South Sydney High

Year 12

Half Yearly

2002

Extension II Mathematics

Time Allowed – 2 Hours
Plus 5 Minutes Reading Time

Directions to Candidates

- Attempt ALL questions
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Start each question on a new page.
- Board approved calculators may be used.

Question 1: (13 marks)

(a) The complex number w is given by $w = -1 + \sqrt{3}i$

i. Show that $w^2 = 2\overline{w}$ **2**

ii. Evaluate $|w|$ and $\arg w$. **2**

iii. Show that w is a root of $w^3 - 8 = 0$ **1**

(b) Sketch the locus of z satisfying:

i. $\text{Re}(z) = |z|$ **2**

ii. both $\text{Im}(z) \geq 2$ and $|z-1| \leq 2$ **3**

(c) Given that *a* and *b* are real numbers and **3**

$$\frac{a}{1+i} + \frac{b}{1+2i} = 1$$

find the values of *a* and *b*.

Question 2: (13 marks)

(a)

i. Express $z = 2 + 2i$ in modulus –argument form. **2**

ii. Hence write z^8 in the form $a + ib$ where *a* and *b* are real. **2**

(b) Find the locus of z if $2|z| = z + \overline{z} + 4$ **2**

(c) If *a, b, c, d* are real, and $ad > bc$, show that $\text{Im}\left(\frac{a+ib}{c+id}\right) < 0$ **2**

(d) If *P* represents the complex number *z*, where *z* satisfies

$$|z-2| = 2 \text{ and } 0 < \arg z < \frac{\pi}{2}$$

i. Show that $|z^2 - 2z| = 2|z|$ **2**

ii. Find the value of *k* (a real number) if $\arg(z-2) = k\arg(z^2 - 2z)$. **3**

Question 3: (15 marks)

(a) The equation $2x^3+5x+1=0$ has roots α,β,γ. Evaluate $\alpha^3+\beta^3+\gamma^3$. **2**

(b) Given the polynomial $P(x)=2x^3-4x^2+mx+n$ where m and n are real numbers.

i. Find the values of m and n if $1+i$ is a root of $P(x)=0$. **3**

ii. Find the zeros of $P(x)$. **1**

(c) A monic cubic polynomial when divided by x^2-9 leaves a remainder of $x+8$ and when divided by x leaves a remainder of -4. Express the polynomial in the form **3**

$$ax^3+bx^2+cx+d.$$

(d)

i. By letting $c=\cos\theta$, show that the equation $\cos 4\theta=\cos 3\theta$ can be expressed in the form $8c^4-4c^3-8c^2+3c+1=0$. **2**

ii. Show that $\theta=\frac{2n\pi}{7}$, where n is an integer, satisfies the equation $\cos 4\theta=\cos 3\theta$. **2**

iii. Using parts i and ii above, find the equation whose roots are $\cos\frac{2\pi}{7},\cos\frac{4\pi}{7},\cos\frac{6\pi}{7}$, expressing your answer in polynomial form. **2**

Question 4: (15 marks)

(a) The equation $x^3+bx^2+x+2=0$ where b is a real number has roots α,β,γ.

i. Obtain an expression in terms of b for $\alpha^2+\beta^2+\gamma^2$. **2**

ii. Hence determine the set of possible values of b if the roots of the above equation are real. **1**

iii. Write down the equation whose roots are **2**

$$2\alpha,2\beta,2\gamma.$$

(b) Given that the polynomial $P(x)=x^4+x^3-3x^2-5x-2$ has a zero of multiplicity 3, find all the zeros of $P(x)$. **3**

(c) If α,β,γ are roots of $x^3-3x^2+2x+1=0$, find the polynomial equation whose roots are **2**

$$\alpha-3,\beta-3,\gamma-3$$

(d) If $px^3+qx^2+rx+s=0$ has one of its roots the reciprocal of the other, show that **5**

$$p^2-s^2=pr-qs$$

Question 5: (19 marks)

(a) An ellipse has equation $\frac{x^2}{9}+\frac{y^2}{4}=1$.

i. Find the eccentricity, co-ordinates of the foci S and S', and the equations of the directrices. **3**

ii. Find the equation of the tangent to the ellipse at a point $P(3\cos\theta, 2\sin\theta)$ on it, where θ is the auxiliary angle. **3**

iii. The ellipse meets the y-axis at the points *A* and *B*. The tangents to the ellipse at *A* and *B* meet the tangent at *P* at the points *C* and *D* respectively. **4**

Prove that $AC \cdot BD = 9$.

