SOUTH SYDNEY HIGH SCHOOL

Name ____________

2007

Year 12

HSC Assessment Task 3

Wednesday 4th April 2007

Extension 2 Mathematics

Weighting: 25%

Working time: 2 hours

Total marks: 90

Topics examined:

Algebra, Circle Geometry, Locus and the Parabola, Integration, Polynomials, Sequences and Series, Harder Trigonometry.

Outcomes assessed:

PE2, PE3, PE4, HE2, HE6

Question	Mark
1	
2	
3	
4	
5	
6	
Bonus	
TOTAL	

Mean:

Standard Deviation:

Ranking within Course:

General Instructions:

- Write using blue or black pen
- Board-approved calculators and templates may be used
- All necessary working should be shown in every question
- Questions are of equal value
- Full marks may not be awarded for careless or badly arranged work
- Questions are not necessarily arranged in order of difficulty
- Begin each question on a new page

Juestion 1 (15 marks)

Question 1 (15 marks)

(a) Two complex numbers are given by $z = 3 - 4i$ and $w = 2 - 2i$.

(i) Find the value of the product $\bar{z}w$. 1

(ii) Find the two square roots of z. 3

(iii) Express w in modulus-argument form and hence find the value of w^4. 2

(b) The locus of a point P which moves in the complex plane is represented by the equation $|z-(3+4i)| = 5$.

(i) Sketch the locus of the point P. 1

(ii) Find the maximum value of the modulus of z and write down the value of $\arg z$ when P is in the position of maximum modulus. 2

(iii) Find the modulus of z when $\arg z = \tan^{-1}\left(\frac{1}{2}\right)$. 1

(c) The complex number z satisfies both equations $|z-1| = \frac{1}{2}|z|$ and $\arg(z-1) - \arg z = \frac{\pi}{3}$. 2

(i) Show that $\frac{z-1}{z} = \frac{1}{2} cis\left(\frac{\pi}{3}\right)$.

(ii) Hence show that $z = \frac{3+i\sqrt{3}}{3}$.

Question 2 (15 marks)

(a) Consider the hyperbola $\mathcal{H}$ defined by $9x^2 - 16y^2 = 144$.

(i) Find the coordinates of the foci 1

(ii) Find the equations of the directrices. 1

(iii) Find the equations of the asymptotes 1

(iv) Sketch $\mathcal{H}$ showing all of the above information and all intercepts with the axes. 3

(b) Find real constants A, B and C such that 3

(c) Let α, β and γ be the roots of $x^3 - 7x^2 + 18x - 7 = 0$.

(i) Find a cubic equation that has roots $1+\alpha^2$, $1+\beta^2$, and $1+\gamma^2$. 2

(ii) Hence, or otherwise find the value of $(1+\alpha^2)(1+\beta^2)(1+\gamma^2)$. 1

(d) If one of the roots of $P(x) = x^3 - x^2 - 6x + 18 = 0$ is $2-\sqrt{2}i$, find the other two roots 3

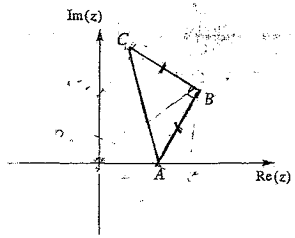
$$\frac{x^2+5x+2}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$$

Question 3 (15 marks)

(a) The polynomial $P(x)=x^3+px^2+qx+r$ has $x=\alpha$ as a double root. 3

Show that $\alpha=\dfrac{-q\pm\sqrt{q^2-3pr}}{p}$

(b)



The diagram above shows the fixed points A, B and C in the Argand plane, where $AB=BC$, $\angle ABC=\dfrac{\pi}{2}$, and A, B and C in anticlockwise order. The point A represents the complex number $z_1=2$ and the point B represents the complex number $z_2=3+\sqrt{5}i$.

(i) Find the complex number z_3 represented by the point C. 2

(ii) D is the point on the Argand plane such that $ABCD$ is a square. Find the complex number z_4 represented by D. 2

(c) (i) Solve $z^3-1=0$, leaving your answers in modulus-argument form. 1

(ii) Let ω be one of the non-real roots of $z^3-1=0$.

