Student name/number: ___________________

SOUTH SYDNEY HIGH SCHOOL

2003 YEAR 12 HALF-YEARLY EXAM

Mathematics Extension 2

Total marks **(75)**
- Attempt Questions 1 – 5
- All questions are of equal value
- Topics: Complex numbers, Polynomials, Conic sections and Graphs.

General Instructions

- Working time – 2 hours
- Board-approved calculators may be used
- All necessary working should be shown in every question

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq 1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2} \right)$$

NOTE: $\ln x = \log_e x, \qquad x > 0$

Question 1 (15 marks) **Marks**

(a) The complex number z and its conjugate $\bar{z}$ satisfy the equation $z\bar{z} + 2iz = 12 + 6i$. Find the possible values of z. **2**

(b) If $z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $z_2 = \sqrt{2}\left(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right)$, find

(i) $\left|\dfrac{z_1}{z_2}\right|$ **2**

(ii) $\arg\left(\dfrac{z_1}{z_2}\right)$ **2**

(c) (i) Find the square root of the complex number $5-12i$. **2**

(ii) Given that $Z = \dfrac{1+\sqrt{5-12i}}{2+2i}$ is purely imaginary, find Z^{400}. **2**

(d) (i) Shade the region in the Argand diagram containing all points representing the complex numbers Z such that **3**

$$|Z-1-i| \le 1 \text{ and } -\frac{\pi}{4} \le Arg(Z-i) \le \frac{\pi}{4}$$

(ii) Let Ψ be the complex number of minimum modulus satisfying the inequalities in (i) **2**

Express Ψ in the form $x+iy$.

Question 2(15 marks)

(a) $P(x)$ is an even monic polynomial of degree 4 with integer coefficients. If $\sqrt{2}$ is a zero, and the constant term is 6. Write down $P(x)$ in factored form over the real field. **3**

(b) If $P(x) = 4x^3 + 12x^2 - 15x + 4$ has a double zero, find all the zeros and factorise $P(x)$ over the real numbers. **3**

(c) When $P(x) = x^4 + ax^2 + 2x$ is divided by x^2+1, the remainder is $2x+3$. Find the value of a. **3**

(d) If $P(x) = x^4 - 2x^3 - x^2 + 6x - 6$ has a zero $1-i$, find all the zeros of $P(x)$. **3**

(e) Two of the roots of $3x^3 + ax^2 + 23x - 6 = 0$ are reciprocals. Find the value of a and the three roots. **3**

Question 3 (15 marks)

(a) (i) Find all the complex roots of the equation $x^6 - 1 = 0$. **3**

(ii) Show that the zeros of $P(x) = x^4 + x^2 + 1$ are included in the zeros of $x^6 - 1$. Hence factorise $P(x)$ over real field. **2**

(b) Sketch the graph of $x^2 - y^2 = 0$ and $xy = 2$ on the same axes. By observing the number of points of intersection of the graphs, deduce that the equation $z^2 = 4i$ has at least one root (in fact two roots). **3**

(c) Use the graphs of $y = x$ and $y = e^{-x}$ to sketch the graph of $y = x - e^{-x}$. (Do not use **Calculus**.) **2**

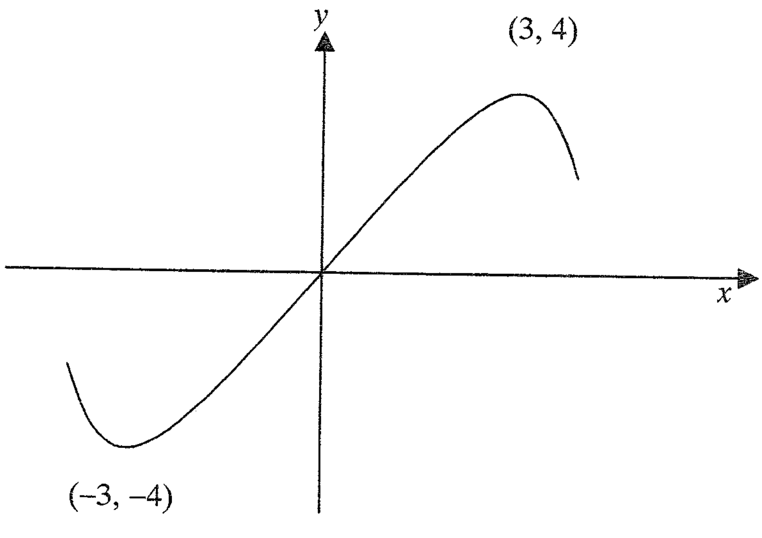
(d) (i) Sketch (showing critical points) the graph of $y = |x| - |x - 2|$. **3**

