The particle comes to rest, firstly at A and then at B. Find

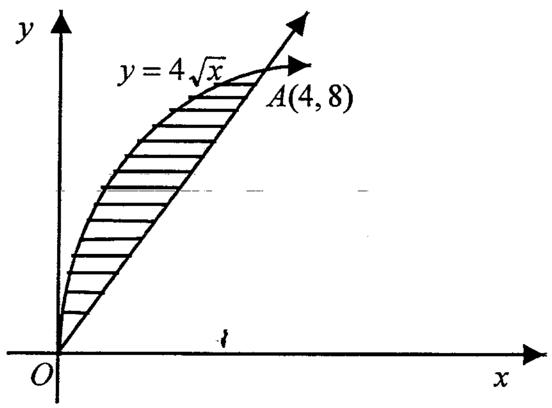
(i) an expression, in terms of t, for the distance of the particle from O at time t. **2**

(ii) the distance AB. **2**

(iii) the total distance travelled in the first 5 seconds after passing through O. **2**

(iv) the initial acceleration. **2**

(b) **4**



The diagram shows part of the curve $y = 4\sqrt{x}$, and the line OA where A is $(4, 8)$. Find the volume, in terms of π, when the shaded region is rotated through 360^0 about the x–axis.

Question 4 (18 marks) **Marks**

(a) Consider the function defined by

$$f(x) = e^{2x}(1 - x) \text{ where } -3 \le x \le 1$$

(i) Copy and complete the table of values on your answer sheet. Give values correct to two decimal places. **1**

x	-3	-2	-1	0	1
$f(x$	0.01	0.05			

(ii) Differentiate $f(x)$ and hence show that the function has only one stationary point. **2**

(iii) Sketch the curve $y = f(x)$ for $-3 \le x \le 1$. **3**

(iv) Without further use of calculus, indicate on your diagram where the curve is concave down. **1**

(v) Show that $\frac{d}{dx}\left[\frac{e^{2x}}{4}(3 - 2x)\right] = e^{2x}(1 - x)$. Hence find the exact value of the area under the curve from $-3 \le x \le 1$. **4**

(vi) Using Simpson's rule with five function values, approximate the area under the curve $y = f(x)$ for $-3 \le x \le 1$. . **2**

(vi) From your diagram, or otherwise decide whether this approximation is an over-estimate or an under-estimate of the true value of the area under the curve. Give a brief reason. **1**

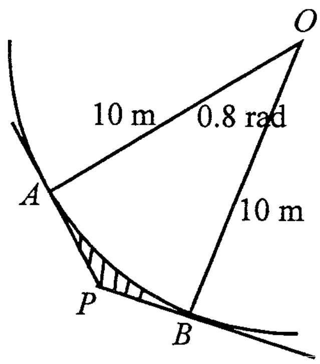
(b) Find the coordinates of the vertex, the focus and the equation of the directrix of the parabola, **4**

$$x^2 - 2x + 12y + 37 = 0$$

Continue next page .../5

Question 5 (11 marks) **Marks**

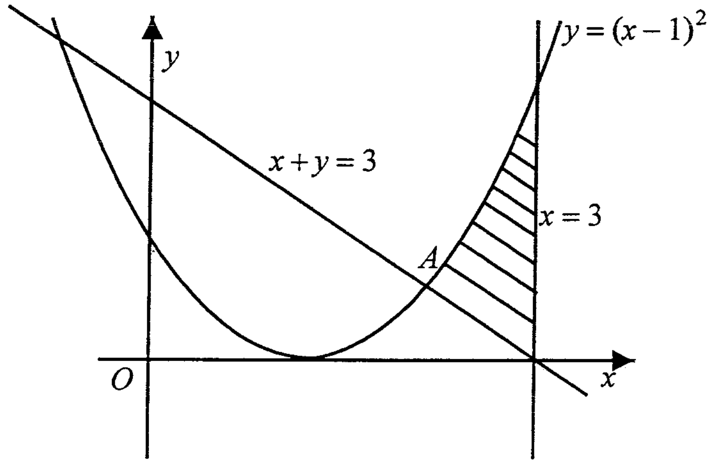
(a) The diagram shows part of a circle, centre O, of radius 10m.
The tangents at A and B on the circumference of the circle meet at the point P and the angle AOB is 0.8 radians.