(b) *P* is any point on the ellipse $3x^2+4y^2=9$ and S and S' are the foci of the ellipse

i. Sketch the curve showing the position of the foci and directrices. **3**

ii. Prove that $SP+S'P=4$ **2**

iii. Prove that the normal at *P* bisects the $\angle SPS'$. **4**

Ext. II Half Yearly 2002 Solutions

Question 1:

(a) $W=-1+\sqrt{3}i$

(i) $w=-1+\sqrt{3}i$

$w^2=(-1+\sqrt{3}i)^2$

$=1-2\sqrt{3}i-3$

$=-2-2\sqrt{3}i$

$2\bar{w}=2(-1-\sqrt{3}i)$

$=-2-2\sqrt{3}i$

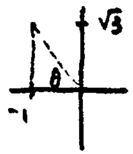
$=w^2$

Hence proven #

2

(ii) $|w|=\sqrt{(-1)^2+(\sqrt{3})^2}$

$=2$



$\arg w=\tan^{-1}\frac{\sqrt{3}}{-1}$

$=\pi-\frac{\pi}{3}$

$=\frac{2\pi}{3}$

2

(iii) $w=2\,\text{cis}\,\frac{2\pi}{3}$

$L.H.S = w^3-8=8\,\text{cis}\,3\left(\frac{2\pi}{3}\right)-8$

$=8\cdot 1-8$

$=0 = R.H.S$

1

(b)

(i) $Re(z)=|z|$

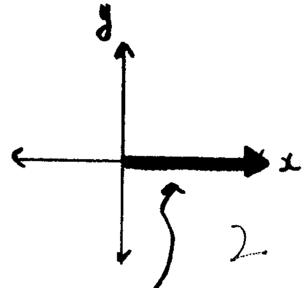
$\therefore x\geq 0$

$x=\sqrt{x^2+y^2}$

$x^2=x^2+y^2$

$y^2=0$

$y=0,\ x\geq 0$



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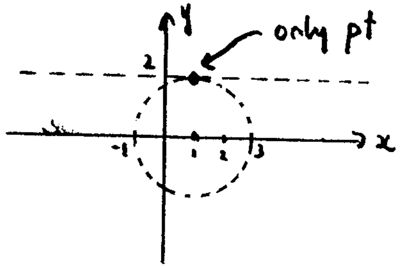
(ii) both $Im(z)\geq 2$ & $|z-1|\leq 2$



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(c) $\frac{a}{1+i}+\frac{b}{1+2i}=1$

$a+2ai+b+bi=(1+i)(1+2i)$

$(a+b)+(2a+b)i=1-2+3i$

equating real & imaginary parts

$a+b=-1$ ... ①

$2a+b=3$ ... ②

②-① $\Rightarrow a=4$

$\therefore$ sub $a=4$ into ①

$4+b=-1$

$b=-5$

$\therefore a=4$

$b=-5$ #

3

Question 2:

(a)

(i) $z=2+2i$

$|z|=\sqrt{2^2+2^2}$

$=\sqrt{8}=2\sqrt{2}$

$\tan\theta=\frac{2}{2}=1\quad\therefore\theta=\frac{\pi}{4}$

$\therefore z=\sqrt{8}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

2

(ii) $z^8=\left[\sqrt{8}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^8$

$=(\sqrt{8})^8\left(\cos\frac{8\pi}{4}+i\sin\frac{8\pi}{4}\right)$

$=4096(\cos 2\pi+i\sin 2\pi)$

$=4096\cdot 1$

$=4096$ #

2

(b) $2|z|=z+\bar{z}+4$

$2\sqrt{x^2+y^2}=x+iy+x-iy+4$

$2\sqrt{x^2+y^2}=2x+4$

$4(x^2+y^2)=4x^2+16x+16$

$4x^2+4y^2=4x^2+16x+16$

$4y^2=16x+16$

$y^2=4(x+1)$ #

2

(c) $\frac{a+ib}{c+id}=\frac{(a+ib)(c-id)}{c^2+d^2}$

$=\frac{ac-bd}{c^2+d^2}+i\,\frac{bc-ad}{c^2+d^2}$

So $Im\left(\frac{a+ib}{c+id}\right)=\frac{bc-ad}{c^2+d^2}$

Now $bc<ad$

$\therefore bc-ad<0$

2

$\therefore \frac{bc-ad}{c^2+d^2}<0$ since c, d are real

$\therefore Im\left(\frac{a+ib}{c+id}\right)<0$ #

(d)