(ii) Hence or otherwise, find the values of x such that **2**

$$|x| - |x - 2| \geq -x$$

Question 4 (15 marks)

(a) The figure below shows a sketch graph of $y = f(x)$



The graph has a maximum point at (3, 4) and a minimum point at (-3, -4).
Sketch the following graphs, using a separate set of axes for each graph. In each case state the co-ordinates of the maximum and minimum points.

(i) $y = 3 + f(x)$ (ii) $y = f(-x)$ **3**

Question 4 (Continued)

(iii) $y = -f\left(\frac{x}{2}\right)$ (iv) $y = \sqrt{f(x)}$ **4**

(v) $y = \left[f(x)\right]^2$ (vi) $y = \log_e f(x)$ **4**

(b) Use the graph of $y = 4\cos x$ to sketch the graphs of: **4**

(i) $y = \sqrt{4\cos x}$ (ii) $y^2 = 4\cos x$

Question 5 (12 marks)

(a) Find the Cartesian equations of : **6**

(i) $x = 5\cos\theta,\ y = 4\sin\theta$;

(ii) $x = 2\sec\theta,\ y = 5\tan\theta$

Draw a neat sketch of each curve showing its foci and directrices.

(b) The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci $S(ae, 0)$ and $S'(-ae, 0)$. Show that **5**

(i) $PS = a(1 - e\cos\theta)$ and $PS' = a(1 + e\cos\theta)$;

(ii) $PS + PS' = 2a$

(c) The point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The tangent at P cuts the x-axis at X and the y-axis at Y. Show that $\frac{PX}{PY} = \sin^2\theta$ and deduce that if P is an extremity of a latus rectum, then $\frac{PX}{PY} = \frac{e^2 - 1}{e^2}$. **4**

END OF ASSESSMENT

S.S.H.S 2003 4u ½ Yrly.

Question 1.