(i) the length of the perimeter of the shaded region. **3**

(ii) the area of the shaded region. **3**

(b) The diagram shows the shaded region bounded by the curve $y = (x-1)^2$ and the lines $x - y = 3$ and $x = 3$.



(i) the co-ordinates of the point A. **2**

(ii) the area of the shaded region. **3**

Continue next page .../6

Question 6 (12 marks) **Marks**

(a) A solid initially at temperature $50^0 C$ cools so that its temperature T at time t seconds is given by

$T = 50e^{-kt}$ where k is a constant.

(i) Show that the above equation satisfies **2**

$$\frac{dT}{dt} = -kT$$

(ii) If the solid cools to $30^0 C$ in one minute, find k. **2**

(iii) Hence find its temperature after a further one minute. **2**

(b) On the 1st of January 2000, Beatrice joins a superannuation fund by investing \$5000 at 7% p.a. compound interest. A similar amount is invested at the beginning of each subsequent year until she retires on 31st December 2030.

(i) Show that the accumulated value of the investment at the date of retirement is given by $\dfrac{5000(1.07)(1.07^{30}-1)}{0.07}$ **3**

(ii) Find the total interest earned in her 30 years of investment. **1**

(iii) If she missed one payment at the beginning of 2020, find the amount that she would have earned. **2**

End of Assessment Task

Stephanie Sui

1

South Sydney High School 1998 3U

78/80

✓, Good effort!

QUESTION 1

A.) $\frac{d}{dx} x\tan x = \tan x + x\sec^2 x$ ✓✓

B) $e^{2x} - e^{x} - 20 = 0$ Let $u = e^x$

$u^2 - u - 20 = 0$ $u^2 = e^{2x}$.

$(u-5)(u+4) = 0$

$u = 5$ or $u = -4$

$e^x = 5$ or $e^x = -4$. Since $e^x > 0$ $\therefore e^x \neq -4$. ✓

$x = \ln 5$ ✓ $\therefore$ no solution

$\therefore x = \ln 5$

c) $5\cos 2\theta - 4\sin 30° = 0$ $0° \le \theta \le 360°$

$5\cos 2\theta - 2 = 0$ $0 \le 2\theta \le 720°$

$\cos 2\theta = \frac{2}{5}$

S	A✓
T	C✓

$2\theta = 66°25', 293°35', 426°25, 653°35'$ ✓

$\theta = 33°13', 146°48', 213°13', 326°48'$ ✓

D) $\int_1^2 \frac{x}{x^2+4}\,dx = \frac{1}{2}\left[\ln(x^2+4)\right]_1^2$

$= \frac{1}{2}(\ln 8 - \ln 5)$ ✓

$= \frac{1}{2}\ln\frac{8}{5}$ ✓

E) i.) $\frac{d}{dx}\cos^2 3x$

$= 2\cos 3x \cdot -3\sin 3x$ ✓

$= -6\cos 3x\sin 3x = -3 \times 2\sin 3x\cos 3x$

$= -3\sin 6x$ ✓

ii.) $\frac{d}{dx} \ln \left(\frac{2x-1}{x+5}\right) = \frac{d}{dx}\left[\ln(2x-1) - \ln(x+5)\right]$

