Student name/number: ________________

SOUTH SYDNEY HIGH SCHOOL

DEC 2002 - HSC TASK 1

Mathematics Extension 1

Total marks **(54)**

- Attempt Questions 1 – 5
- All questions are **NOT** of equal value
- Topics: Parametric representation of the parabola, Permutations and Combinations, Polynomials.

General Instructions

- Working time – 2 periods
- Board-approved calculators may be used
- All necessary working should be shown in every question

Question 1 (12 marks)

(a) Find the Cartesian equation to each of these parametric equations:

(i) $x = 2t, y = 4t^2$ *(2m)*

(ii) $x = 4\cos\theta, y = 3\sin\theta$ *(2m)*

(b) (i) Find all the arrangements of the word "PARAMETRIC". *(1m)*

(ii) In how many of these arrangements are the vowels together? *(1m)*

(iii) In how many of these arrangements are the vowels positioned at the beginning, in the middle and at the end of the word , as such, A _ _ _ AE _ _ _ I ? *(2m)*

(c) There are 52 playing cards made up of Red, Yellow, Green and Blue cards. Each set of coloured cards are numbered from 1 to 13. How many different 5 card hands can I deal which contains

(i) four of kind e.g. 1, 1, 1, 1, 2? *(2m)*

(ii) a full house e.g. 1, 1, 1, 2, 2? *(1m)*

(iii) all of the same colour? *(1m)*

Question 2 (9 marks)

(a) For the polynomial $P(x) = x^2(x^2 - 3x + 2)$, write down the

(i) degree of the polynomial *(1m)*

(ii) constant term *(1m)*

(iii) coefficient of the term in x^3 *(1m)*

(b) (i) Is the polynomial in part Q2(a) monic? *(1m)*

(ii) Factorise the polynomial in part Q2(a) completely. *(1m)*

(iii) Provide a neat sketch of the polynomial function:

$$y = x^2(x^2 - 3x + 2)$$ *(2m)*

(iv) Hence, solve the inequality $x^2(x^2 - 3x + 2) \geq 0$ *(2m)*

Question 3 (13 marks)

(a) (i) Show that $x = -3$ is a zero of the polynomial $P(x) = 2x^3 - 5x^2 - 28x + 15$ *(1m)*

(ii) Hence or otherwise, solve the equation $2x^3 - 5x^2 - 28x + 15 = 0$ *(3m)*

(b) If α, β and γ are the roots of the equation $x^3 - 6x^2 - x + 5 = 0$, evaluate

(i) $\alpha + \beta + \gamma$ *(1m)*

(ii) $\alpha\beta + \beta\gamma + \gamma\alpha$ *(1m)*

(iii) $\alpha\beta\gamma$ *(1m)*

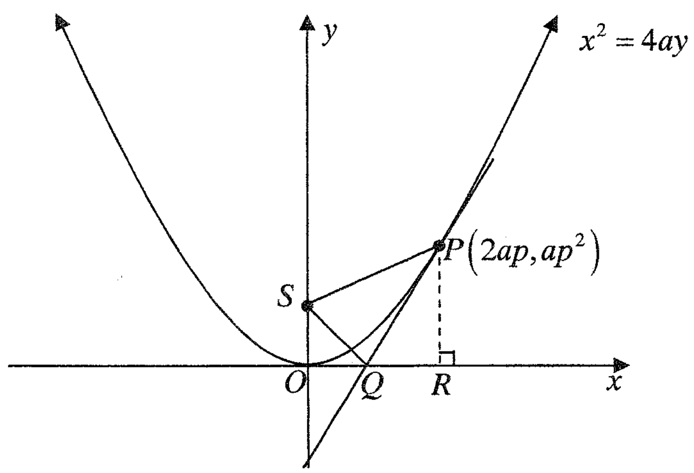
(iv) $\alpha^2 + \beta^2 + \gamma^2$ *(2m)*

(v) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$ *(2m)*

(vi) $\dfrac{\alpha\beta}{\gamma} + \dfrac{\beta\gamma}{\alpha} + \dfrac{\gamma\alpha}{\beta}$ *(2m)*

Question 4 (11 marks)

(a)



P is a point on $x^2 = 4ay$. The tangent at P meets the x-axis at Q, R is the foot of the ordinate from P, S is the focus and O the vertex.
Prove that :

(i) Q is the midpoint of OR. *(3m)*

(ii) PQ is perpendicular to SQ. *(2m)*

(iii) $(SQ)^2 = OS \times SP$. *(2m)*

(b) The letters of the word **SURROUND** are written at random on the circumference of a circle.

