South Sydney High

Year 12

Half Yearly

2002

Extension I Mathematics

Time Allowed – 2 Hours
Plus 5 Minutes Reading Time

Directions to Candidates

- Attempt ALL questions
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Start each question on a new page.
- Board approved calculators may be used.

Question 1: (12 marks)

(a) Solve $x-3<\frac{4}{x}$ **2**

(b) Given the lines $L_1: 3y = x+1$ and $L_2: 3x-4y=12$ **2**
Show that the acute angle formed by them is equal to the acute angle formed by L_1 and the x-axis.

(c) If $(x-2)$ is a factor of $P(x) \equiv x^3+2x^2+kx-6$, find the remainder when $P(x)$ is divided by $(x-1)$. **2**

(d) From the set of six letters A, B, C, D, E, F **2**

i. How many different sets of three letters may be chosen?
ii. How many of these include the letter A?

(e) Simplify $\dfrac{\tan A}{\sin A-\sin^3 A}$ **2**

(f) Differentiate with respect to x, $x^2\ln(x^2+1)$ **2**

Question 2: (12 marks)

(a) Use the substitution $u=x^2+1$ to evaluate **3**

$$\int_0^{\sqrt{3}} x\sqrt{x^2+1}\ dx$$

(b) Find $\displaystyle\int \frac{e^x+1}{e^x}\ dx$ **1**

(c) $I_1=\displaystyle\int_0^{\frac{\pi}{4}} \frac{\cos x}{\sin x+\cos x}\ dx \qquad I_2=\int_0^{\frac{\pi}{4}} \frac{\sin x}{\sin x+\cos x}\ dx$

i. Prove that $I_1+I_2=\frac{\pi}{4}$ **1**

ii. Prove that $I_1-I_2=\frac{1}{2}\ln 2.$ **2**

iii. Find the value of Prove that I_1 and Prove that I_2 **2**

(d) Use the substitution $u=\sin x$ to evaluate $\displaystyle\int_0^{\frac{\pi}{2}}\cos^3 x\ dx$ **3**

Question 3: (12 marks)

(a) Two parallel tangents to a circle whose centre is O are cut by a third tangent at P and Q. Show that angle POQ is $90°$. **4**

(b) Find the coordinates of the focus and the equation of the directrix for the curve $x^2 = 4(x+y)$ and hence sketch. **2**

(c) **6**

i. Find the equation of the tangent to the curve B: $y = 2x^2$ at the point P given by $(t, 2t^2)$.

ii. The point Q lies on the curve C: $y = x^2 + 1$ with the same abscissae as the point P.

Show that the equation of the tangent to C at Q is $y = 2tx + (1 - t^2)$.

iii. As "t" varies find the locus of the point of intersection of these two tangents.
Sketch this locus.

Question 4: (12 marks)

(a) Prove the identity $\dfrac{2\cos A}{\operatorname{cosec} A - 2\sin A} = \tan 2A$ **3**

(b) R is the region enclosed by the coordinate axis and the curve

$$y = \sqrt{2}\cos x - 1 \text{ for } 0 \le x \le \frac{\pi}{4}$$

i. Sketch the curve for $0 \le x \le 2\pi$, showing where it crosses the x-axis and the minimum point. **2**

ii. Prove that the area of R is $(1 - \frac{\pi}{4})$ square units. **2**

iii. If R makes a revolution about the x-axis, prove that the volume of the resulting solid is $\dfrac{\pi(\pi - 3)}{2}$ cubic units. **3**

(c) If α and β are the roots of $2x^2 + 4x + 1 = 0$, find the values of: **2**

i. $a^2\beta + \alpha\beta^2$

ii. $\dfrac{1}{\alpha\beta^3} + \dfrac{1}{\alpha^3\beta}$

Question 5: (12 marks)

(a) Show that $\cot\theta + \tan\frac{\theta}{2} = \operatorname{cosec}\theta$ **2**

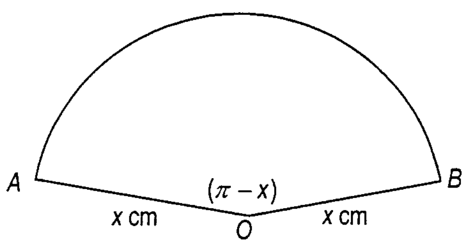
(b) Find constants *A, B, C* and *D* such that **2**

$$x^3 + 2x^2 - x \equiv Ax^3 + B(x-1)^2 + C(x-2) + D$$

(c) Consider the function $f(x) = \dfrac{2x}{1+3x^2}$. **8**

i. Show that $f(x)$ is an odd function.
ii. Find the coordinates and nature of any stationary points.
iii. Find the coordinates of any points of inflection.
iv. Identify any asymptotes of $f(x)$.
v. Sketch the curve showing all essential features.

Question 6: (12 marks)



A sector *OAB*, of a circle is such that, when its radii are *x* cm, then $\angle AOB = (\pi - x)$) radians, and *x* varies from 0 to π.

(a) Prove that the area of such sectors has a maximum value of $\dfrac{2\pi^3}{27} cm^2$ **3**

(b) Prove that the maximum perimeter of such sectors $\left(\dfrac{\pi+2}{2}\right)^2 cm$ **3**

and discuss the minimum.