13

$= \frac{2}{2x-1} - \frac{1}{x+5}$ ✓

$= \frac{2(x+5) - 2x + 1}{(2x-1)(x+5)} = \frac{2x+10-2x+1}{(2x-1)(x+5)}$

$= \frac{11}{(2x-1)(x+5)}$ ✓

Question 2

A.) $\log_e x - \log_e(x^2+3) + 2\log_e 2 = 0$

$\log_e \frac{x}{x^2+3} = -2\log_e 2$

$\log_e \frac{x}{x^2+3} = \log_e \frac{1}{4}$

✓

$\frac{x}{x^2+3} = \frac{1}{4}.$

$4x = x^2+3$; $x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$ ✓

$x = 3$ or $x = 1$

Check

$\log_e 3 - \log_e 12 + \log_e 4 = 0$ ✓✓

$\log_e 1 - \log_e 4 + \log_e 4 = 0$ ✓✓ $\therefore x = 3$ or $x = 1$

B) $\displaystyle\int_1^9 \frac{1}{\sqrt{1+7x}}\,dx = \int_1^9 (1+7x)^{-\frac{1}{2}}\,dx$

$$= \left(\frac{(1+7x)^{\frac{1}{2}}}{\frac{1}{2}\times 7}\right)_1^9 \checkmark$$

$$= \left(\frac{2}{7}\sqrt{1+7x}\right)_1^9 \checkmark$$

$$= \frac{2}{7}\left(\sqrt{1+63} - \sqrt{1+7}\right)$$

$$= \underline{\frac{2}{7}\left(8 - 2\sqrt{2}\right)} \checkmark$$

14

c) $\displaystyle\int_0^\pi \sin^3 x\,dx = \int_0^\pi \frac{3\sin x - \sin 3x}{4}\,dx$ ✓

$$= \frac{1}{4}\int_0^\pi 3\sin x - \sin 3x\,dx$$

$$= \frac{1}{4}\left(-3\cos x + \frac{1}{3}\cos 3x\right)_0^\pi \checkmark$$

$$= \frac{1}{4}\left(-3\cos\pi + \frac{1}{3}\cos 3\pi + 3\cos 0 - \frac{1}{3}\cos 0\right)$$

$$= \frac{1}{4}\left(3 + \frac{1}{3} + 3 - \frac{1}{3}\right) = \frac{1}{4}\times 6 = \underline{\underline{\frac{3}{2}}} \checkmark$$

D) $y = 1 + \ln x$

i.) At A, $y = 0$.

$1 + \ln x = 0$

$\ln x = -1$; $x = e^{-1}$ $\therefore A = \underline{\left(\frac{1}{e}, 1\right)}$ ✓

ii.) $\dfrac{d}{dx}(x\ln x) = \ln x + x\cdot\dfrac{1}{x} = \ln x + 1$ ✓

iii.) $y = 1 + \ln x$

$\ln x = y - 1$; $\underline{x = e^{y-1}}$. At $x = 2$, $y = 1 + \ln 2$.

Area $= 2(1+\ln 2) - \displaystyle\int_0^{1+\ln 2} e^{y-1}\,dy$ ✓

$= 2 + 2\ln 2 - \left(e^{y-1}\right)_0^{1+\ln 2}$

$= 2 + 2\ln 2 - \left(e^{\ln 2} - e^{-1}\right)$

$= \cancel{2} + 2\ln 2 - \cancel{2} + \frac{1}{e}$

$= \left(2\ln 2 + \frac{1}{e}\right) u^3$

Alternately,

$\int_{\frac{1}{e}}^{2} 1 + \ln x \, dx$

$= \left[x \ln x\right]_{\frac{1}{e}}^{2}$

$= 2\ln 2 - \frac{1}{e} \ln e^{-1}$

$= \left(2\ln 2 + \frac{1}{e}\right) u^3$

Question 3

A) $V = t^2 - 6t + 5$.