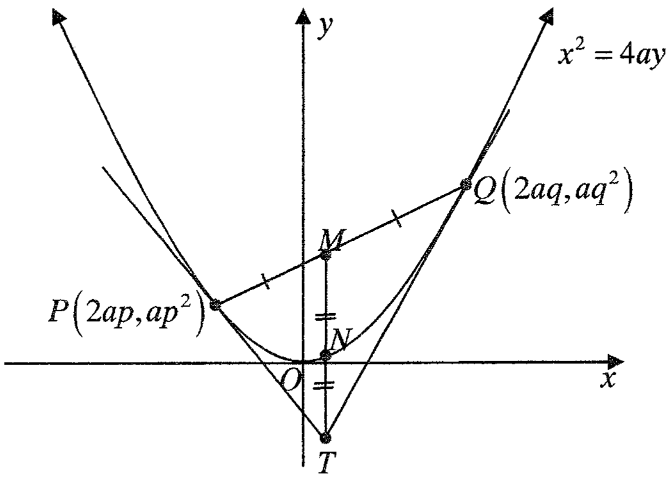
(i) How many different arrangements are possible? *(1m)*

(ii) How many different arrangements if the **R**'s and **U**'s are together? *(1m)*

(iii) How many different arrangements if the **vowels** are together? *(2m)*

Question 5 (12 marks)

(a)



In the diagram, $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are distinct variable points on the parabola $x^2 = 4ay$.

(i) Show that the tangent at P has equation $y = px - ap^2$. *(2m)*

(ii) The tangents at P and Q meet at T. Show that T is the point $(a(p+q), apq)$. *(2m)*

(iii) M is the midpoint of the chord PQ. Show that MT is parallel to the axis of symmetry of the parabola. *(2m)*

(iv) N is the midpoint of MT. Show that as P and Q vary on the parabola $x^2 = 4ay$, N also varies on the parabola $x^2 = 4ay$. *(2m)*

(b) In each of the following parts, use the information to obtain the required real polynomial in the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ where $n, a_0, a_1, \ldots, a_n$ are to be given numerical values.

(i) $P(x)$ is quadratic, $P(0) = 32$, and $P(2^m) = 0$ has roots at $m = 1$ and $m = 3$. *(2m)*

(ii) $P(x)$ has degree 4, has factors $(x+2)^2$ and $(x-2)^2$, and has a remainder 50 on division by $x - 3$. *(2m)*

END OF ASSESSMENT

Mathematics Extension 1

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$\cos^2\theta + \sin^2\theta = 1$

Good work!

Question 1

a. (i). $x = 2t$, $y = 4t^2$

$\frac{x}{2} = t$ $y = 4\left(\frac{x^2}{4}\right)$

$\frac{x^2}{4} = t^2$ ✓ $y = x^2$

$x^2 = y$ ✓

(ii). $x = 4\cos\theta$, $y = 3\sin\theta$

$x^2 = 16\cos^2\theta$ $y^2 = 9\sin^2\theta$

$x^2 = 16\left(\frac{9-y^2}{9}\right)$ $y^2 = 9(1-\cos^2\theta)$

$x^2 = \frac{144}{9} - \frac{16y^2}{9}$ $y^2 = 9 - 9\cos^2\theta$

$x^2 = 16 - \frac{16y^2}{9}$ ×9 $9\cos^2\theta = 9 - y^2$

$9x^2 = 144 - 16y^2$ ✓ $\cos^2\theta = \frac{9-y^2}{9}$

$9x^2 = 16(9-y^2)$

b. (i). PARAMETRIC → $\frac{10!}{2!\,2!} = 907200$ ✓

AAEI

(ii). (A A E I) P R M T R C.

$\frac{7!}{2!\,2!} = 1260$ ✗

PARAMETRIC

(iii).