(i) show $|z^2-2z|=2z$

$^*|z-2|=2$

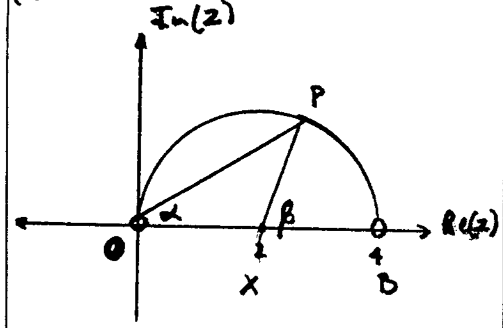
$|z^2-2z|=|z(z-2)|$

$=|z||z-2|$

$=2|z|$ #

2

(ii)



$\angle POX = \alpha = \arg z$

$\angle PXB = \beta = \arg(z-2)$

$OX = XP$ (radii)

$\therefore \triangle OXP$ isosceles (2 sides equal)

$\therefore \angle XPO = \alpha$ (base $\angle$s of isosceles $\triangle$ are equal)

$\therefore \angle PXB = \beta = 2\alpha$ (external $\angle$ of $\triangle$)

ie $\arg(z-2) = 2\arg z$

Now $\arg(z-2) = k\arg(z^2-2z)$

$2\arg z = k\arg[z(z-2)]$

$2\arg z = k[\arg z + \arg(z-2)]$

$2\arg z = k(\arg z + 2\arg z)$

$2 = 3k$

$\therefore k = \frac{2}{3}$ #

3

QUESTION 3:

(a) $2x^3+5x+1=0$ Roots α, β, γ

$\alpha+\beta+\gamma = -\frac{b}{a} = 0$

$2\alpha^3+5\alpha+1 = 0$

$2\beta^3+5\beta+1 = 0$

$2\gamma^3+5\gamma+1 = 0$ +

$2(\alpha^3+\beta^3+\gamma^3)+5(\alpha+\beta+\gamma)+3=0$

$2(\alpha^3+\beta^3+\gamma^3)+5(0)+3=0$

2

$\alpha^3+\beta^3+\gamma^3 = -\frac{3}{2}$ #

(b) $P(x) = 2x^3-4x^2+mx+n$

Since the co-efficients of $P(x)$ are real, its complex roots occur in conjugate pairs.

So $1-i$ is a root since $1+i$ is a root.

Let the third root be α.

$\alpha+(1+i)+(1-i) = -\frac{-4}{2} = 2$ (sum of roots)

$\alpha+2=2$

$\alpha=0$

$P(\alpha)=0$

$P(0)=0$

$P(0)=n$

3

so $n=0$

Sum of roots 2 at a time

$0(1+i)+0(1-i)+(1-i)(1+i) = \frac{m}{2}$

$1+1 = \frac{m}{2}$

$m=4$

$\therefore n=0,\ m=4$ #

(ii) The zeros of $P(x)$ are $(1-i)$ $(1+i)$ and 0. 1

(c) Let the polynomial be $P(x)$

$P(x) = (x^2-9)(x+A) + (x+B) \ldots ①$

$P(x) = (x)(x^2+Bx+c) + (-4) \ldots ②$

① = ②

$(x^2-9)(x+A) + (x+B) = x(x^2+Bx+c)+-4$

$x^3+Ax^2-9x-9A+x+B$

$= x^3+Bx^2+cx-4$

$x^3+Ax^2-8x+(-9A+B)$

$= x^3+Bx^2+cx-4$

Equating coefficients

$A = B$

$-8 = c$

$-9A+8 = -4$

$-9A = -12$

$A = \frac{4}{3}$ $\therefore B = 4/3$

$\therefore P(x) = x^3+\frac{4}{3}x^2-8x-4$

$\therefore$ In required form poly is

$3x^3+4x^2-24x-12$ #

3

(d) $\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$

$= c^4+4c^3is-6c^2s^2-4cis^3+s^4$

Equating real parts

$\cos 4\theta = c^4-6c^2s^2+s^4$

$= c^4-6c^2(1-c^2)+(1-c^2)^2$

[since $s^2+c^2=1$]