i.) $\frac{dx}{dt} = t^2 - 6t + 5$

$x = \int t^2 - 6t + 5 \, dt$

$= \frac{t^3}{3} - 3t^2 + 5t + c$

ii.) Find t when $v = 0$

$t^2 - 6t + 5 = 0$

$(t-5)(t-1) = 0 \quad t = 5 \text{ or } t = 1$

$\therefore$ particle is stationary at $t = 1, 5$ sec

when $t = 1$, $x = \frac{1}{3} - 3 + 5 + c = \frac{1}{3} + 2 + c$

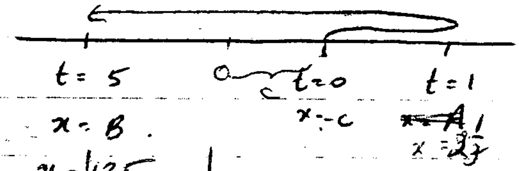
when $t = 5$, $x = \frac{125}{3} - 75 + 25 + c = \frac{125}{3} - 50 + c$

Distance AB = Distance $(x)_{t_5 - t_1}$

$= \left|\frac{125}{3} - 50 - \frac{1}{3} - 2 + \cancel{c - c}\right|$

$= \left|\frac{124}{3} - 52\right| = \left|\frac{-32}{3}\right| = \frac{32}{3}$ cm

iii.) ~~When $t=0$, $x=c$ and $v=5$.~~
~~When $t=1$, $x=A$ and $v=0$~~
~~When $t=2$,~~



Total distance travelled $x = \left|\frac{125}{3} - 50\right|$

$= \cancel{\left(\frac{1}{3}+2+c-c\right) + \left(\frac{1}{3}+2+c+\frac{125}{3}-50\right)}$

$= 2\frac{1}{3} + 2\frac{1}{3} + \cancel{\frac{125}{3}} \cancel{-50}\ 50 - \frac{125}{3}$ ✓

$= \frac{7}{3} + \frac{7}{3} - \frac{125}{3} + 50 = \underline{13\text{m}}$ ✓

iv.) $\ddot{x} = 2t - 6$. ✓
When $t=0$, $\underline{\ddot{x} = -6\text{cm}/s^2}$ ✓

$\therefore$ initial acceleration $= -6\text{cm}/s^2$

B.) Grad. of OA $= \frac{8}{4} = 2$

$\therefore$ eqt of OA is $y = 2x$ ✓

Volume of shaded region $= \pi \int_0^4 16x \, dx - \pi \int_0^4 4x^2 \, dx$ ✓

$= \pi \left(8x^2\right)_0^4 - \pi \left(\frac{4x^3}{3}\right)_0^4$ ✓

12

$= \pi (128) - \pi \left(\frac{256}{3}\right)$

$= \pi \left(128 - \frac{256}{3}\right) = \left(\frac{128\pi}{3}\right) u^3$ ✓

QUESTION 4

A.) $f(x) = e^{2x}(1-x)$ $\quad -3 \le x \le 1$

i.)

x	-3	-2	-1	0	1
$f(x)$	0.01	0.05	0.27	1	0

✓

ii.) $f(x) = e^{2x} - xe^{2x}$

$f'(x) = 2e^{2x} - \left(e^{2x} + x.2e^{2x}\right)$ ✓

$= 2e^{2x} - e^{2x} - 2xe^{2x} = e^{2x} - 2xe^{2x}$

$= e^{2x}(1-2x)$

At stationary pts, $f'(x) = 0$ ✓

$e^{2x}(1-2x) = 0$

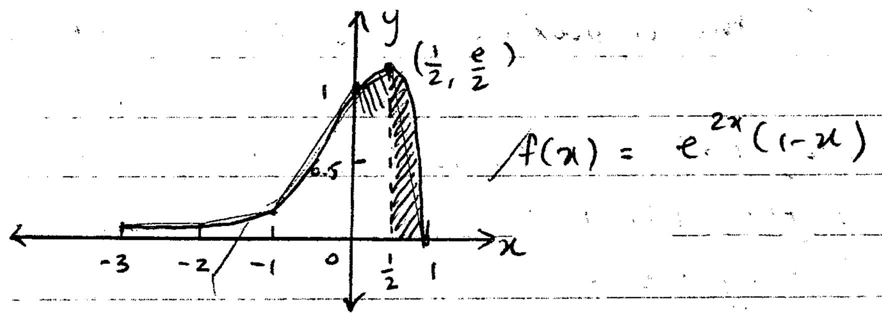
$e^{2x} > 0$ for all x.