a. let $z = x+iy$

$(x+iy)(x-iy) + 2iz = 12+6i$

$x^2+y^2+2iz = 16+6i$

$x^2+y^2+2i(x+iy) = 16+6i$

$x^2+y^2+2ix-2y = 16+6i$

$\therefore x^2+y^2-2y = 16$ — ①

$2x = 6$ — ②

$\therefore x = 3$ — ③

sub ③ into ①

$9+y^2-2y = 16$

$y^2-2y = 7$ ✓

$y^2-2y-7=0$

$\cancel{(y-1)(?)}$

$y^2-2y+1-8=0$

$(y-1)^2 = 8$

$\therefore y = \pm\sqrt{8}$

$y = \pm\sqrt{8}+1$ ✓

$= 1\pm\sqrt{8}$

$\therefore z = \cancel{3+2i}$

$= \cancel{3+(1\pm\sqrt{8})y}$

$= 3+(1\pm\sqrt{8})i$ ✓

Ans: $(3+3i, 3-i)$

b. i: $z_1 = 2\text{cis}\frac{\pi}{3}$

$z_2 = \sqrt{2}\text{cis}(-\frac{\pi}{4})$

$= \dfrac{2\text{cis}\frac{\pi}{3}}{\sqrt{2}\text{cis}(-\frac{\pi}{4})}$

$= \frac{2}{\sqrt{2}}\text{cis}(\frac{\pi}{3}-{-\frac{\pi}{4}})$

$= \sqrt{2}\text{cis}(\frac{7\pi}{12})$ ✓

~~ii) arg $\frac{z_1}{z_2}$~~

$\therefore \left|\frac{z_1}{z_2}\right| = \sqrt{2}$ ✓

ii) $\arg\left(\frac{z_1}{z_2}\right) = \frac{7\pi}{12}$ ✓

c i) $z = 5-12i$

$5-12i = (x+iy)^2$

$= x^2+2ixy-y^2$

$\therefore x^2-y^2 = 5$ — ①

$2xy = -12$ — ②

From ②

$x = \frac{-6}{y}$ — ③

sub ③ into ①

$\frac{36}{y^2} - y^2 = 5$ ✓

$36-y^4 = 5y^2$

let $u = y^2$

$36-u^2 = 5u$

$0 = u^2+5u-36$

$= (u+9)(u-4)$ ✓

$\therefore u = 4, -9$

$u^2 = 4, -9$

y has to be real.

$\therefore y = \pm 2$ — ④

sub ④ into ③

$x = -\frac{6}{2}, \ -\frac{6}{-2}$ ✓

$= -3, 3$ ✓

$\therefore x+iy = -3+2i, 3-2i$

ii) $\cancel{\dfrac{1+3-2i}{2+2i}}$

$= \cancel{\dfrac{4-2i}{2+2i}}$

when $x+iy = -3+2i$

$\dfrac{1-3+2i}{2+2i}$

$= \dfrac{-2+2i}{2+2i} \times \dfrac{2-2i}{2-2i}$

$= \dfrac{-4+4i+4i+4}{4+4}$ ✓

$= \frac{8i}{8}$

$= i$

$\therefore z = i$

$i^{400} = i^0$ ✗

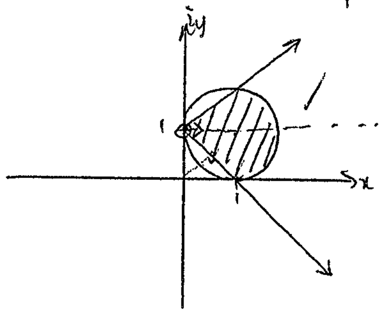
$(i^2)^{200} = 1$

$= (-1)^{200}$

$= \underline{+1}$.

d i) $|z-(1+i)| \le 1$

$-\frac{\pi}{4} \le \arg(z-i) \le \frac{\pi}{4}$ ✓

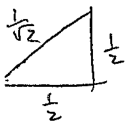


ii) Find ⊥ distance from $(0,0)$ and $0 = x+y-1$

$\perp \text{dist} = \dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$

$= \dfrac{|1(0)+1(0)-1|}{\sqrt{1^2+1^2}}$

$= \dfrac{1}{\sqrt{2}}$



$\therefore \Psi\left(\frac{1}{2}, \frac{1}{2}\right)$ ✓

$\therefore \Psi$: $\frac{1}{2} + i\frac{1}{2}$

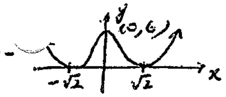
or alternatively, solve between $y = x$ and $x+y-1=0$

$\therefore 2x-1=0$

$x = \frac{1}{2}, \ y = \frac{1}{2}$

Question 2

Note:-



a. $P(x) = \cancel{(x+\sqrt{2})^2(x-\alpha)^2+6}$

$\cancel{= (x-\sqrt{2})^2(x+\sqrt{2})^2+4}$

$= (x-\sqrt{3})^2(x+\sqrt{2})^2 + k$ $\cancel{(x-\sqrt{2})(x+\sqrt{2})(x-\alpha)(x+\alpha)+6}$

$\therefore k = 2$

$P(x) = (x-\sqrt{2})^2(x+\sqrt{2})^2+2$ $\cancel{(x^2-2)(x-\alpha)(x+\alpha)+6}$

b. $P(x) = 4x^3+12x^2-15x+4 = 0$

$P'(x) = 12x^2+24x-15 = 0$

$P''(x) = 24x+24 = 0$

$\therefore 12x^2+24x-15=0$

$4x^2+8x-5=0$

$(2x+5)(2x-1) \ne 0$

$\therefore x = \frac{1}{2}, -\frac{5}{2}$

Test $P\left(\frac{1}{2}\right) = 0$

$P\left(-\frac{5}{2}\right) \ne 0$

$\therefore$ Double root at $\frac{1}{2}$ ✓

$\therefore P(x) = (2x-1)^2(Q(x))$.