(c) The area of ΔAOB is denoted by $f(x)$. **6**

i. Prove that $f(x) = \dfrac{x^2 \sin x}{2}$.
ii. Show that, when $f(x)$ is maximum, $x + 2\tan x = 0$.
iii. By sketching a graph of $y = \tan x$ and a suitable straight line, show that the solution of this equation is close to $\dfrac{3\pi}{4}$.
iv. Taking $\dfrac{3\pi}{4}$ as a first approximation to this root, use Newtons Method to obtain the better approximation $\dfrac{3\pi+2}{5}$.

Question 7: (12 marks)

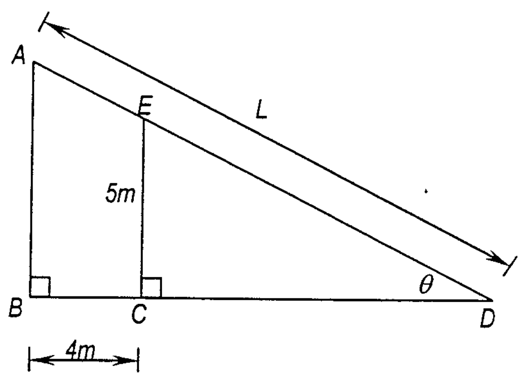
(a) Evaluate $\lim_{x \to 0} \dfrac{\sin^2 4x}{x^2}$ **3**

(b) Given $y = a\cos\left(\frac{\pi}{2}x\right) + b\sin\left(\frac{\pi}{2}x\right) + c$, use mathematical induction to show that for all positive integers of n: **4**

$$\frac{d^n y}{dx^n} = \left(\frac{\pi}{2}\right)^n \left[a\cos\left(\frac{\pi}{2}x + n\frac{\pi}{2}\right) + b\sin\left(\frac{\pi}{2}x + n\frac{\pi}{2}\right)\right]$$

where $\dfrac{d^n y}{dx^n}$ is the n^{th} derivative of y.

(c) A 5m fence stands 4m from the wall of a house. A farmer wishes to reach a point A on the wall by the use of a ladder L that can reach from the ground outside the fence to the wall as shown in the diagram below. Let $\angle ADB = \theta$. **5**



i. Show that $L = \dfrac{5}{\sin\theta} + \dfrac{4}{\cos\theta}$

ii. Hence find the length of the shortest ladder that can reach from the ground outside the fence to the wall. Express your answer correct to 1 decimal place.

Ext. I Half Yearly 2002
Solutions

Question 1:

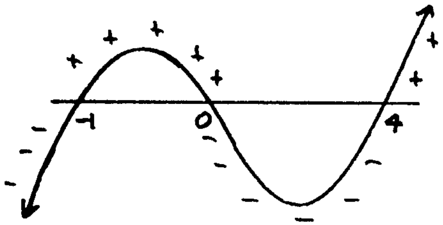
(a) $x-3 < \dfrac{4}{x}$ × both sides by x^2

$x^3-3x^2 < 4x$

$x^3-3x^2-4x < 0$

$x(x^2-3x-4) < 0$

$x(x-4)(x+1) < 0$



$\therefore \quad x < -1 \qquad 0 < x < 4$ #

(b) $L_1 \Rightarrow 3y = x+1$ $L_2 \Rightarrow 3x-4y=12$

$y = \frac{x}{3}+\frac{1}{3}$ $y = \frac{3x}{4}-3$

$\therefore m_1 = \frac{1}{3}$ $\therefore m_2 = \frac{3}{4}$

Now $\tan\theta = \left|\dfrac{m_1-m_2}{1+m_1m_2}\right|$

$= \left|\dfrac{\frac{1}{3}-\frac{3}{4}}{1+\frac{1}{3}\cdot\frac{3}{4}}\right|$

$= \left|\dfrac{-\frac{5}{12}}{\frac{5}{4}}\right| = \frac{1}{3} = m_1$

#

(c) $P(x) = x^3+2x^2+kx-6$

$(x-2)$ is a factor

$\therefore P(2)=0$

$\therefore 2^3+2\cdot 2^2+2k-6=0$

$8+8+2k-6=0$

$\therefore k=-5$

$\therefore P(x) = x^3+2x^2-5x-6$

$\therefore$ Remainder when $P(x)$ is divided by $(x-1)$

$\Rightarrow P(1) = 1+2-5-6$

$= -8$

$\therefore$ Remainder $= -8$ #

(d) A, B, C, D, E, F

(i) ${}^6C_3 = \dfrac{6!}{3!\cdot 3!} = \dfrac{6\times5\times4}{3\times2\times1} = 20$

(ii) Find how many ways 3 letters can be chosen not including A

$\Rightarrow {}^5C_3 = \dfrac{5!}{3!\cdot 2!} = \dfrac{5\cdot 4}{2} = 10$

$\therefore$ With letter A

$= 20-10$

$= 10$ ways #

(e) Simplify $\dfrac{\tan A}{\sin A - \sin^3 A}$

$\dfrac{\tan A}{\sin A-\sin^3 A} = \dfrac{\frac{\sin A}{\cos A}}{\sin A(1-\sin^2 A)}$