$\therefore 1 - 2x = 0$

$x = \frac{1}{2}$, $y = \frac{1}{2}e$ ✓

$\therefore$ $f(x)$ has only 1 stationary pt at $\left(\frac{1}{2}, \frac{e}{2}\right)$

iii)



iv.) Curve is concave

v.) $$\frac{d}{dx}\left(\frac{e^{2x}}{4}(3-2x)\right) = \frac{d}{dx}\left(\frac{3}{4}.e^{2x} - \frac{xe^{2x}}{2}\right)$$

$$= \left(\frac{3}{4}\times 2e^{2x}\right) - \frac{1}{2}\left(e^{2x} + x.2e^{2x}\right)$$

$$= \frac{3}{2}e^{2x} - \frac{e^{2x}}{2} - xe^{2x}$$

$$= e^{2x} - xe^{2x} = \underline{e^{2x}}(1-x)$$

$$\text{Area} = \int_{-3}^{1} e^{2x}(1-x)\,dx$$

$$= \left(\frac{e^{2x}}{4}(3-2x)\right)_{-3}^{1} = \frac{e^2}{4} - \frac{e^{-6}}{4}(3+6)$$

$$= \frac{e^2}{4} - \frac{9e^{-6}}{4}$$

$$= \left(\frac{e^2}{4} - \frac{9}{4e^6}\right) u^2$$

vi.) $$\text{Area} \doteqdot \frac{1}{3}\left(0.01+4(0.05)+2(0.27)+4+0\right)$$

$$\doteqdot \frac{19}{12} \doteqdot 1.583\, u^2$$

vi.) The exact area, using integration $= 1.841686832u^2$.
Area, using ~~~~ simpson's rule $= 1.583u^2 < 1.842u^2$

$\therefore$ the approximation is an UNDER-estimate

B) $x^2 - 2x + 12y + 37 = 0$ (complete the square)
$(x^2 - 2x + 1) + 12y + 37 - 1 = 0$
$(x-1)^2 + 12y + 36 = 0$
$(x-1)^2 = -12(y+3)$ ✓

18

$\therefore$ Vertex $= (1, -3)$ ✓
focal length; $4a = 12$
$a = 3$ $\therefore$ focus $= (1, -6)$

Directrix is $y = 0$ ie x-axis ✓

QUESTION 5

A) i.) arc $AB = r\theta = 10 \times 0.8 = 8m$ ✓
$AP = PB$ (tgts drawn from an exterior point are equal)
In ΔOPB, $\tan 0.4$ radians $= \frac{PB}{10}$; $PB = 4.23m$ ✓

$\therefore$ perimeter $= 2 \times 4.23 + 8$
$= 16.5m$ (1dp).

ii.) In ΔOPB, using pythag theorem,
$10^2 + 4.23^2 = OP^2$
$OP = 10.857$ ($OP > 0$ b/c it is a length)
Area of $\Delta OPB = \frac{1}{2} \times 10 \times$

ii.) Area of shaded region = area of quad AOBP − area of sector AOB.

Area of AOBP = 2 × Area ΔOPB ($\Delta OPB \equiv \Delta APO$ (SSS))

$= 2 \times \frac{1}{2} \times PB \times 10$

$= 4.23 \times 10 = \underline{42.279\ m^2}$

Area of sector AOB $= \frac{0.8}{2\pi} \times \pi(10^2) = 40m^2$

$\therefore$ area of shaded region $= 42.279 - 40$

$= \underline{\underline{2.28m^2}}$ (2dp)

B) i.) $y = (x-1)^2$ — ①

$y = 3 - x$ — ②

to find A, solve ① and ② simult.

$(x-1)^2 = 3 - x$

$x^2 - 2x + 1 = 3 - x$.