By inspection:

$P(x) = (4x^2-4x+1)(x+4)$

$\therefore P(x) = (2x-1)^2(x+4)$, over $\mathbb{R}$ ✓

c. $P(x) = x^4+ax^2+2x$

$P(x) = (x^2+1)Q(x) + (2x+3)$ ✓

$P(i) = 2i+3$

$\therefore 2i+3 = i^4+ai^2+2i$ ✓

$= 1-a+2i$

Equate real parts

$3 = 1-a$ ✓

$\therefore a = -2$ ✓

d. $P(x) = x^4-2x^3-x^2+6x-6$

root: $1+i, 1-i$ because $P(x)$ has real coeffs.

~~(x^2+1) is a root of $P(x)$~~

$(x-(1+i))(x-(1-i))$ ⟹

Easier to use $\alpha = 1+i, \ \beta = 1-i$

$\therefore x^2-(\alpha+\beta)x+\alpha\beta = 0$

$x^2-2x+2=0$

$= (x-1-i)(x-1+i)$

$= (x^2-x+ix-x+1-i-ix+i-i^2)$

$= x^2-2x+2$ ✓

$\therefore x^2-2x+2$ is a root of $P(x)$

$$\begin{array}{r|l} & x^2+0x-3 \\ \hline x^2-2x+2 & x^4-2x^3-x^2+6x-6 \\ & x^4-2x^3+2x^2 \\ \hline & 0x^3-3x^2+6x \\ & 0x^3-0x^2+0x \\ \hline & -3x^2+6x-6 \\ & -3x^2+6x-6 \\ \hline & 0 \end{array}$$

$\therefore$ zeros: $1+i, \ 1-i, \ \sqrt{3}, \ -\sqrt{3}$ ✓

2.

$P(x) = 3x^3 + ax^2 + 23x - 6$

roots: $\alpha, \frac{1}{\alpha}, \beta$

$\alpha + \frac{1}{\alpha} + \beta = -\frac{a}{3}$ — ①

$1 + \alpha\beta + \frac{\beta}{\alpha} = \frac{23}{3}$ — ②

$\beta = \frac{6}{3} = 2$ — ③

Sub ③ into ②

$1 + 2\alpha + \frac{2}{\alpha} = \frac{23}{3}$

$\alpha + 2\alpha^2 + 2 = \frac{23\alpha}{3}$

$3\alpha + 6\alpha^2 + 6 = 23\alpha$

$0 = 6\alpha^2 + 3\alpha - 23\alpha + 6$

$= 6\alpha^2 - 20\alpha + 6$

$= 3\alpha^2 - 10\alpha + 3$

$\alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{10 \pm \sqrt{10^2 - 4(3)(3)}}{2(3)}$

$= \frac{10 \pm 8}{6} = 3, \frac{1}{3}$ — ④

$\therefore$ roots: $3, \frac{1}{3}, 2$

~~∴ sub ④ into~~

Sub roots into ①

$3 + \frac{1}{3} + 2 = -\frac{a}{3}$

$16 = -a$

$\therefore a = -16$

$\therefore$ $\boxed{\begin{array}{l} a = -16 \\ \text{roots: } 3, \frac{1}{3}, 2 \end{array}}$