A _ _ _ A E _ _ _ I

1 6×5×4 1 1×3×2×1 1 = 720 ✗

(1)

c. R 1 2 3 4 5 6 7 8 9 10 11 12 13

52 — Y 1 2 3 4 5 6 7 8 9 10 11 12 13 (i) 5 card — $^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{40}C_1 = 1370920$

G 1 2 3 4 5 6 7 8 9 10 11 12 13 (ii). $^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{49}C_2$ ✗ $= 2503672$

B 1 2 3 4 5 6 7 8 9 10 11 12 13 (iii). all same colour $= 4 \times {}^{13}C_5 = 5148$ ✓

Question 2.

a. $P(x) = x^2(x^2 - 3x + 2)$

$= x^4 - 3x^3 + 2x^2$

(i). Degree of polynomial $= 4$ ✓

(ii). Constant term $= 0$ ✓

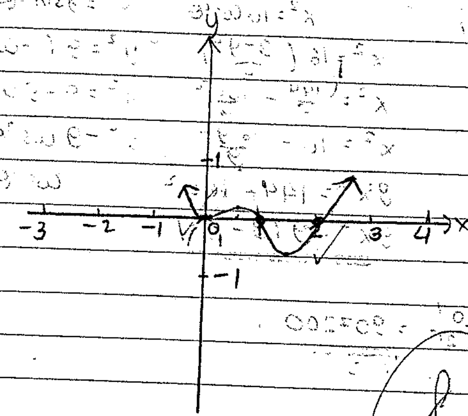
(iii). Coefficient of the term in $x^3 = -3$ ✓

b (i). Yes, it is monic, since the leading coefficients is 1.

(ii). $P(x) = (x^2)(x^2 - 3x + 2) = (x^2)(x-1)(x-2)$ ✓

(iii). $P(x) = x^2(x-1)(x-2)$

$x = 0$, $x = 1$, $x = 2$



Test:

$\frac{1}{2} \to \frac{1}{4}\left(-\frac{1}{2}\right)(-1.5) = 0$

$\frac{3}{2} \to \frac{9}{4}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) = -0.5$

$-\frac{1}{2} \Rightarrow \frac{1}{4}\left(-\frac{3}{2}\right)\left(-2\frac{1}{2}\right) > 0$.

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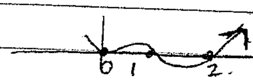
(iv). $x^2(x^2 - 3x + 2) \geq 0$

$x^2(x-1)(x-2) \geq 0$



Test $2 \Rightarrow 4(1)(0) = 0$

$\therefore 0 \leq x \leq 1$, $x \geq 0$ ✓

Question 3

a. (i). $P(x) = 2x^3 - 5x^2 - 20x + 15$

$P(-3) = -54 - 5(-3)(-3) - 20(-3) + 15$

$= -54 - 45 + 84 + 15 = 0$ ✓

$\therefore x = -3$ is a zero of the polynomial $P(x) = 2x^3 - 5x^2 - 20x + 15$

(ii). Since $x = -3$ is a root, $\therefore (x+3)$ is a factor of the polynomial $P(x) = 2x^3 - 5x^2 - 20x + 15$

$$\begin{array}{r} 2x^2 - 11x + 5 \\ x+3 \,\big)\, 2x^3 - 5x^2 - 20x + 15 \\ 2x^3 + 6x^2 \\ -11x^2 - 20x \\ -11x^2 - 33x \\ 5x + 15 \\ 5x + 15 \\ 0 \end{array}$$

$\therefore P(x) = 2x^3 - 5x^2 - 20x + 15 = (x+3)(2x^2 - 11x + 5)$ ✓

$= (x+3)(2x-1)(x-5)$ ✓

$x = -3$ $2x - 1 = 0$ $x = 5$

$2x = 1$ ✓

$x = \frac{1}{2}$

$\alpha^2+\alpha\beta+\alpha\gamma+\alpha\beta+\beta^2+\beta\gamma+\alpha\gamma+\beta\gamma+\gamma^2$

b. $x^3-6x^2-x+5=0$ $\qquad \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\alpha\gamma+\beta\gamma)$

(i). $\alpha+\beta+\gamma = -\frac{b}{a} = -\frac{(-6)}{1} = 6$ ✓ $\qquad (\alpha\beta\gamma)(\alpha\beta\gamma) = (-5)(-5)$

