SOUTH SYDNEY HIGH SCHOOL

YEAR 12

MATHEMATICS

2006

ASSESSMENT 2

Time Allowed—2 PERIODS

Directions to Candidates

- Attempt ALL questions
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work.
- Board approved calculators maybe used.

QUESTION 1. (14 marks) **Marks**

Consider the curve given by $y = x^3 - 12x + 4$.

(a) Find the coordinates of any stationary points and determine their nature. 6

(b) Find the coordinates of any points of inflexion. 3

(c) Sketch the curve for the domain $-3 \le x \le 4$. 4

(d) For what value(s) of x in the domain $-3 \le x \le 4$ does y have its maximum value? 1

QUESTION 2. (13 marks)

(a) (i) What is the condition (in terms of $\frac{dy}{dx}$) for a function to be decreasing? 4

(ii) Find the values of x for which the function $y = 4 + 36x - 3x^2 - 2x^3$ is decreasing.

(b) Three circles, two with radii r and one with radius R are formed such that the sum of their radii is 18 cm.

(i) Show that the sum, S, of the areas of the three circles is $S = \pi(6r^2 - 72r + 324)$. 2

(ii) Hence find the radii of the circles if the sum of the areas is a minimum. 3

(c) For the parabola $x^2 = 10y + 5$ find the: 4

(i) coordinates of the vertex,

(ii) focal length,

(iii) coordinates of the focus,

(iv) equation of the directrix.

QUESTION 3. (16 marks)

(a) Find: 6

(i) $\int (1 - x^2)\,dx$

(ii) $\int \left(x + \frac{1}{x^2}\right) dx$

(iii) $\int (x^{\frac{3}{2}} + x^{-\frac{1}{3}})\,dx$.

(b) Show that $\int_0^4 \sqrt{2x+1}\,dx = 8\frac{2}{3}$. 4

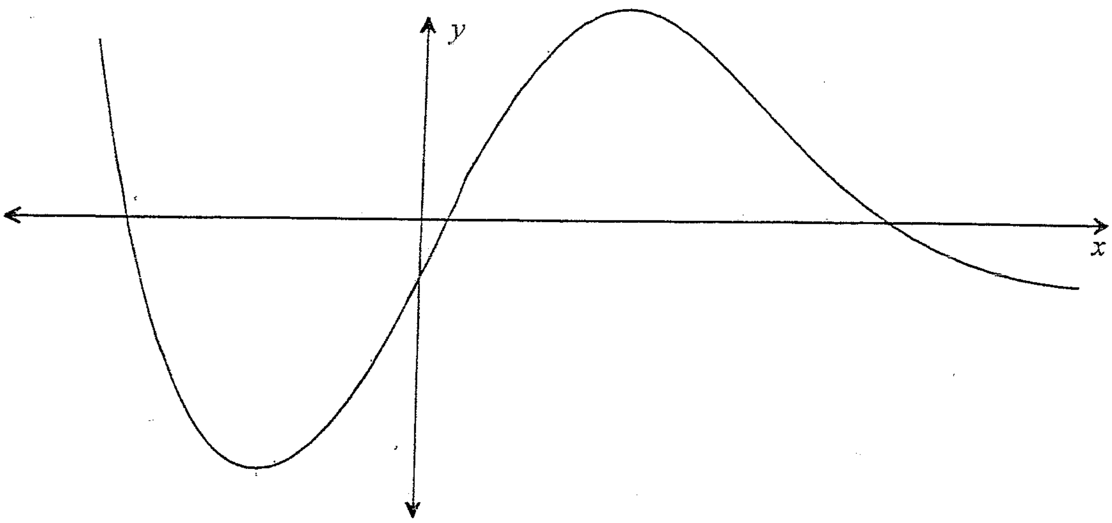
(c) Consider the function $y = x^2 - 6x + 5$.