$\cancel{x^2} - \cancel{2x} + 1 - 3 + \cancel{x} = 0$

$x^2 - x - 2 = 0$; $(x-2)(x+1) = 0$

$x = 2$ or $x = -1$

<u>But</u> the x-coordinate of A is positive

$\therefore \underline{x = 2}$

when $x = 2$, $y = (2-1)^2 = 1$ $\therefore$ $\underline{A = (2,1)}$

ii.) Area $= \int_2^3 (x-1)^2 - (3-x)\, dx$

$= \int_2^3 \cancel{x^2} - \cancel{2x} + 1 - 3 + \cancel{x}\, dx = \int_2^3 x^2 - x - 2\, dx$

$= \left(\frac{x^3}{3} - \frac{x^2}{2} - 2x\right)_2^3$

$= \left(\frac{27}{3} - \frac{9}{2} - 6 - \frac{8}{3} + \frac{4}{2} + 4\right)$

$= \left(9 - \frac{9}{2} - \cancel{6} - \frac{8}{3} + \cancel{6}\right) = \left(\frac{9 - 27 - 16}{6}\right) = \left(\frac{11}{6}\right)$

area $= \frac{11}{\ }u^2$

QUESTION 6

A.) i.) $T = 50e^{-kt}$ — ①

$\frac{dT}{dt} = 50 \cdot -ke^{-kt}$ (sub in ①)

$= -kT$

ii.) when $t=0$, $T=50$

$50 = 50e^{0}$

when $t=60$, $T=30$.

$30 = 50e^{-60k}$

$e^{-60k} = 0.6$

$-60k = \ln 0.6$

$k = \frac{\ln 0.6}{-60}$

iii.) when $t=120$, Find T.

$T = 50e^{\frac{\ln 0.6}{60} \times 120}$

$= 18°C$

B) Let A_n be amount in fund after n years.

$A_1 = \$5000(1.07)$

$A_2 = \$5000(1.07)^2 + \$5000(1.07)$

$A_3 = \$5000(1.07)^3 + \$5000(1.07)^2 + \$5000(1.07)$

$= \$5000(1.07 + 1.07^2 + 1.07^3)$

i.) $A_{30} = \$5000(1.07 + 1.07^2 + 1.07^3 + \ldots 1.07^{30})$

$= \$5000\left(\frac{1.07(1.07^{30}-1)}{1.07-1}\right)$

$= \$5000\left(\frac{1.07(1.07^{30}-1)}{0.07}\right)$

6

ii.) Total interest earned $= \$5000\left(\frac{1.07(1.07^{30}-1)}{0.07}\right) - \$5000(30)$

$= \$355,365$ ✓

iii.) $A_{19} = \$5000\left(\frac{1.07(1.07^{19}-1)}{0.07}\right)$

$A_{20} = \$5000\left(\frac{1.07(1.07^{19}-1)}{0.07}\right) \times 1.07$

$\times 1.07^2$

$A_{21} = \$5000\left(\frac{1.07(1.07^{19}-1)}{0.07}\right) + \$5000\,(1.07)$

10

$A_{22} = \$5000 \times 1.07^3\left(\frac{1.07(1.07^{19}-1)}{0.07}\right) + \$5000(1.07+1.07^2)$

$A_{30} = \$5000 \times 1.07^{11}\left(\frac{1.07(1.07^{19}-1)}{0.07}\right) + \$5000\,(1.07+\dots 1.07^{10})$

$= \$5000 \times \frac{1.07^{12}(1.07^{19}-1)}{0.07} + \$5000\left(\frac{1.07(1.07^{10}-1)}{0.07}\right)$

$= \$494,840.95$ X ~~or~~ subtract A_{10} from the total in (i)

i.e. $A_{10} = PR^{10}$

$= 5000\,(1.07)^{10}$

$= \$9835.76$

$\therefore$ Total amount invested $= \$505365.21 - \$9835.76$

$= \cancel{\$49555}$

$= \$495\,529$ (to nearest dollar).