Question 3

a. i) $x^6 = 1$

$= \operatorname{cis} 0$

$\therefore x = (\operatorname{cis} 0)^{\frac{1}{6}}$

$= (\operatorname{cis} 2k\pi)^{\frac{1}{6}}$, where $k \in \mathbb{Z}$

$= \operatorname{cis} \frac{k\pi}{3}$

$\therefore x_0 = \operatorname{cis} 0$

$x_1 = \operatorname{cis} \frac{\pi}{3}$

$x_2 = \operatorname{cis} \frac{2\pi}{3}$

$x_3 = \operatorname{cis} \pi$

$x_4 = \operatorname{cis} \frac{4\pi}{3}$

$x_5 = \operatorname{cis} \frac{5\pi}{3}$

ii) ~~$P(x) = x^4 + x^2 + 1$, let $u = x^2$, $\therefore 0 = u^2 + u + 1$, $u = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$~~

real roots of (i) are ± 1

$\therefore (x^2 - 1)$ is a factor of $x^6 - 1 = 0$

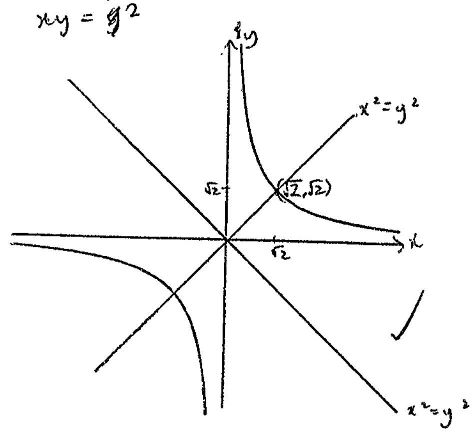
$$\begin{array}{r}
x^4 + 0x^3 + x^2 + 0x + 1 \\
x^2 + 0x - 1 \enclose{longdiv}{x^6 + 0x^5 + 0x^4 + 0x^3 + 0x^2 + 0x - 1} \\
x^6 + 0x^5 - x^4 \\
\hline
0x^5 + x^4 + 0x^3 \\
0x^5 + 0x^4 - 0x^3 \\
\hline
x^4 + 0x^3 + 0x^2 \\
x^4 + 0x^3 - x^2 \\
\hline
0x^3 + x^2 + 0x \\
0x^3 + 0x^2 + 0x \\
\hline
x^2 + 0x - 1
\end{array}$$

$\therefore x^6 - 1 = (x^2 - 1)(x^4 + x^2 + 1)$ $\Rightarrow$ factorised over $\mathbb{R}$

$\therefore$ zeros of $P(x)$ are included in zeros of $x^6 - 1 = 0$.

$x^2 = y^2$

$xy = 2$



$x^2 - y^2 = 0$ —①

$xy = 2$ —②

let $z = x + iy$

$\therefore z^2 = x^2 - y^2 + 2ixy$

$= (x^2 - y^2) + 2i(2)$

$= 4i$

$\therefore$ there are 2 roots $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$

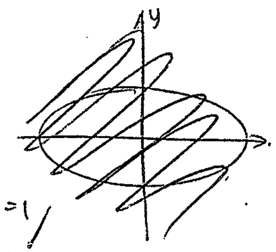
Question 5.

a. i) $\frac{x}{5} = \cos\theta$

$\frac{y}{4} = \sin\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\therefore \frac{x^2}{25} + \frac{y^2}{16} = 1$



ii) $\frac{x}{2} = \sec\theta$

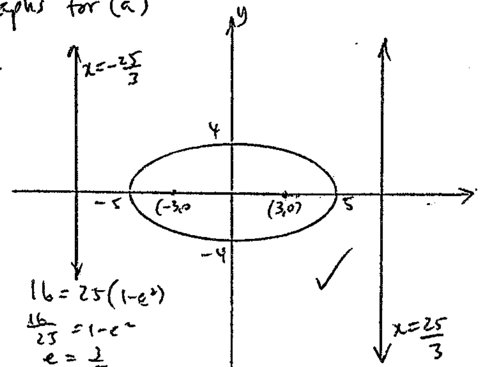
$\frac{y}{5} = \tan\theta$

$\tan^2\theta + 1 = \sec^2\theta$

$\frac{y^2}{25} + 1 = \frac{x^2}{4}$

$\therefore 1 = \frac{x^2}{4} - \frac{y^2}{25}$

Graphs for (a)