(i) Sketch the curve for the domain $0 \le x \le 7$, showing the intercepts on the axes, and the endpoints. **3**

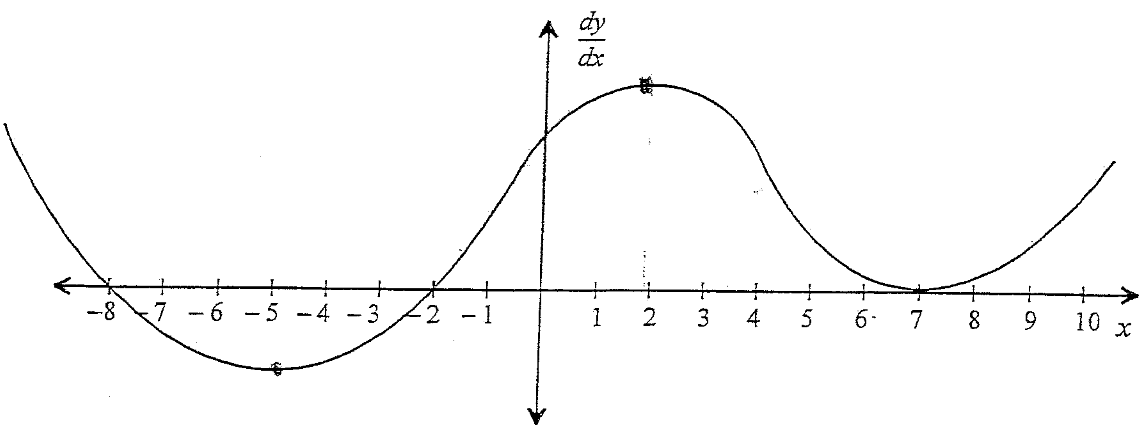
(ii) Find the area between the x-axis and the section of the curve below the x-axis. **3**

QUESTION 4. (7 marks) **Marks**

(a) Copy or trace the curve of $y = f(x)$ and on the same set of axes sketch the curve of the gradient function $f'(x)$. **3**



(b) The given curve represents a gradient function $\frac{dy}{dx}$ relative to x of a function $y = f(x)$. **4**



Use this graph to determine the values of x at which the graph of the function $y = f(x)$:

(i) has a maximum turning point,

(ii) has a horizontal point of inflexion,

(iii) is concave down.

(14 m) Mathematics Solution. Y12 Assessment 2 (2006)

SC
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(a) $y = x^3 - 12x + 4$

$\frac{dy}{dx} = 3x^2 - 12$

Stationary points when $\frac{dy}{dx} = 0$.

$3x^2 - 12 = 0$ 1

$3(x-2)(x+2) = 0$

$x = 2$ or $x = -2$ 1

When $x = 2$, $y = (2)^3 - 12 \times (2) + 4$

$= -12$ 1

When $x = -2$, $y = (-2)^3 - 12 \times (-2) + 4$

$= 20$ 1

Stationary points at $(-2, 20)$ and $(2, -12)$.

$\frac{d^2y}{dx^2} = 6x$

When $x = -2$, $\frac{d^2y}{dx^2} = -12 < 0$ $\therefore$ concave down. 1

$\therefore$ Maximum turning point at $(-2, 20)$.

When $x = 2$, $\frac{d^2y}{dx^2} = 12 > 0$ $\therefore$ concave up 1

$\therefore$ Minimum turning point at $(2, -12)$.

Total = 6

(b) A point of inflection occurs when $\frac{d^2y}{dx^2} = 0$ and concavity changes. 1

$6x = 0$ $\therefore$ $x = 0$

At $x = 0 - \varepsilon$, $\frac{d^2y}{dx^2} = 6 \times (-\varepsilon) < 0$ (concave down)

At $x = 0 + \varepsilon$, $\frac{d^2y}{dx^2} = 6 \times \varepsilon > 0$ (concave up) 1

Concavity changes.
$\therefore$ Point of inflection is at $(0, 4)$. 1

Note: To check for change of concavity on either side of $x = 0$, rather than use ε (a small positive number), you may substitute a numerical value, say $x = -0.1$ and $x = 0.1$.

Note: $y = 4$ is found by substituting $x = 0$ into the equation of the curve $y = x^3 - 12x + 4$.