(α) Show that $1+\omega+\omega^2=0$. 2

(β) Hence simplify $(1+\omega)^8$. 1

(d) In the Argand diagram points A, B, C, D represent the complex numbers α, β, γ, δ respectively.

(i) If $\alpha+\gamma=\beta+\delta$ show that $ABCD$ is a parallelogram. 2

(ii) If $ABCD$ is a square with vertices in anticlockwise order, show that $\gamma+i\alpha=\beta+i\beta$. 2

Question 4 (15 marks)

(a) $P(a\cos\alpha, b\sin\alpha)$ and $Q(a\cos\beta, b\sin\beta)$ lie on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

(i) Verify that the coordinates of P satisfy $\dfrac{x}{a}\cos\dfrac{\alpha+\beta}{2}+\dfrac{y}{b}\sin\dfrac{\alpha+\beta}{2}=\cos\dfrac{\alpha-\beta}{2}$. 1

(ii) Deduce that the chord PQ has equation $\dfrac{x}{a}\cos\dfrac{\alpha+\beta}{2}+\dfrac{y}{b}\sin\dfrac{\alpha+\beta}{2}=\cos\dfrac{\alpha-\beta}{2}$. 1

(iii) Hence or otherwise show that if PQ is a focal chord of the ellipse, then 1

$$\cos\frac{\alpha-\beta}{2}=\pm e\cos\frac{\alpha+b}{2}$$

(b) (i) Show that $\cos(p+q)+\cos(p-q)=2\cos p\cos q$, and deduce that 2

$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}.$$

(ii) Show that if PQ is a focal chord of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, where 3

$P(a\cos\alpha, b\sin\alpha)$ and $Q(a\cos\beta, b\sin\beta)$, then $PQ=2a\left\{1-e^2\cos^2\left(\dfrac{\alpha+\beta}{2}\right)\right\}$.

(c) $P(a\cos\alpha, b\sin\alpha)$, $Q(a\cos\beta, b\sin\beta)$ and $R(a\cos\theta+b\sin\theta)$, lie on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

PQ is a focal chord of the ellipse, RT is the diameter through R, and PQ is parallel to RT.

(i) Show this information on a sketch. 1

(ii) Deduce that T has coordinates $(-a\cos\theta, -b\sin\theta)$ and $RT=2RO$. 1

(iii) Show that $RT^2=4a^2\left(1-e^2\sin^2\theta\right)$. 2

(iv) Show that $\tan\theta\tan\dfrac{\alpha+\beta}{2}=-1$. 1

(v) Show that $RT^2=2aPQ$. 2

SOLUTIONS TO ASSESSMENT EXT 2 4th March 2007 Task 3

QUESTION 1

a) (i) $z = 3-4i, \quad w = 2-2i$

$\therefore \bar{z}w = (3+4i)(2-2i)$

$= 6 - 6i + 8i - 8i^2$

$= \underline{14+2i}$

1

(ii) Let $(a+ib)^2 = 3-4i$

ie $a^2 - b^2 + 2abi = 3-4i$

$\therefore a^2 - b^2 = 3 \quad —(1)$

$2ab = -4 \quad —(2)$

From (2): $b = -\frac{4}{2a} = \frac{-2}{a}$

Subst into (1): $a^2 - \left(\frac{-4}{2a}\right)^2 = 3$

3

$a^2 + \frac{16}{4a^2} = 3$

$4a^4 - 16 = 12a^2$

$4a^4 - 12a^2 - 16 = 0$

$a^4 - 3a^2 - 4 = 0$

$(a^2-4)(a^2+1) = 0$

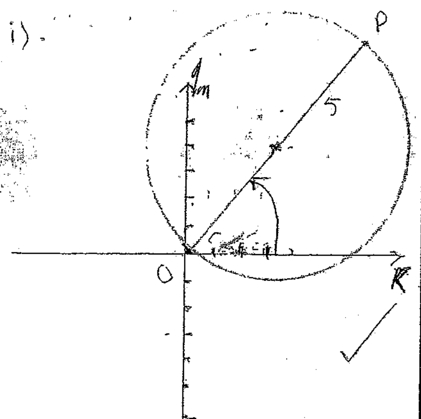
$a = \pm 2$

$b = \pm 1$

$\therefore \sqrt{3-4i} = \underline{\underline{2-i \text{ or } -(2-i)}}$

b) i)

