Student Name / Number ____________________

SOUTH SYDNEY HIGH SCHOOL

Year 12 Half-yearly Examination
May 2001

MATHEMATICS
EXTENSION 2

Instructions :

Time Allowed: *2 hours*
(Plus 5 mins reading time)

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown.
- Marks may be deducted for poorly arranged or missing working.
- Write your Name / Student Number on every page.
- Board-approved calculators may be used.
- A table of standard integrals is included.

Question 1 (15 marks) **Marks**

(a) If $z = \dfrac{3+7i}{2-5i}$ find : **4**

(i) $\text{Re}(z)$ (ii) $\text{Im}(z)$

(iii) $|z|$ (iv) $\arg(z)$

(v) $\arg\left(\bar{z}\right)$

(b) Express the following complex numbers in the form $a+ib$. **4**

(i) $\left(\sqrt{3}-i\right)^6$

(ii) $\left(\cos\dfrac{\pi}{12}+i\sin\dfrac{\pi}{12}\right)^{-4}$

(c) Sketch on an Argand diagram the region satisfying the following conditions simultaneously **4**

$|\arg z| \le \dfrac{\pi}{3}$, $|z| < 2$ and $|\text{Im}(z)| \le 1$

(d) $ABCD$ is a rectangle described in an anticlockwise direction. $BC = 2AB$. If A and B represent $-2-3i$ and $-3+2i$ respectively, find the complex numbers representing points C and D. **3**

Continue next page....

Question 2 (15 marks) **Marks**

(a) Show that $2+i$ is a root of $P(x)=x^3-2x^2-3x+10$. **3**

Hence factorise $P(x)$ completely.

(b) If $u=\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}$ and $v=\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)$ are two complex numbers. **4**

Using De Moivre's theorem:

(i) show that $u^4+v^4+2=0$.

(ii) find the values of u^2+v^2 and u^2-v^2, giving answers in the form $a+ib$.

(c) (i) Find in mod-arg form the five roots of the equation $z^5=1$ and and graph them on the Argand diagram. **3**

(ii) Let α be the root in the first quadrant. Show that the other four roots are all powers of α. **1**

(iii) By grouping the unreal roots in conjugate pairs, factorise the polynomial z^5-1 into the product of three factors with real coefficients. **2**

(iv) Hence show that $\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}=-\frac{1}{2}$ **2**

Continue next page....

Question 3 (15 marks) **Marks**

(a) Find $\int_0^{\frac{\pi}{12}} \sin^2 3x \, dx$ **3**

(b) Using the given substitution, find the integral **3**

$$\int \frac{1}{x\sqrt{\log_e x}} dx \qquad \text{if } u = \log_e x$$

(c) Find the following integrals: **3**

(i) $\int x\sqrt{x^2 - 5} \, dx$

(ii) $\int \sin^3 x \cos^2 x \, dx$

(d) Find the following integrals: **6**

(i) $\int \frac{dx}{(x-2)(x+1)}$

(ii) $\int \frac{x^2 + 5}{x^2 + 4} dx$

Continue next page....

Question 4 (15 marks) **Marks**

(a) Find the following integrals: **4**

(i) $\int x\cos x\,dx$ (ii) $\int(\log_e x)^2\,dx$

(b) If $I_n = \int x^n e^x\,dx$, use integration by parts to show that **4**

$I_n = x^n e^x - nI_{n-1}$ for $n \geq 1$.

Hence evaluate $\int_0^1 x^4 e^x\,dx$.

(c) Use the substitution $t = \tan\frac{\theta}{2}$ to show that **3**

$$\int_0^{\frac{\pi}{3}} \frac{dx}{2-\cos\theta} = \frac{\sqrt{3}\pi}{6}$$

(Hint: Use the standard integral $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$)

(d) Express $\frac{-4}{x^2(x-2)}$ in partial fractions, and show that **4**

$$\int_3^4 \frac{-4}{x^2(x-2)}dx = \frac{1}{6} + \log_e\frac{2}{3}$$

Continue next page....