$= \dfrac{\frac{\sin A}{\cos A}}{\sin A\cdot\cos^2 A}$

$= \dfrac{\sin A}{\cos A} \times \dfrac{1}{\sin A\cdot\cos^2 A}$

$= \dfrac{1}{\cos^3 A}$

$= \sec^3 A$ #

(f) $x^2\ln(x^2+1)$

$f = x^2 \qquad g = \ln(x^2+1)$

$f' = 2x \qquad g' = \dfrac{2x}{x^2+1}$

$\therefore \dfrac{d}{dx}\, x^2\ln(x^2+1)$

$= 2x\ln(x^2+1) + x^2\cdot\dfrac{2x}{x^2+1}$

$= 2x\ln(x^2+1) + \dfrac{2x^3}{x^2+1}$ #

Question 2:

(a) $\displaystyle\int_0^{\sqrt{3}} x\sqrt{x^2+1}\,dx$ given $u = x^2+1$

$u = x^2+1 \Rightarrow du = 2x\,dx$

$\therefore dx = \dfrac{du}{2x}$

Now when $x=0$

$u = 0^2+1$

$u = 1$

Now when $x=\sqrt{3}$

$u = (\sqrt{3})^2+1$

$= 3+1$

$\therefore u = 4$

$\therefore I = \displaystyle\int_1^4 x\cdot\sqrt{u}\cdot\frac{du}{2x}$

$= \dfrac{1}{2}\displaystyle\int_1^4 u^{\frac{1}{2}}\cdot du$

$= \dfrac{1}{2}\cdot\dfrac{2}{3}\left[u^{\frac{3}{2}}\right]_1^4$

$= \dfrac{1}{3}\left[4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$

$= \dfrac{1}{3}[8-1] = \dfrac{7}{3}$ #

QUESTION 2:

(b) $\int \frac{e^x+1}{e^x}\,dx$

$= \int \frac{e^x}{e^x} + e^{-x}\,dx$

$= \int 1 + e^{-x}\,dx$

$= x - e^{-x} + c$ #

(c) $I_1 = \int_0^{\frac{\pi}{4}} \frac{\cos x}{\sin x+\cos x}\,dx$

$I_2 = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\sin x+\cos x}\,dx$

(i) $I_1 + I_2$

$= \int_0^{\frac{\pi}{4}} \frac{\cos x}{\sin x+\cos x} + \frac{\sin x}{\sin x+\cos x}\,dx$

$= \int_0^{\frac{\pi}{4}} \frac{\cos x+\sin x}{\sin x+\cos x}\,dx$

$= \int_0^{\frac{\pi}{4}} 1\,dx$

$= \left[x\right]_0^{\frac{\pi}{4}} = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

Hence proven #

(ii) $I_1 - I_2$

$= \int_0^{\frac{\pi}{4}} \frac{\cos x - \sin x}{\sin x + \cos x}\,dx$

$= \int_0^{\frac{\pi}{2}} \frac{f'(x)}{f(x)}\,dx$

$= \left[\ln(\sin x + \cos x)\right]_0^{\frac{\pi}{4}}$

$= \left[\ln\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - \ln(\sin 0 - \cos 0)\right]$

$= \ln\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - \ln(0+1)$

$= \ln\left(\frac{2}{\sqrt{2}}\right) - \ln(1)$

$= \ln 2 - \ln 2^{\frac{1}{2}}$

$= \ln 2 - \frac{1}{2}\ln 2$

$= \frac{1}{2}\ln 2$

HENCE PROVEN #

(iii) FIND I_1 & I_2

FROM (i) & (ii)

$I_1 + I_2 = \frac{\pi}{4}$ ... ①

$I_1 - I_2 = \frac{1}{2}\ln 2$ ... ②

$\therefore$ ① + ②

$2I_1 = \frac{\pi}{4} + \frac{1}{2}\ln 2$

$\therefore I_1 = \frac{\pi}{8} + \frac{1}{4}\ln 2$ #

① − ②

$2I_2 = \frac{\pi}{4} - \frac{1}{2}\ln 2$

$\therefore I_2 = \frac{\pi}{8} - \frac{1}{4}\ln 2$ #

(d) $\int_0^{\frac{\pi}{2}} \cos^3 x\,dx$

$u = \sin x$

$du = \cos x\,dx$

$\therefore dx = \frac{du}{\cos x}$

Also

$u = \sin x$

$\therefore u^2 = \sin^2 x$

$\therefore u^2 = 1 - \cos^2 x$

when $x = 0$, $u = \sin 0$, $\therefore u = 0$

$x = \frac{\pi}{2}$, $u = \sin\frac{\pi}{2}$, $u = 1$

$\therefore I = \int_0^1 (\cos x)^3 \cdot \frac{du}{\cos x}$

$= \int_0^1 \cos^2 x \cdot du$

$= \int_0^1 1 - u^2\,du$

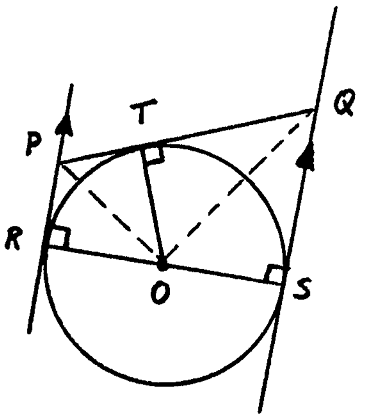
$= \left[u - \frac{u^3}{3}\right]_0^1$

$= \left(1 - \frac{1}{3}\right) - 0$

$= \frac{2}{3}$ #

QUESTION 3:

(a)