1



(ii) From the diagram max value for $|z|$ is OP

ie max value $\underline{\underline{|z| = 10}}$

$\arg z = \tan^{-1}\frac{4}{3}$ [From diagram]

$=$

(iii) $|z| = \sqrt{1^2+2^2}$

$= \underline{\sqrt{5}}$

c) i) $\arg(z-1) - \arg z = \frac{\pi}{3}$

$\arg\left(\frac{z-1}{z}\right) = \frac{\pi}{3}$

Also $|z-1| = \frac{1}{2}|z|$

$\frac{|z-1|}{|z|} = \frac{1}{2}$

$\left|\frac{z-1}{z}\right| = \frac{1}{2}$

$\therefore \underline{\underline{\frac{z-1}{z} = \frac{1}{2}\operatorname{cis}\frac{\pi}{3}}}$

(ii) $\frac{z-1}{z} = \frac{1}{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$1 - \frac{1}{z} = \frac{1+\sqrt{3}i}{4}$

$\frac{1}{z} = 1 - \frac{1+\sqrt{3}i}{4}$

$\frac{1}{z} = \frac{4-1-\sqrt{3}i}{4}$

$\frac{1}{z} = \frac{3-\sqrt{3}i}{4}$

$z = \frac{4}{3-\sqrt{3}i} \times \frac{3+\sqrt{3}i}{3+\sqrt{3}i}$

$= \frac{12+4\sqrt{3}i}{12} = \underline{\underline{\frac{3+\sqrt{3}i}{3}}}$

QUESTION 2

(a) $9x^2 - 16y^2 = 144$

$\frac{9x^2}{144} - \frac{16y^2}{144} = 1$

$\frac{x^2}{16} - \frac{y^2}{9} = 1$

$\therefore a = 4,\ b = 3$

$b^2 = a^2(e^2 - 1)$

$9 = 16(e^2 - 1)$

$\frac{9}{16} = e^2 - 1$

$e^2 = \frac{25}{16}$

$e = \frac{5}{4}$

1 (i) $\therefore$ foci $(\pm ae, 0) \rightarrow \underline{(\pm 5, 0)}$

1 (ii) $\therefore$ directrices $x = \pm\frac{a}{e} \Rightarrow \underline{x = \pm\frac{16}{5}}$

1 (iii) $\therefore$ asymptotes $y = \pm\frac{b}{a}x \Rightarrow \underline{y = \pm\frac{3}{4}x}$

3 For graph

b) $\frac{x^2+5x+2}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$

$x^2+5x+2 = (Ax+B)(x+1) + C(x^2+1)$

$= Ax^2 + Ax + Bx + B + Cx^2 + C$

$= (A+C)x^2 + (A+B)x + B + C$

when $x = -1$: $1 - 5 + 2 = 2C$

$-2 = 2C \Rightarrow \underline{C = -1}$

Also $A + C = 1$

$A - 1 = 1 \rightarrow \underline{A = 2}$

$B + C = 2$

$B - 1 = 2 \rightarrow \underline{B = 3}$

c)(i) Let $y = 1 + x^2 \Rightarrow x = \sqrt{y-1}$

$\therefore P(\sqrt{x-1}) = 0$

$(\sqrt{x-1})^3 - 7(\sqrt{x-1})^2 + 18(\sqrt{x-1}) - 7 = 0$

$(x-1)\sqrt{x-1} - 7(x-1) + 18\sqrt{x-1} - 7 = 0$

$\sqrt{x-1}\,(x - 1 + 18) = 7(x-1) + 7$

$(\sqrt{x-1})(x+17) = 7x$

$(x-1)(x+17)^2 = 49x^2$

$(x-1)(x^2 + 34x + 289) = 49x^2$

$x^3 + 34x^2 + 289x - x^2 - 34x - 289 = 49x^2$

$\underline{x^3 + 16x^2 + 255x - 289 = 0}$

(ii) $(1+\alpha^2)(1+\beta^2)(1+\gamma^2) = -\frac{d}{a}$

$= \frac{289}{1}$

$= \underline{289}$

d) $P(x) = x^3 - x^2 - 6x + 18 = 0$

If $2 - \sqrt{2}i$ is a root then so is $2 + \sqrt{2}i$

Also $P(-3) = 0$

$\therefore 3$ is a root

The two roots are $\underline{3,\ \ 2 + \sqrt{2}i}$

QUESTION 3

(a) $P(x) = x^3 + px^2 + qx + r$

$P(\alpha) = 0$

$\therefore \alpha^3 + p\alpha^2 + q\alpha + r = 0$ — (1)