(ii) $\alpha\beta+\beta\gamma+\alpha\gamma = \frac{c}{a} = \frac{-1}{1} = -1$ ✓ $\qquad \alpha^2\beta^2\gamma^2 \quad 25$

(iii). $\alpha\beta\gamma = -\frac{d}{a} = -\frac{5}{1} = -5$ ✓

(iv). $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= (6)^2 - 2(-1) = 36+2 = 38$ ✓

(v). $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2 = \alpha\beta\gamma(\alpha+\beta+\gamma)$

$= -5(6) = -30$ ✓

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(vi). $\frac{\alpha\beta}{\gamma} + \frac{\beta\gamma}{\alpha} + \frac{\gamma\alpha}{\beta} = \frac{\alpha\beta(\alpha\beta)+(\beta\gamma)(\beta\gamma)+(\gamma\alpha)(\gamma\alpha)}{\alpha\beta\gamma}$

$= \frac{(-\frac{5}{\gamma})(-\frac{5}{\gamma})+(-\frac{5}{\alpha})(-\frac{5}{\alpha})+(-\frac{5}{\beta})(-\frac{5}{\beta})}{-5}$

$\alpha^2 = 38-\beta^2-\gamma^2$

$= \left(\frac{25}{\gamma^2}+\frac{25}{\alpha^2}+\frac{25}{\beta^2}\right)\times -\frac{1}{5}$ $\qquad \frac{\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2}{\alpha^2\beta^2\gamma^2}$

$= 25\left(\frac{1}{\gamma^2}+\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\times -\frac{1}{5}$ $\qquad (-5)(-5) = 25$

$= -5\left(\frac{\alpha^2\beta^2+\gamma^2\alpha^2\beta^2+\gamma^2\alpha^2}{\alpha^2\beta^2\gamma^2}\right)$ $\qquad (\alpha\beta\gamma)(\alpha\beta\gamma)$

$= -5\left(\frac{2(\alpha^2+\beta^2+\gamma^2)}{25}\right)$ $\qquad \alpha^2\beta^2\gamma^2$

$= \frac{2(38)}{-5} = -15\frac{1}{5}$

Question 4.

a. $x^2 = 4ay$ $\qquad$ Point $R \rightarrow x = 2ap$

$\frac{x^2}{4a} = y$ $\qquad y = 0$

$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$ $\qquad O(0,0)\quad R(2ap,0)$.

$m_p = \frac{2ap}{2a} = p$ $\qquad$ Midpoint: $x = \frac{0+2ap}{2} = ap$

$y-ap^2 = p(x-2ap)$ ✓ $\qquad y = \frac{0+0}{2} = 0$

$y-ap^2 = px-2ap^2$ $\qquad \therefore Q(ap,0)$ ✓

$0 = px-y-ap^2$ $\qquad \therefore Q$ is midpoint of OR.

Point $Q \rightarrow y=0 \quad 0 = px-0-ap^2$

$0 = px-ap^2$

$ap^2 = px$ ✓

$\frac{ap^2}{p} = x$

$x = ap$

$\therefore Q(ap,0)$

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(ii). $P(2ap, ap^2)\quad Q(ap,0)$