$= c^4-6c^2+6c^4+1-2c^2+c^4$

$= 8c^4-8c^2+1$

Now

$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$

$= c^3+3c^2is-3cs^2-is^3$

Equating real parts

$\cos 3\theta = c^3-3cs^2$

$= c^3-3c(1-c^2)$

$= c^3-3c+3c^3$

$= 4c^3-3c$

2

Now $\cos 4\theta = \cos 3\theta$ then becomes

$8c^4-8c^2+1 = 4c^3-3c$

$8c^4-4c^3-8c^2+3c+1=0$ #

(ii) when $n=1$, $\theta = \frac{2\pi}{7}$

$$\cos 4\theta = \cos\left(\frac{2\pi}{7}\times 4\right)$$
$$= \cos\left(\frac{8\pi}{7}\right)$$
$$= \cos\left(2\pi - \frac{8\pi}{7}\right)$$
$$= \cos\frac{6\pi}{7}$$
$$= \cos 3\theta \ \#$$

when $n=2$ $\theta = \frac{4\pi}{7}$

$$\cos 4\theta = \cos\frac{16\pi}{7} = \cos\frac{2\pi}{7}$$
$$= \cos\left(-\frac{2\pi}{7}\right)$$
$$= \cos\frac{12\pi}{7}$$
$$= \cos 3\theta \ \#$$

when $n=3$, $\theta = \frac{6\pi}{7}$

$$\cos 4\theta = \cos\frac{24\pi}{7} = \cos\left(-\frac{3\pi}{7}\right)$$
$$= \cos\left(\frac{3\pi}{7}\right) \quad 2$$
$$= \cos\frac{18\pi}{7}$$
$$= \cos 3\theta \ \#$$

when $n=4$, $\theta = \frac{8\pi}{7}$

$$\cos 4\theta = \cos\frac{32\pi}{7} = \cos\frac{4\pi}{7}$$
$$= \cos\left(-\frac{4\pi}{7}\right)$$
$$= \cos\frac{24\pi}{7}$$
$$= \cos 3\theta \ \#$$

$\cos 4\theta = \cos 3\theta$ can be expressed as a quartic, so these 4 soln's are the only solutions so $\theta = \frac{2n\pi}{7}$ satisfies $\cos 4\theta = \cos 3\theta$ #

(iii) $\cos\frac{2\pi}{7}$, $\cos\frac{4\pi}{7}$, $\cos\frac{6\pi}{7}$ are 3 roots of $8c^4 - 4c^3 - 8c^2 + 3c + 1 = 0$ since $\theta = \frac{2n\pi}{7}$ satisfies the equivalent $\cos 4\theta = \cos 3\theta$.

The 4th root of $8c^4 - 4c^3 - 8c^2 + 3c + 1 = 0$ is $c=1$ by inspection, So to find the equation whose roots are $\cos\frac{2\pi}{7}$, $\cos\frac{4\pi}{7}$ & $\cos\frac{6\pi}{7}$, divide by $(c-1)$

$$\begin{array}{r|l} & 8c^3 + 4c^2 - 4c - 1 \\ \hline c-1 & 8c^4 - 4c^3 - 8c^2 + 3c + 1 \\ & 8c^4 - 8c^3 \\ \hline & 4c^3 - 8c^2 \\ & 4c^3 - 4c^2 \\ \hline & -4c^2 + 3c \\ & -4c^2 + 4c \\ \hline & -c + 1 \\ & -c + 1 \\ \hline & 0 \end{array}$$

∴ The eqn is

$$8c^3 + 4c^2 - 4c - 1 = 0 \ \# \quad 2$$

QUESTION 4:

(a) $x^3 + bx^2 + x + 2 = 0$ $\quad \alpha\,\beta\,\gamma$

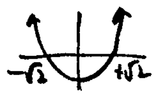
$\Sigma\alpha = -\frac{b}{a} = -b$

$\Sigma\alpha\beta = \frac{c}{a} = 1$

$\Sigma\alpha\beta\gamma = -\frac{d}{a} = -2$

(i) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$
$$= (-b)^2 - 2(1)$$
$$= b^2 - 2 \ \# \quad 2$$

(ii) all real if $\alpha^2 + \beta^2 + \gamma^2 \geq 0$
$$\therefore b^2 - 2 \geq 0 \quad 1$$



$\therefore +\sqrt{2} \leq b \leq \sqrt{2}$ #

(iii) if $2\alpha, 2\beta, 2\gamma$ are roots of new eqn y, then $\alpha = \frac{y}{2}$ is a root of the original eqn $\quad 2$

$$\therefore \left(\frac{y}{2}\right)^3 + b\left(\frac{y}{2}\right)^2 + \frac{y}{2} + 2 = 0$$
$$y^3 + 2by^2 + 4y + 16 = 0 \ \#$$

(b) let α be the root of mult 3.

$\therefore P(\alpha) = P'(\alpha) = P''(\alpha) = 0$

$P'(x) = 4x^3 + 3x^2 - 6x - 5$

$P''(x) = 12x^2 + 6x - 6$

Let $P''(x) = 0$

$\therefore 2x^2 + x - 1 = 0$

$(2x-1)(x+1) = 0$

$\therefore x = \frac{1}{2}$ or -1

$P'(-1) = -4 + 3 + 6 - 5 = 0$

$\therefore -1$ is the triple root

Since $P(x)$ is a quartic there are at most 4 roots.

Product of roots $= -2$
$$= (-1)^3 \cdot \beta \quad 3$$

$\therefore \beta = 2$

$\therefore$ zeros of $P(x)$ are $-1, -1, -1, 2$ #

(c) $y = x - 3 \Rightarrow x = y + 3$

$(y+3)^3 - 3(y+3)^2 + 2(y+3) + 1 = 0$

$y^3 + 9y^2 + 27y + 27 - 3y^2 - 18y - 27 + 2y + 6 + 1 = 0$

$y^3 + 6y^2 + 11y + 7 = 0$ #

ie $x^3 + 6x^2 + 11x + 7 = 0$ # $\quad 2$

(d) $px^3+qx^2+rx+s=0$

ie $x^3+\frac{q}{p}x^2+\frac{r}{p}x+\frac{s}{p}=0$

Let the roots be $\alpha, \frac{1}{\alpha}, \beta$

$\alpha\cdot\frac{1}{\alpha}\cdot\beta = -\frac{s}{p}$

$\therefore \beta = -\frac{s}{p}$

$\alpha+\frac{1}{\alpha}+\beta = -q/p$

$\therefore \alpha+\frac{1}{\alpha} = -q/p + \frac{s}{p}$

$\alpha\cdot\frac{1}{\alpha} + \alpha\beta + \frac{1}{\alpha}\beta = \frac{r}{p}$

$\therefore 1+\beta\left(\alpha+\frac{1}{\alpha}\right) = r/p$

$\therefore \alpha+\frac{1}{\alpha} = \left(\frac{r}{p}-1\right)\div s/p$

$= -r/s + p/s$

5

$\therefore -r/s + p/s = -q/p + s/p$

$-rp+p^2 = -sq+s^2$

$\therefore p^2-s^2 = pr-qs$ #

QUESTION 5:

(a) $\frac{x^2}{9}+\frac{y^2}{4}=1$

(i) $a^2=9$ $b^2=4$

$b^2=a^2(1-e^2)$

$4=9(1-e^2)$

$\frac{4}{9}=1-e^2$

$e^2=5/9$

$e=\frac{\sqrt{5}}{3}$ #

foci at $(\pm ae, 0)$ $a=3$

$\therefore S(\sqrt{5},0)$ # $S'(-\sqrt{5},0)$ #

3

Directrices at $x=\pm\frac{a}{e}$

$x=\pm\frac{9}{\sqrt{5}}$ #

(ii) $\frac{x^2}{9}+\frac{y^2}{4}=1$

$\frac{2x}{9}+\frac{2y}{4}\frac{dy}{dx}=0$

$\therefore \frac{dy}{dx} = -\frac{4x}{9y}$

3

at P

$\frac{dy}{dx} = \frac{-4\cdot 3\cos\theta}{9\cdot 2\sin\theta} = -\frac{2}{3}\frac{\cos\theta}{\sin\theta}$