$16 = 25(1-e^2)$

$\frac{16}{25} = 1 - e^2$

$e = \frac{3}{5}$

foci: $(\pm ae, 0)$

$(\pm\frac{3}{5}(5), 0)$

$(\pm 3, 0)$

directrices: $x = \pm\frac{a}{e}$

$= \pm 5 \times \frac{5}{3}$

$= \pm\frac{25}{3}$

ii) $1 = \frac{x^2}{4} - \frac{y^2}{25}$

$b^2 = a^2(e^2 - 1)$

$25 = 4(e^2 - 1)$

$\frac{25}{4} + 1 = e^2$

$\therefore e = \frac{\sqrt{29}}{2}$ $(e > 1)$

$\therefore$ foci: $(\pm ae, 0)$

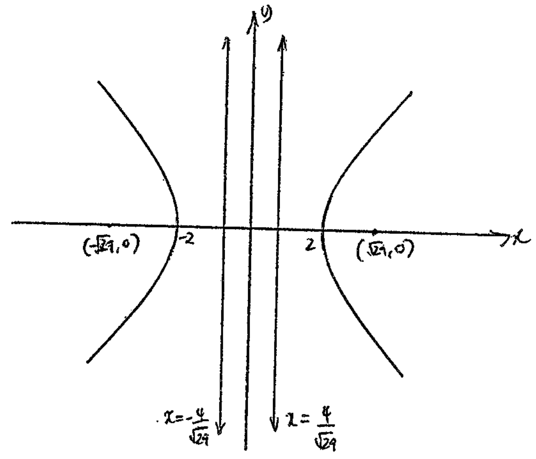
$(\pm\frac{\sqrt{29}}{2}\times 2, 0)$

$(\pm\sqrt{29}, 0)$

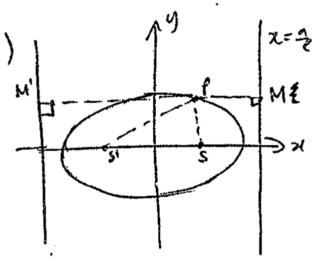
directrices: $x = \pm\frac{a}{e}$

$= \pm 2 \div \frac{\sqrt{29}}{2}$

$x = \pm\frac{4}{\sqrt{29}}$



b. i)



$PS = ePM$

$= e\left(\frac{a}{e} - a\cos\theta\right)$

$= a - ae\cos\theta$

$= a(1 - e\cos\theta)$

$PS' = ePM'$

$= e\left(a\cos\theta + \frac{a}{e}\right)$

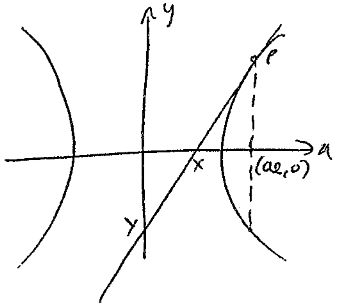
$= ae\cos\theta + a$

$= a(1 + e\cos\theta)$

ii) $PS + PS' = a - \cancel{ae\cos\theta} + a + \cancel{ae\cos\theta}$

$= 2a$

c.