$P'(x) = 3x^2 + 2px + q$

$P'(\alpha) = 0$ (since α is a double root)

$\therefore 3\alpha^2 + 2p\alpha + q = 0$ — (2)

(1) $\times 3$ − (2) $\times \alpha$ gives:

$(3\alpha^3 - 3\alpha^3) + (3p\alpha^2 - 2p\alpha^2) + (3q\alpha - q\alpha) + r = 0$

$\therefore p\alpha^2 + 2q\alpha + 3r = 0$

$\alpha = \frac{-2q \pm \sqrt{4q^2 - 12rp}}{2p}$

$= \frac{-2q \pm 2\sqrt{q^2 - 3rp}}{2p} = \frac{-q \pm \sqrt{q^2 - 3pr}}{p}$

b)(i) $\vec{OC} = \vec{OA} + \vec{AB} + \vec{BC}$

$= 2 + (1 + \sqrt{5}i) + i(1 + \sqrt{5}i)$

$\therefore z_3 = \underline{[(3 - \sqrt{5}) + (1 + \sqrt{5})i]}$

(ii) $\vec{OD} = \vec{OA} + \vec{BC}$

$= 2 + i(1 + \sqrt{5}i)$

$\therefore \underline{z_4 = (2 - \sqrt{5}) + i}$

c)(i) $z^3 - 1 = 0$

Let $z = \text{cis}\,\theta$ [clearly $|z| = 1$]

$z^3 = 1$

$\therefore (\cos\theta + i\sin\theta)^3 = 1$

$\cos 3\theta + i\sin 3\theta = 1$

$\cos 3\theta = 1$

$3\theta = 0, 2\pi, 4\pi$

$\therefore z = 1,\ \text{cis}\frac{2\pi}{3},\ \text{cis}\frac{4\pi}{3}$ or $\text{cis}\,-\frac{2\pi}{3}$

(ii) The roots of $z^3 - 1 = 0$ are $1, \omega, \omega^2$

α) sum of roots $1 + \omega + \omega^2 = 0$

or $1 + \omega + \omega^2 = 1 + \text{cis}\frac{2\pi}{3} + \text{cis}\left(-\frac{2\pi}{3}\right)$

$= 1 + \text{cis}\frac{2\pi}{3} + \text{cis}\left(\frac{2\pi}{3}\right)$

$= 1 + 2\text{cis}\frac{2\pi}{3}$

$= 1 + 2 \times \left(-\frac{1}{2}\right)$

$= 1 - 1$

$= \underline{0}$

1 β) $(1 + \omega)^8 = (-\omega^2)^8$

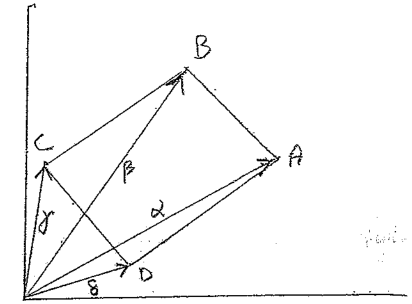
$= \omega^{16}$

$= (\omega^3)^5 \times \omega$

$= 1^5 \times \omega$

$= \underline{\omega}$

d)



(i) Given $\alpha + \gamma = \beta + \delta$

$\Longrightarrow \alpha - \beta = \delta - \gamma$

From the diagram we see that

$\alpha - \beta$ is the vector of $\vec{AB}$

and $\delta - \gamma$ is the vector of $\vec{DC}$

So if $\alpha - \beta = \delta - \gamma$

then $\vec{AB} = \vec{DC}$

i.e. $AB \parallel RS$ and $|\vec{AB}| = |\vec{DC}|$

i.e. one pair of opposite sides of quadrilateral are equal & parallel.