$S(0,a)\quad Q(ap,0)$

$m_{PQ} = \frac{0-ap^2}{ap-2ap} = \frac{-ap^2}{-ap} = p$ ✓

$m_{SQ} = \frac{0-a}{ap-0} = \frac{-a}{ap} = -\frac{1}{p}$

$\therefore m_{PQ}\cdot m_{SQ} = p \times -\frac{1}{p} = -1$ ✓

$\therefore PQ \perp SQ$

(iii). $S(0,a)\quad Q(ap,0)\quad O(0,0)\quad P(2ap,ap^2)$

LHS $= (SQ)^2$ $\qquad S(0,a)$

$SQ = \sqrt{(0-a)^2+(ap-0)^2} = \sqrt{a^2+a^2p^2} = \sqrt{a^2(1+p^2)}$

$(SQ)^2 = (\sqrt{a^2(1+p^2)})(\sqrt{a^2(1+p^2)})$

$= a^2(1+p^2)$

$= a^2(p^2+1)$ ✓

RHS $=$ $OS = \sqrt{(a-0)^2+(0-0)^2} = \sqrt{a^2} = a$

$SP = \sqrt{(ap^2-a)^2+(2ap)^2} = \sqrt{(ap^2-a)(ap^2-a)+4a^2p^2}$

$= \sqrt{a^2p^4-a^2p^2-a^2p^2+a^2+4a^2p^2}$

$= \sqrt{a^2p^4+2a^2p^2+a^2}$

$= \sqrt{a^2(p^4+2p^2+1)}$

$= a\sqrt{(p^2+1)(p^2+1)}$

$= a(p^2+1)$ ✓

$\therefore OS \times SP = a[a(p^2+1)]$

$= a(ap^2+a) = a^2p^2+a^2 = a^2(p^2+1)$

$\therefore$ LHS $=$ RHS $= a^2(p^2+1)$

b. (i). SURROUND. $\quad \frac{(8-1)!}{2!2!} = 1260$ ✓

(ii). RRUU ODNS $\quad \frac{(4!)\times 4!}{2!2!} = 144$ ✓

(iii). RRUU ODNS $\quad \frac{5!}{2!2!}\times 3! = 180$ ✓

Mira Chow

Question 5.

(8)

a.(i). $x^2 = 4ay$

$\frac{x^2}{4a} = y$

$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$

At P, $m = \frac{2ap}{2a} = p$

$y - y_1 = m(x - x_1)$

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$0 = px - y - ap^2$

$\therefore y = px - ap^2$ ... (i)

(ii). $x^2 = 4ay$

$\frac{x^2}{4a} = y$

$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$

At Q, $m = \frac{2aq}{2a} = q$

$y - y_1 = m(x - x_1)$

$y - aq^2 = q(x - 2aq)$

$y - aq^2 = qx - 2aq^2$

$y = qx - aq^2$ ... (ii)

(i) and (ii) $y = px - ap^2$

$y = qx - aq^2$

$0 = px - qx - ap^2 + aq^2$

$0 = (p-q)x - a(p^2 - q^2)$

$a(p+q)(p-q) = (p-q)x$

$\therefore x = a(p+q)$.

$qy = pqx - aqp^2$

$py = pqx - apq^2$

$qy - py = -aqp^2 + apq^2$

$(q-p)y = apq(-p+q)$

$y = \frac{apq(q-p)}{(q-p)}$

$\therefore y = apq$.

$P(2ap, ap^2)$ $Q(2aq, aq^2)$. $\therefore T(a(p+q), apq)$

(iii). Midpoint: $M \rightarrow x = \frac{2ap + 2aq}{2} = \frac{2(ap+aq)}{2} = a(p+q)$

$y = \frac{ap^2 + aq^2}{2} = \frac{a(p^2+q^2)}{2}$

$\therefore M\left(a(p+q), \frac{1}{2}a(p^2+q^2)\right)$

$T(a(p+q), apq)$.

Point $M \rightarrow x = a(p+q)$

Point $T \rightarrow x = a(p+q)$.

$\therefore x = a(p+q)$.

$\therefore$ MT is parallel to the axis of symmetry of parabola.

(MT is a straight line // axis symmetry of parabola).

Midpoint (N)

(iv). $x = \frac{ap + aq + ap + aq}{2} = \frac{2ap + 2aq}{2} = ap + aq$.

$y = \frac{\frac{ap^2 + aq^2}{2} + apq}{2} = \frac{\frac{ap^2 + aq^2 + 2apq}{2}}{2}$

$y = \frac{ap^2 + aq^2 + 2apq}{4}$

$4y = a(p^2 + q^2 + 2pq)$.

$4y = a(p+q)^2$

$4y = a\left(\frac{x^2}{a^2}\right)$

$4y = \frac{x^2}{a}$

$x = a(p+q)$

$\frac{x}{a} = p + q$.

$\frac{x^2}{a^2} = (p+q)^2$

$\rightarrow \therefore 4ay = x^2 \Rightarrow x^2 = 4ay$

locus of P and Q $= x^2 = 4ay$ same as locus of N / N also varies on parabola $x^2 = 4ay$.