$\angle ORP = \angle OSQ = 90^\circ$

$\angle ORP + \angle OSQ = 180^\circ$

Now $RP \parallel SQ$ & co-interior angles are supplementary

But $\angle ORP$ & $\angle OSQ$ are supplementary

$\therefore$ ROS is a straight line ... (i)

CONSIDER ΔTOQ & ΔQOS

$TO = OS$ (radii)

OQ IS COMMON

$\angle OTQ = \angle OSQ$

(radii are perpendicular to tangents)

$\therefore \Delta TOQ \equiv \Delta QOS$ (R.H.S)

(hypotenuse & one side of right Δ's are equal)

$\therefore \angle TOQ = \angle QOS$

Similarly

$\angle TOP = \angle ROP$

$\therefore \angle TOP + \angle TOQ = \frac{1}{2} \angle ROS$

$= 90^\circ$

($\angle ROS = 180^\circ$ from (i))

ie $\angle POQ = 90^\circ$

(b) $x^2 = 4(x+y)$

$= 4x + 4y$

$x^2 - 4x = 4y$

$x^2 - 4x + 4 = 4y + 4$

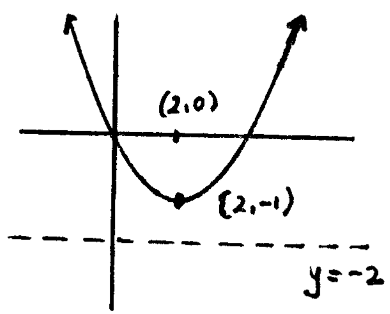
$(x-2)^2 = 4(y+1)$

Comparing with $(x-h)^2 = 4a(y-k)$

Vertex is $(2, -1)$ $\quad a = 1$

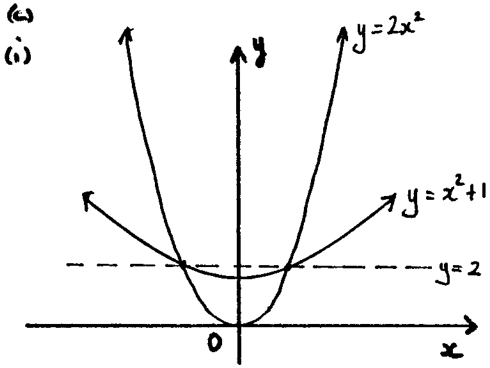
$\therefore$ focus is $(2, 0)$

Directrix is $y = -2$



(c)

(i)



$y = 2x^2$

$y' = 4x$

At $x = t$, $y' = 4t$

The eqn of the tangent at $(t, 2t^2)$ is

$y - 2t^2 = 4t(x - t)$

$y - 2t^2 = 4tx - 4t^2$

$\therefore y = 4tx - 2t^2 \ldots ①$

(ii) If $x = t$ then $y = t^2 + 1$ on the curve $y = x^2 + 1$

$y = x^2 + 1$

$y' = 2x$

At $x = t$, $y' = 2t$.

The eqn of the tangent at (t, t^2+1) is

$y - t^2 - 1 = 2t(x - t)$

$y - t^2 - 1 = 2tx - 2t^2$

$y = 2tx + (1 - t^2) \ldots ②$

(iii) To find the point of intersection, the tangents ① & ② must be solved simultaneously

ie ① = ②

$4tx - 2t^2 = 2tx + (1 - t^2)$

$2tx = 2t^2 + 1 - t^2$

$x = \dfrac{t^2+1}{2t}$

sub x into ①

$y = 4t\left(\dfrac{t^2+1}{2t}\right) - 2t^2$

$= 2t^2 + 2 - 2t^2$

$= 2$

As t varies & x varies, y is always 2.

The locus of (x, y), the point of intersection of the tangents is the line $y = 2$, as in diagram.

QUESTION 4:

(a) Prove $\dfrac{2\cos A}{\operatorname{cosec} A - 2\sin A} = \tan 2A$

$$L.H.S = \frac{2\cos A}{\operatorname{cosec} A - 2\sin A}$$

$$= \frac{2\cos A}{\frac{1}{\sin A} - 2\sin A}$$

$$= \frac{2\sin A\cos A}{1 - 2\sin^2 A}$$

$$= \frac{\sin 2A}{\cos 2A} \quad *$$

$$= \tan 2A$$

$$= R.H.S$$

$\therefore L.H.S = R.H.S$

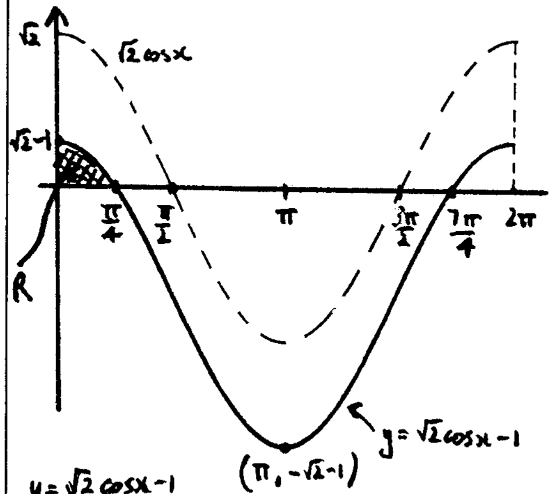
HENCE PROVEN #

$$\left[\begin{aligned} * \ \sin 2A &= 2\sin A\cos A \\ \cos 2A &= \cos^2 A - \sin^2 A \\ &= 1 - 2\sin^2 A \end{aligned}\right]$$