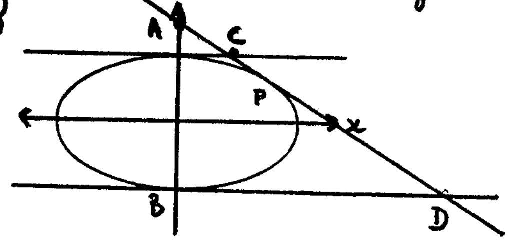
$y-y_1=m(x-x_1)$

$y-2\sin\theta = -\frac{2}{3}\frac{\cos\theta}{\sin\theta}(x-3\cos\theta)$

$3y\sin\theta-6\sin^2\theta = -2x\cos\theta+6\cos^2\theta$

$3y\sin\theta+2x\cos\theta=6$ is the tangent.

(iii)



A is the y-int of the ellipse ie (0,2)

so tangent at A is $y=2$

IIIly, tangent at B is $y=-2$

Find C,

C has y-coord 2

sub into eqn of tangent at P

$6\sin\theta+2x\cos\theta=6$

$2x\cos\theta=6-6\sin\theta$

$x=\frac{3-3\sin\theta}{\cos\theta}$

$\therefore$ C is $\left(\frac{3-3\sin\theta}{\cos\theta}, 2\right)$

$\therefore AC = \left|\frac{3-3\sin\theta}{\cos\theta}\right|$ 4

Similarly for D

$-6\sin\theta+2x\cos\theta=6$

$x=\frac{3+3\sin\theta}{\cos\theta}$

$\therefore BD = \left|\frac{3+3\sin\theta}{\cos\theta}\right|$

$\therefore AC\cdot BD$

$= \left|\frac{3-3\sin\theta}{\cos\theta}\right| \left|\frac{3+3\sin\theta}{\cos\theta}\right|$

$= \left|\frac{9-9\sin^2\theta}{\cos^2\theta}\right|$

$= \left|\frac{9\cos^2\theta}{\cos^2\theta}\right| = 9$ #

(b) $3x^2+4y^2=9$

(i) $\frac{x^2}{3}+\frac{y^2}{9/4}=1$

$a=\sqrt{3}$ $b=3/2$

$9/4=3(1-e^2)$

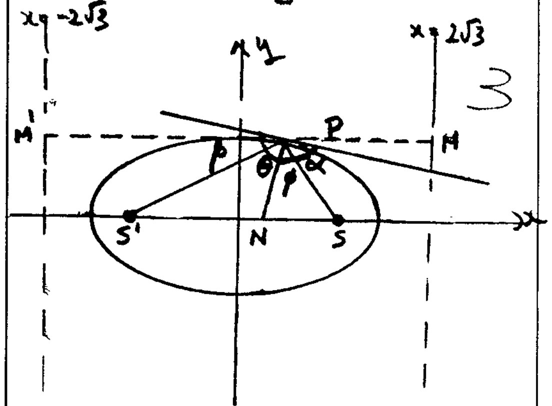
$3/4=1-e^2$

$e^2=1/4$

$\therefore e=1/2$

$\therefore S(\pm ae,0) \Rightarrow S(\pm\sqrt{3}/2, 0)$.

Directrices $x=\pm\frac{a}{e} = \pm 2\sqrt{3}$ #



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(ii) By definition of the ellipse

$\frac{SP}{PM}=e$ & $\frac{S'P}{PM'}=e$ where M & M' lie on the directrices

$\therefore SP+S'P = e(MP+PM') = e\cdot MM'$

but $MM' = \frac{a}{e}+\frac{a}{e}$

$\therefore SP+S'P = e\cdot\frac{2a}{e} = 2\times a$

$=2\times\sqrt{3}$

$=2\sqrt{3}$ #

2