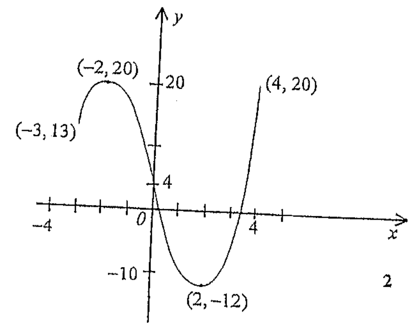
Total 3

(c) When $x = -3$, $y = (-3)^3 - 12 \times (-3) + 4$

$= 13$ 1

When $x = 4$, $y = 4^3 - 12 \times 4 + 4$

$= 20$ 1



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Total = 4

) y has a maximum value when $x = -2$, $x = 4$. 1

Note: Determined from the graph.

QUESTION 2 (13 m)

(a) (i) A function is decreasing when $\frac{dy}{dx} < 0$. 1

(ii) $y = 4 + 36x - 3x^2 - 2x^3$

$\frac{dy}{dx} = 36 - 6x - 6x^2$

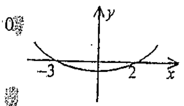
When the function is decreasing,

$36 - 6x - 6x^2 < 0$ 1

$6x^2 + 6x - 36 > 0$

$x^2 + x - 6 > 0$

$(x+3)(x-2) > 0$ 1



$x < -3$, $x > 2$ 1

Note: Multiply by -1, reverse the inequality.

Note: Quadratic inequality – always sketch a graph of the quadratic function to determine the solution.

Total = 3

(b) (i) $2r + R = 18$ $\therefore$ $R = 18 - 2r$

$S = \pi r^2 + \pi r^2 + \pi R^2$ 1

$= 2\pi r^2 + \pi(18 - 2r)^2$ 1

$= 2\pi r^2 + \pi\left(324 - 72r + 4r^2\right)$

$= \pi\left(6r^2 - 72r + 324\right)$

Note: Before differentiating, we must express the right-hand side in terms of one pronumeral. We use the relation $R = 18 - 2r$.

Total = 2

(ii) For minimum sum of areas, $\frac{dS}{dr} = 0$ and $\frac{d^2S}{dr^2} > 0$.

$\frac{dS}{dr} = \pi(12r - 72) = 0$

$r = 6$ 1

$\frac{d^2S}{dr^2} = \pi \times 12 > 0$ $\therefore$ concave up 1

$\therefore$ sum of areas of circles is minimum when the radii are all 6 cm 1

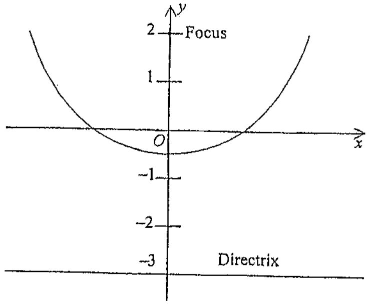
Note: $R = 18 - 2(6) = 18 - 12 = 6$

Total = 3

(c) Rewrite the parabola in the form $(x-h)^2 = 4a(y-k)$. *Note :* $(x-h)^2 = 4a(y-k)$ has vertex at (h, k).

$$x^2 = 10\left(y+\frac{1}{2}\right)$$

i.e. $x^2 = 4 \times \frac{5}{2}\left(y+\frac{1}{2}\right)$



(i) Vertex is $(0, -\frac{1}{2})$. 1

(ii) Focal length is $2\frac{1}{2}$ units. 1

(iii) Coordinates of the focus are $(0, 2)$. 1

(iv) Equation of the directrix is $y = -3$. 1 **Total = 4**

QUESTION 3 (16M)

(a) (i) $\int\left(1-x^2\right) dx = x - \frac{1}{3}x^3 + C$ 2 *Note :* $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$.

(ii) $\int\left(x+\frac{1}{x^2}\right)dx = \int\left(x+x^{-2}\right)dx$ 1 *Note :* "$+C$" needs to be included in all indefinite integrals.

$= \frac{1}{2}x^2 - x^{-1} + C$ 1

or $\frac{1}{2}x^2 - \frac{1}{x} + C$ **Total = 2**

(iii) $\int\left(x^{\frac{3}{2}} + x^{-\frac{1}{3}}\right)dx = \frac{2}{5}x^{\frac{5}{2}} + \frac{3}{2}x^{\frac{2}{3}} + C$ 2

(b) $\int_0^4 \sqrt{2x+1}\, dx = \int_0^4 (2x+1)^{\frac{1}{2}} dx$ 1 *Note :* $\int(ax+b)^n dx = \frac{1}{n+1} \times \frac{1}{a}(ax+b)^{n+1} + C$.