(b) $y = \sqrt{2}\cos x - 1$

(i) Amp $= \sqrt{2}$

$T = 2\pi$



$y = \sqrt{2}\cos x - 1$

x-int occurs when $y = 0$

$\sqrt{2}\cos x - 1 = 0$

$\cos x = \frac{1}{\sqrt{2}}$

S | A ✓ / T | C ✓

$\therefore$ First sol^n $x = \frac{\pi}{4}$

$\therefore$ 2nd sol^n $x = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

MIN OCCURS AT π

$\therefore$ MIN Pt $(\pi, -\sqrt{2}-1)$

(ii) $$A = \int_0^{\frac{\pi}{4}} \sqrt{2}\cos x - 1 \, dx$$

$$= \left[\sqrt{2}\sin x - x\right]_0^{\frac{\pi}{4}}$$

$$= \left[\left(\sqrt{2}\sin\frac{\pi}{4} - \frac{\pi}{4}\right) - \left(\sqrt{2}\sin 0 - 0\right)\right]$$

$$= \left[\left(\sqrt{2}\cdot\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right) - 0\right]$$

$$= \left(1 - \frac{\pi}{4}\right) u^2 \quad \#$$

HENCE PROVEN

(iii) $$V = \pi\int_0^{\frac{\pi}{4}} y^2 \, dx$$

$$= \pi\int_0^{\frac{\pi}{4}} (\sqrt{2}\cos x - 1)^2 \, dx$$

$$= \pi\int_0^{\frac{\pi}{4}} 2\cos^2 x - 2\sqrt{2}\cos x + 1 \, dx$$

BUT

$\cos 2x = \cos^2 x - \sin^2 x$

$\therefore \cos 2x = \cos^2 x - (1 - \cos^2 x)$

$\therefore \cos 2x = 2\cos^2 x - 1$

$\therefore 2\cos^2 x = \cos 2x + 1$

$\therefore V =$

$$\pi\int \cos 2x + 1 - 2\sqrt{2}\cos x + 1 \, dx$$

$$= \pi\int_0^{\frac{\pi}{4}} \cos 2x - 2\sqrt{2}\cos x + 2 \, dx$$

$$= \pi\left[\frac{1}{2}\sin 2x - 2\sqrt{2}\sin x + 2x\right]_0^{\frac{\pi}{4}}$$

$$= \pi\left[\left(\frac{1}{2}\sin\frac{\pi}{2} - 2\sqrt{2}\sin\frac{\pi}{4} + \frac{\pi}{2}\right) - 0\right]$$

$$= \pi\left[\frac{1}{2} - 2 + \frac{\pi}{2}\right]$$

$$= \pi\left[\frac{\pi}{2} - \frac{3}{2}\right]$$

$$= \pi\left(\frac{\pi - 3}{2}\right)$$

$$= \frac{\pi(\pi - 3)}{2} \, u^3 \quad \#$$

(c) $2x^2 + 4x + 1 = 0$ Roots α, β

$\alpha + \beta = \frac{-b}{a} = -2$

$\alpha\beta = \frac{c}{a} = \frac{1}{2}$

(i) $\alpha^2\beta + \alpha\beta^2$

$= \alpha\beta(\alpha + \beta)$

$= \frac{1}{2} \times -2$

$= -1$ #

(ii) $\frac{1}{\alpha\beta^3} + \frac{1}{\alpha^3\beta}$

$$= \frac{\alpha^2 + \beta^2}{(\alpha\beta)^3} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^3}$$

$$= \frac{(-2)^2 - 2 \times \frac{1}{2}}{(\frac{1}{2})^3} = \frac{4-1}{\frac{1}{8}}$$

$= 24$ #

QUESTION 5:

(a) Show $\cot\theta + \tan\frac{\theta}{2} = \operatorname{cosec}\theta$

using $\tan\frac{\theta}{2} = t$

$\sin\theta = \frac{2t}{1+t^2}$ $\quad\cos\theta = \frac{1-t^2}{1+t^2}$

$\tan\theta = \frac{2t}{1-t^2}$

$\therefore$ L.H.S $= \frac{1-t^2}{2t} + t$

$= \frac{1-t^2+2t^2}{2t}$

$= \frac{1+t^2}{2t}$

$= \frac{1}{\sin\theta}$

$= \operatorname{cosec}\theta$

$=$ R.H.S

$\therefore$ L.H.S $=$ R.H.S

Hence proven #

(b)
$x^3 + 2x^2 - x \equiv Ax^3 + B(x-1)^2 + C(x-2) + D$

$\equiv Ax^3 + B(x^2-2x+1) + Cx - 2C + D$

$\equiv Ax^3 + Bx^2 - 2Bx + B + Cx - 2C + D$

$\equiv Ax^3 + Bx^2 - (2B-C)x + B - 2C + D$

Now equating co-efficients

$A = 1 \quad B = 2 \quad 2B - C = 1$

$\Downarrow$

$4 - C = 1$

$\therefore C = 3$

Also $B - 2C + D = 0$

$2 - 6 + D = 0$

$\therefore D = 4$

$\therefore A = 1,\ B = 2,\ C = 3,\ D = 4$ #

(c) $f(x) = \frac{2x}{1+3x^2}$

(i) $f(-x) = \frac{-2x}{1+3x^2} = -f(-x)$

$\therefore f(x) = -f(-x)$

$\therefore f(x)$ is odd #

(ii) $f(x) = \frac{2x}{1+3x^2}$

$f'(x) = \frac{2(1+3x^2) - 6x \cdot 2x}{(1+3x)^2}$

$f'(x) = \frac{2+6x^2-12x^2}{(1+3x)^2}$

$= \frac{2-6x^2}{(1+3x^2)^2}$

Now Stat pts occur at $f'(x) = 0$

ie $2 - 6x^2 = 0$

$2 = 6x^2$

$\therefore x = \pm\frac{1}{\sqrt{3}}$

Now

$f''(x) = \frac{-12x(1+3x^2)^2 - 2(1+3x^2)\cdot 6x(2-6x^2)}{(1+3x^2)^4}$

$= \frac{-12x(1+3x^2)\left[(1+3x^2) + (2-6x^2)\right]}{(1+3x^2)^4}$

$= \frac{-12x(1+3x^2)(3-3x^2)}{(1+3x^2)^4}$

$= \frac{-12x(3-3x^2)}{(1+3x^2)^3} = \frac{-36x(1-x^2)}{(1+3x^2)^3}$

At $x = \frac{1}{\sqrt{3}}$, $y = \frac{1}{\sqrt{3}}$ & $f''(x) < 0$

$\therefore \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is A MAX T.P

At $x = -\frac{1}{\sqrt{3}}$, $y = -\frac{1}{\sqrt{3}}$ & $f''(x) > 0$

$\therefore \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ is a MIN T.P

(iii) let $f''(x) = 0$ to find possible points of inflexion

ie $-36x(1-x^2) = 0$

ie $x = 0, -1, 1$.

To verify pts of inflexion at these points, check the sign of $f''(x)$ at $(x=0, y=0)$, $(x=1, y=\frac{1}{2})$, $(x=-1, y=-\frac{1}{2})$

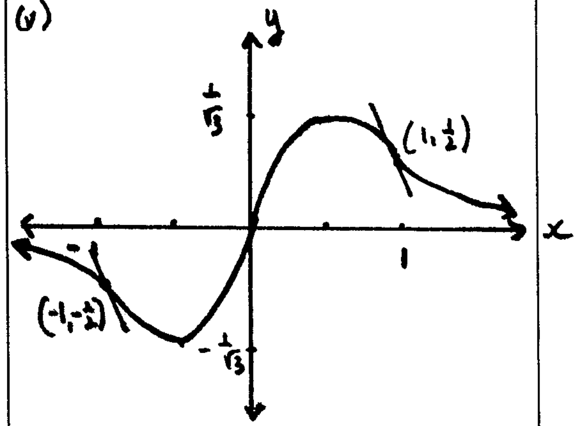
x	-1.5	-1	-0.5	0	0.5	1	1.5
$f''(x)$	$-ve$	0	$+ve$	0	$-ve$	0	$+ve$

since $f''(x)$ changes sign in passing through all these points $(0,0)$, $(-1, -\frac{1}{2})$ $(1, \frac{1}{2})$ are all points of inflexion

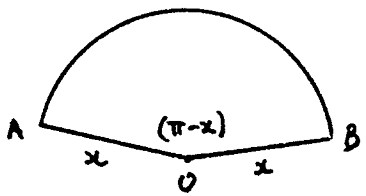
(iv) $\lim_{x\to\pm\infty} \frac{2x}{1+3x^2} = \lim_{x\to\pm\infty} \frac{\frac{2}{x}}{\frac{1}{x^2}+3} = 0$

$\therefore$ x-axis is a horizontal asymptote #

(v)



QUESTION 6:



(a) $A = \frac{1}{2} r^2 \theta$

$= \frac{1}{2} x^2 (\pi - x)$

$\therefore A = \frac{1}{2}\pi x^2 - \frac{1}{2}x^3$

$\frac{dA}{dx} = \pi x - \frac{3}{2}x^2$

Now stat pts occur when $\frac{dA}{dx} = 0$

ie $\pi x - \frac{3}{2}x^2 = 0$

$x(\pi - \frac{3}{2}x) = 0$

ie $x = 0$ $\quad \pi - \frac{3}{2}x = 0$

$\Downarrow$

$x = \frac{2\pi}{3}$

Now $\frac{d^2A}{dx^2} = \pi - 3x$

when $x = \frac{2\pi}{3}$,

$\frac{d^2A}{dx^2} = \pi - 3 \cdot \frac{2\pi}{3}$

$= \pi - 2\pi = -\pi < 0$

$\therefore$ concave down

$\therefore$ MAX AREA Occurs WHEN $x = \frac{2\pi}{3}$

$\therefore$ MAX AREA

$= \frac{1}{2} x^2 (\pi - x)$

$= \frac{1}{2}\left(\frac{2\pi}{3}\right)^2\left(\pi - \frac{2\pi}{3}\right)$

$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{4}\pi^2}{9} \cdot \frac{\pi}{3}$

$= \frac{2\pi^3}{27}$ cm^2 #

(b) PERIMETER $= x + x +$ Arc length

$= 2x + r\theta$

$\therefore P = 2x + x(\pi - x)$

$\frac{dP}{dx} = 2 + \pi - 2x$

MAX/MIN occur At $\frac{dP}{dx} = 0$

ie $2 + \pi - 2x = 0$

$2x = 2 + \pi$

$x = \frac{\pi + 2}{2}$

$\frac{d^2P}{dx^2} = -2 \Rightarrow$ Always $-$ve $\therefore \cap$

$\therefore$ MAX Perimeter at $x = \frac{\pi+2}{2}$

$\therefore P = 2\left(\frac{\pi+2}{2}\right) + \pi\left(\frac{\pi+2}{2}\right) - \left(\frac{\pi+2}{2}\right)^2$