$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Diff implicitly w.r.t x

$\frac{2x}{a^2}-\frac{2y\frac{dy}{dx}}{b^2}=0$

$\therefore \frac{dy}{dx}=\frac{\cancel{2}x}{a^2}\cdot\frac{b^2}{\cancel{2}y}$

$=\frac{xb^2}{a^2y}$ ✓

At P, $\frac{dy}{dx}=\frac{a\sec\theta\, b^2}{a^2 \cancel{b}\tan\theta}$

$=\frac{b\sec\theta}{a\tan\theta}$ ✓

$\therefore$ tgt at P, $y-y_1=m(x-x_1)$

$y-b\tan\theta=\frac{b\sec\theta}{a\tan\theta}(x-a\sec\theta)$ ✓

X, when $y=0$

$-b\tan\theta=\frac{b\sec\theta}{a\tan\theta}(x-a\sec\theta)$

$-ab\tan^2\theta=xb\sec\theta-ab\sec^2\theta$

$ab\sec^2\theta-ab\tan^2\theta=xb\sec\theta$

$ab=xb\sec\theta$

$\therefore x=\frac{a}{\sec\theta}$ ✓

$\therefore X\left(\frac{a}{\sec\theta},0\right)$

sec = tan+1

sec - tan = 1

Y, when $x=0$

$y-b\tan\theta=\frac{b\sec\theta}{a\tan\theta}(-a\sec\theta)$ ✓

$ya\tan\theta-ab\tan^2\theta=-ab\sec^2\theta$

$-ay\tan\theta=(\sec^2\theta-\tan^2\theta)ab$

$\therefore y=\frac{-b}{\tan\theta}$

$\therefore Y\left(0,\frac{-b}{\tan\theta}\right)$

$P(a\sec\theta, b\tan\theta)$
$X\left(\frac{a}{\sec\theta},0\right)$
$Y\left(0,\frac{-b}{\tan\theta}\right)$

$\therefore \frac{PX^2}{PY^2}=\frac{\left(a\sec\theta-\frac{a}{\sec\theta}\right)^2+(b\tan\theta)^2}{(a\sec\theta)^2+\left(b\tan\theta+\frac{b}{\tan\theta}\right)^2}$ ✓

sec = tan+1

$=\frac{\left(\frac{a\sec^2\theta-a}{\sec\theta}\right)^2+b^2\tan^2\theta}{a^2\sec^2\theta+\left(\frac{b\tan^2\theta+b}{\tan\theta}\right)^2}$ ✓

$=\frac{\frac{a^2(\sec^2\theta-1)^2}{\sec^2\theta}+b\tan^2\theta}{a^2\sec^2\theta+\frac{b^2(\tan^2\theta+1)^2}{\tan^2\theta}}$ ✓

$$\frac{PX^2}{PY^2} = \frac{\dfrac{a^2\tan^4\theta}{\sec^2\theta} + b\tan^2\theta}{a^2\sec^2\theta + b^2\dfrac{\sec^4\theta}{\cancel{t}\tan^2\theta}}$$

$$= \frac{\dfrac{a^2\tan^4\theta + b\tan^2\theta\sec^2\theta}{\sec^2\theta}}{\dfrac{a^2\sec^2\theta\tan^2\theta + b^2\sec^4\theta}{\tan^2\theta}}$$

$$= \frac{\dfrac{\tan^2\theta\,\cancel{(a^2\tan^2\theta + b^2\sec^2\theta)}}{\sec^2\theta}}{\dfrac{\sec^2\theta\,\cancel{(a^2\tan^2\theta + b^2\sec^2\theta)}}{\tan^2\theta}}$$

$$= \frac{\tan^2\theta}{\sec^2\theta} \div \frac{\sec^2\theta}{\tan^2\theta}$$

$$= \frac{\tan^4\theta}{\sec^4\theta}$$

$$= \frac{\sin^4\theta}{\cancel{\cos^4\theta}} \times \cancel{\cos^4\theta}$$

$$= \sin^4\theta$$ ✓

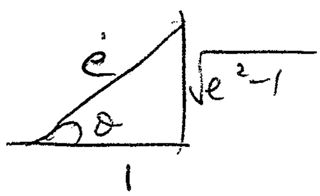
$$\therefore \frac{PX}{PY} = \sin^2\theta$$

when P is extremity of latus rectum,

$a\sec\theta = ae$

$\therefore$ ~~cos θ = e~~

$\sec\theta = e$



V. Good!

$$\therefore \sin^2\theta = \left(\frac{\sqrt{e^2-1}}{e}\right)^2$$ ✓

$$= \frac{e^2-1}{e^2} = \frac{PX}{PY}$$ as req'd.