$= \left[\frac{2}{3}(2x+1)^{\frac{3}{2}} \times \frac{1}{2}\right]_0^4$

$= \left[\frac{1}{3}(2x+1)^{\frac{3}{2}}\right]_0^4$ 1

$= \left[\frac{1}{3}(2\times4+1)^{\frac{3}{2}}\right] - \left[\frac{1}{3}(2\times0+1)^{\frac{3}{2}}\right]$ 1

$= \left[\frac{1}{3} \times 9^{\frac{3}{2}}\right] - \left[\frac{1}{3} \times 1^{\frac{3}{2}}\right]$ *Note :* $x^{\frac{m}{n}} = (\sqrt[n]{x})^m$

$\therefore 9^{\frac{3}{2}} = \left(\sqrt{9}\right)^3 = 3^3 = 27$.

$= 9 - \frac{1}{3}$

$= 8\frac{2}{3}$ 1 **Total = 4**

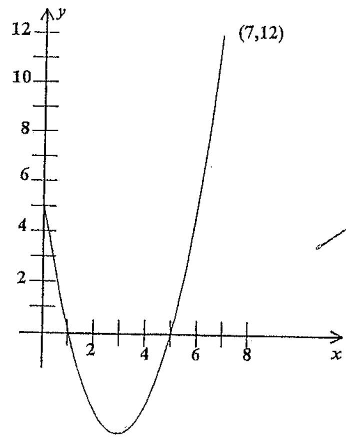
(c) (i) $y = x^2 - 6x + 5$

$y = (x-1)(x-5)$

Cuts x-axis at $x = 1$, $x = 5$ 1

When $x = 0$, $y = 5$

When $x = 7$, $y = 12$. 1



1 **Total = 3**

(ii) $A = \left|\int_1^5 (x^2 - 6x + 5)dx\right|$ 1

$= \left|\left[\frac{1}{3}x^3 - 3x^2 + 5x\right]_1^5\right|$

$= \left|\left(41\frac{2}{3} - 75 + 25\right) - \left(\frac{1}{3} - 3 + 5\right)\right|$ 1

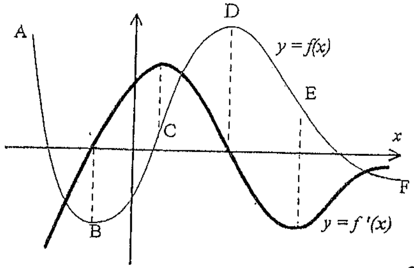
$= \left|-8\frac{1}{3} - 2\frac{1}{3}\right|$ *Note :* The definite integral is a negative number because $y < 0$ in this domain.

$= \left|-10\frac{2}{3}\right|$

Area is $10\frac{2}{3}$ unit^2. 1 **Total = 3**

QUESTION 4 (7m)

(a)



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Notes : From A $\rightarrow$ B, gradient is negative.
At B, gradient is zero (stationary point).
From B $\rightarrow$ C, gradient is positive and increases to maximum value at C (point of inflection).
From C $\rightarrow$ D, gradient is still positive but becomes zero at D (stationary point).
From D $\rightarrow$ E, gradient is negative, and has maximum negative value at E (a point of inflection).
From E $\rightarrow$ F, gradient is still negative, but approaching zero.

(b) (i) Maximum turning point:

$\frac{dy}{dx} = 0$

$\frac{dy}{dx} > 0$ / — \ $\frac{dy}{dx} < 0$

Maximum value at $x = -8$. ✓ 1

(ii) Horizontal point of inflexion where $\frac{dy}{dx} = 0$

and $\frac{dy}{dx}$ has the same sign on either side.

$\therefore$ horizontal point of inflection at $x = 7$. 1 ✓

(iii) Concave down when

$\frac{d^2y}{dx^2} < 0$, i.e. $\frac{d}{dx}\left(\frac{dy}{dx}\right) < 0$

i.e. the derivative of $\frac{dy}{dx} < 0$

i.e. the $\frac{dy}{dx}$ curve is decreasing.

✓ ✓

Concave down for $x < -5$, $2 < x < 7$. 1, 1 **Total = 2**