$= \frac{\pi+2}{2}\left[2 + \pi - \left(\frac{\pi+2}{2}\right)\right]$

$= \frac{\pi+2}{2}\left[\frac{4 + 2\pi - \pi - 2}{2}\right]$

$= \left(\frac{\pi+2}{2}\right)\left(\frac{\pi+2}{2}\right) = \left(\frac{\pi+2}{2}\right)^2$ cm

$\therefore$ Max Perimeter is $\left(\frac{\pi+2}{2}\right)^2$ cm #

(c)

(i) AREA OF $\triangle$ AOB $= \frac{1}{2} \cdot x \cdot x \sin(\pi - x)$

$= \frac{1}{2}x^2 \sin(\pi - x)$ $\quad \left[\begin{array}{l}\sin(180 - \theta) \\ = \sin\theta\end{array}\right]$

$= \frac{1}{2}x^2 \sin x$

$\therefore f(x) = \frac{x^2 \sin x}{2}$ #

(ii) $f(x) = \frac{1}{2}x^2 \sin x$

$f'(x) = x \sin x + \frac{x^2}{2}\cos x$

Now $f(x)$ is MAX when $f'(x) = 0$

ie $x \sin x + \frac{x^2}{2}\cos x = 0$

$x\left(\sin x + \frac{x}{2}\cos x\right) = 0$

$\sin x + \frac{x}{2}\cos x = 0$ $\qquad x = 0$

$2\sin x + x\cos x = 0$

$2\sin x = -x\cos x$

$2\tan x = -x$

$x + 2\tan x = 0$ #

(iii) $x + 2\tan x = 0$

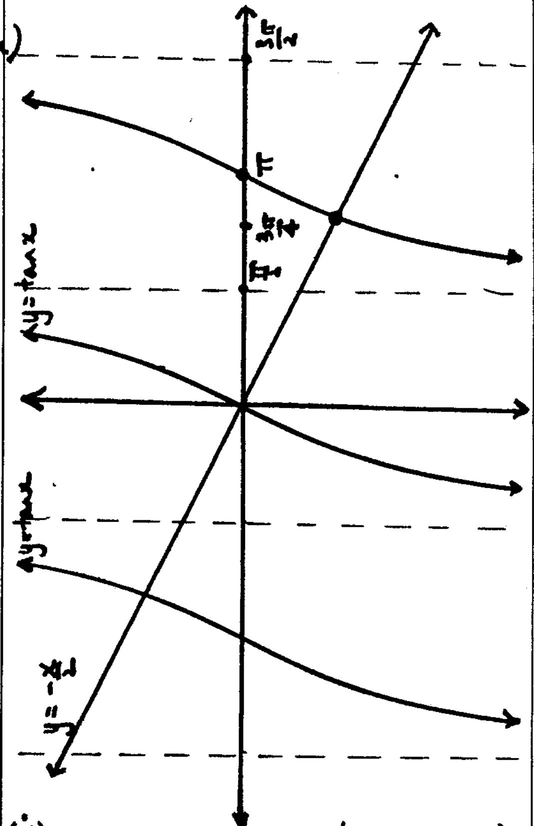
$\therefore -x = 2\tan x$

$\therefore -\frac{x}{2} = \tan x$

$\therefore$ Must sketch $y = \tan x$

$y = -\frac{1}{2}x$

on same graph to solve



(iv) $f(x) = x + 2\tan x$ $\qquad f'(x) = 1 + 2\sec^2 x$

$f\left(\frac{3\pi}{4}\right) = \frac{3\pi}{4} + 2\tan\frac{3\pi}{4}$

$= \frac{3\pi}{4} - 2$

$= \frac{3\pi - 8}{4}$

$f'\left(\frac{3\pi}{4}\right) = 1 + \frac{2}{\left(\cos\frac{3\pi}{4}\right)^2}$

$= 1 + \frac{2}{\frac{1}{2}}$

$= 5$

$\therefore$ Using newtons method

$x_{NEW} = x_{OLD} - \frac{f(x_{OLD})}{f'(x_{OLD})}$

$= \frac{3\pi}{4} - \frac{\frac{3\pi - 8}{4}}{5}$

$= \frac{3\pi}{4} - \frac{3\pi - 8}{20}$

$= \frac{15\pi - 3\pi + 8}{20}$

$= \frac{12\pi + 8}{20}$

$= \frac{3\pi + 2}{5}$ #

QUESTION 7:

(a) $\lim_{x\to 0} \dfrac{\sin^2 4x}{x^2}$

Now $\sin 4x$

$= \sin 2(2x)$ let $2x = X$

$= \sin 2X$

$= 2\sin X \cos X$

$= 2\sin 2x \cos 2x$

$\therefore \lim_{x\to 0} \dfrac{\sin^2 4x}{x^2}$

$= \lim_{x\to 0} \dfrac{(2\sin 2x \cos 2x)^2}{x^2}$

$= \lim_{x\to 0} \dfrac{4\sin^2 2x \cos^2 2x}{x^2}$

$= \lim_{x\to 0} \dfrac{4\cdot 4\sin^2 x \cos^2 x \cos^2 2x}{x^2}$

$= \lim_{x\to 0} \left[16 \cdot \dfrac{\sin^2 x}{x^2} \cdot \cos^2 x \cdot \cos^2 2x\right]$