Question 5 (15 marks) **Marks**

(a) For the polynomial equation $2x^3 - 4x^2 + x - 6 = 0$, find the value of **7**

(i) $\sum \alpha = \alpha + \beta + \gamma$ (ii) $\sum \alpha\beta$ (iii) $\alpha\beta\gamma$

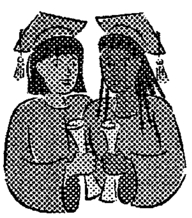
(iv) $\dfrac{\alpha}{\beta\gamma} + \dfrac{\beta}{\alpha\gamma} + \dfrac{\gamma}{\alpha\beta}$ (v) $\alpha^3 + \beta^3 + \gamma^3$

(b) When a polynomial $P(x)$ is divisible by $(x-2)$ and $(x-3)$, the respective remainders are 4 and 9, determine what the remainder must be when it is divided by $(x-2)(x-3)$. **3**

(c) If the equation $x^3 + 3ax^2 + 3bx + c = 0$ has a repeated root, show that the root also satisfies the equation $x^2 + 2ax + b = 0$. **5**

Hence show that the value of the repeated root in terms of *a*, *b* and *c* is given by

$$\frac{c - ab}{2\left(a^2 - b\right)}.$$



End of Half-yearly Exam

(1) (a) (i) -1 (ii) 1

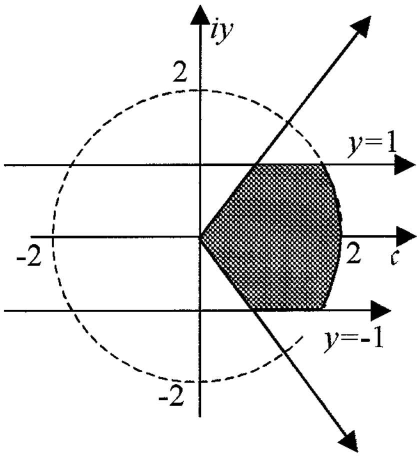
(iii) $\sqrt{2}$ (iv) $\frac{3\pi}{4}$

(v) $-\frac{3\pi}{4}$

(b) (i) -64

(ii) $\frac{1}{2}\left(1-\sqrt{3}i\right)$

(c)



(d) $C(-13,0)$; $D(-12,-5)$

(2) (a) $\left(x^2-4x+5\right)\left(x+2\right)$

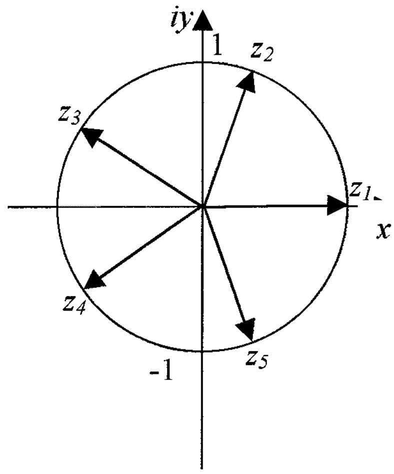
(b) (i) Proof

(ii) 0; $2i$

(c) (i) $1, cis\left(\frac{2\pi}{5}\right), cis\left(\frac{4\pi}{5}\right)$

$cis\left(\frac{6\pi}{5}\right)=cis\left(-\frac{2\pi}{5}\right)$

$cis\left(\frac{8\pi}{5}\right)=cis\left(-\frac{2\pi}{5}\right)$



(ii) $\alpha = cis\left(\frac{2\pi}{5}\right)$

$cis\left(\frac{4\pi}{5}\right)=cis\left(\frac{2\pi}{5}\right)^2=\alpha^2$

and so on.

(iii)

$$(z-1)\left(z^2-2\cos\frac{2\pi}{5}z+1\right)\left(z^2-2\cos\frac{4\pi}{5}+1\right)$$

(iv) Proof

(3) (a) $\frac{1}{2}\left(\frac{\pi}{12}-\frac{1}{6}\right)$

(b) $2\sqrt{\log_e x}+c$

(c) (i) $\dfrac{\sqrt{\left(x^2-5\right)^3}}{3}+c$

(ii) $-\dfrac{\cos^3 x}{3}+\dfrac{\cos^5 x}{5}+c$

(d) (i) $\dfrac{1}{3}\ln\left|\dfrac{x-2}{x-1}\right|+c$

(ii) $x+\dfrac{1}{2}\tan^{-1}\dfrac{x}{2}+c$

(4) (a) (i) $x\sin x+\cos x+c$

(ii) $x\left(\ln x\right)^2-2x\ln x+2x+c$

(b) $9e-24$

(c) Proof

(d) Proof

(5) (a) (i) 3 (ii) $\dfrac{1}{2}$

(iii) 3 (iv) 1

(v) 10

(b) $5x-6$

(c) Proof