$= \lim_{x\to 0} \left[16 \cdot \left(\dfrac{\sin x}{x}\right)^2 \cdot \cos^2 x \cdot \cos^2 2x\right]$

$= 16$ #

(b) STEP(i)

Test result for $n = 1$

$y = a\cos\left(\frac{\pi}{2}x\right) + b\sin\left(\frac{\pi}{2}x\right) + c$

$\dfrac{dy}{dx} = -a\cdot\frac{\pi}{2}\sin\left(\frac{\pi}{2}x\right) + b\cdot\frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right)$

$= \left(\frac{\pi}{2}\right)\left[a\cos\left(\frac{\pi}{2}x + 1\cdot\frac{\pi}{2}\right) + b\sin\left(\frac{\pi}{2}x + 1\cdot\frac{\pi}{2}\right)\right]$

$\therefore$ True for $n = 1$

STEP(ii)

Assume true for $n = k$

ie $\dfrac{d^k y}{dx^k} = \left(\frac{\pi}{2}\right)^k\left[a\cos\left(\frac{\pi}{2}x + k\frac{\pi}{2}\right) + b\sin\left(\frac{\pi}{2}x + k\frac{\pi}{2}\right)\right]$

STEP(iii)

Prove true for $n = k+1$

$\dfrac{d}{dx}\left(\dfrac{d^k y}{dx^k}\right)$

$= \dfrac{d}{dx}\left[\left(\frac{\pi}{2}\right)^k\left[a\cos\left(\frac{\pi}{2}x + k\frac{\pi}{2}\right) + b\sin\left(\frac{\pi}{2}x + k\frac{\pi}{2}\right)\right]\right]$

$= \left(\frac{\pi}{2}\right)^k\left[-a\cdot\frac{\pi}{2}\sin\left(\frac{\pi}{2}x + k\frac{\pi}{2}\right) + b\cdot\frac{\pi}{2}\cos\left(\frac{\pi}{2}x + k\frac{\pi}{2}\right)\right]$

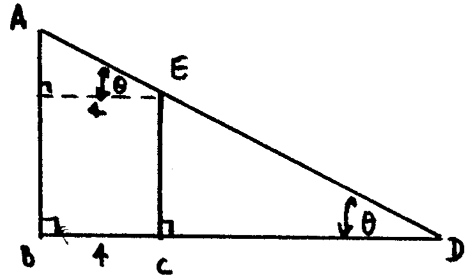
$= \left(\frac{\pi}{2}\right)^{k+1}\left[a\cos\left(\frac{\pi}{2}x + k\frac{\pi}{2} + \frac{\pi}{2}\right) + b\sin\left(\frac{\pi}{2}x + k\frac{\pi}{2} + \frac{\pi}{2}\right)\right]$

$= \left(\frac{\pi}{2}\right)^{k+1}\left[a\cos\left(\frac{\pi}{2}x + (k+1)\frac{\pi}{2}\right) + b\sin\left(\frac{\pi}{2}x + (k+1)\frac{\pi}{2}\right)\right]$

$= \dfrac{d^{k+1} y}{dx^{k+1}}$

STEP(iv) If true for $n = k$, its true for $n = k+1$. As shown true for $n = 1$, then true for $n = 2$ & true then for $n = 3$ & so on $\therefore$ true for all +ve integers.

(c)



(i) Looking at ΔAEP

$\cos\theta = \dfrac{4}{AE} \Rightarrow AE = \dfrac{4}{\cos\theta}$

Looking at ΔEDC

$\sin\theta = \dfrac{5}{DE} \Rightarrow DE = \dfrac{5}{\sin\theta}$

$\therefore L = AE + DE = \dfrac{4}{\cos\theta} + \dfrac{5}{\sin\theta}$ #

(ii) $L = \dfrac{4}{\cos\theta} + \dfrac{5}{\sin\theta}$

$= 4(\cos\theta)^{-1} + 5(\sin\theta)^{-1}$

$\therefore \dfrac{dL}{d\theta} = \dfrac{-4\cdot -\sin\theta}{\cos^2\theta} - \dfrac{5\cos\theta}{\sin^2\theta}$

$= \dfrac{4\sin\theta}{\cos^2\theta} - \dfrac{5\cos\theta}{\sin^2\theta}$

$= \dfrac{4\sin^3\theta - 5\cos^3\theta}{\cos^2\theta \sin^2\theta}$

let $\dfrac{dL}{d\theta} = 0$ to find MAX/MIN

ie $4\sin^3\theta - 5\cos^3\theta = 0$

$4\sin^3\theta = 5\cos^3\theta$

$\therefore \tan^3\theta = \dfrac{5}{4}$

$\therefore \tan\theta = \sqrt[3]{\dfrac{5}{4}}$

$\therefore \theta \doteqdot 47°8'$

θ	45°	47°	50°
$\dfrac{dL}{d\theta}$	−ve	0	+ve

$\therefore \theta = 47°8'$ creates min

$\therefore$ shortest ladder L is given by

$L = \dfrac{4}{\cos 47°8'} + \dfrac{5}{\sin 47°8'}$

$= 12.7$ m #