(ii) $* \ \alpha + \gamma i = \beta + i\beta \ *$

$\gamma - \beta = i(\beta - \alpha)$

Now $\vec{DC} = i\,\vec{DA}$ (it is ⊥ to DA)

but $\vec{DC} = \vec{AB}$

$\vec{DA} = \vec{CB}$

$\therefore \vec{AB} = i\,\vec{CB}$

$\beta - \alpha = i(\gamma - \beta)$

$\beta - \alpha = i\gamma - i\beta$

$\underline{\alpha + \gamma i = \beta + i\beta}$

Question 1 a (iii)

$\omega = 2\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

$\therefore \omega^4 = 64\,\text{cis}(-\pi)$

$= \underline{64}$

QUESTION 4

(i) $\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$

thro' $P(a\cos\alpha, b\sin\alpha)$

then $\frac{a\cos\alpha}{a}\cos\frac{\alpha+\beta}{2} + \frac{b\sin\alpha}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$

$\cos\alpha\cos\frac{\alpha+\beta}{2} + \sin\alpha\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$

$\cos\left[\alpha - \frac{\alpha+\beta}{2}\right] = \cos\frac{\alpha-\beta}{2}$

$\cos\left[\frac{2\alpha-\alpha-\beta}{2}\right] = \cos\frac{\alpha-\beta}{2}$

$\cos\left[\frac{\alpha-\beta}{2}\right] = \cos\left[\frac{\alpha-\beta}{2}\right]$ ✓

∴ P satisfies above eqn

(ii) thro' $Q(a\cos\beta, b\sin\beta)$

then $\frac{a\cos\beta}{a}\cos\frac{\alpha+\beta}{2} + \frac{b\sin\beta}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$

$\cos\beta\cos\frac{\alpha+\beta}{2} + \sin\beta\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$

$\cos\left[\beta - \left(\frac{\alpha+\beta}{2}\right)\right] = \cos\frac{\alpha-\beta}{2}$

$\cos\left[\frac{2\beta-\alpha-\beta}{2}\right] = \cos\frac{\alpha-\beta}{2}$

$\cos\left(\frac{\beta-\alpha}{2}\right) = \cos\frac{\alpha-\beta}{2}$

$\cos\left(\frac{\alpha-\beta}{2}\right) = \cos\frac{\alpha-\beta}{2}$

since $\cos(-\theta) = \cos\theta$

∴ Q satisfies above eqn

hence PQ has eqn ✓

$\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$

(iii) $\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$

thro' $(\pm ae, 0)$ ✓

$\frac{\pm ae}{a}\cos\frac{\alpha+\beta}{2} + 0 = \cos\frac{\alpha-\beta}{2}$

$\cos\frac{\alpha-\beta}{2} = \pm e\cos\frac{\alpha+\beta}{2}$

b) i) $\cos(p+q) + \cos(p-q)$

$= \cos p\cos q - \sin p\sin q + \cos p\cos q + \sin p\sin q$

$= 2\cos p\cos q$ ✓

$\left.\begin{array}{l}\alpha = p+q\\ \beta = p-q\end{array}\right\} \Rightarrow p = \frac{\alpha+\beta}{2},\ q = \frac{\alpha-\beta}{2}$

hence $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

ii) If $S(ae, 0)$ lies on PQ then $PQ = PS + QS$

Now $PS^2 = (a\cos\alpha - ae)^2 + b^2\sin^2\alpha$

$= a^2\cos^2\alpha - 2a^2e\cos\alpha + a^2e^2 + b^2(1-\cos^2\alpha)$

$= a^2\cos^2\alpha - 2a^2e\cos\alpha + a^2e^2 + b^2 - b^2\cos^2\alpha$

$= (a^2 - b^2)\cos^2\alpha + b^2 - 2a^2e\cos\alpha + a^2e^2$

$= (a^2 - a^2 + a^2e^2)\cos^2\alpha + a^2 - a^2e^2 - 2a^2e\cos\alpha + a^2e^2$

$= a^2e^2\cos^2\alpha + a^2 - 2a^2e\cos\alpha$

$= a^2 - 2a^2e\cos\alpha + a^2e^2\cos^2\alpha$

$= (a - ae\cos\alpha)^2$

∴ $PS = (a - ae\cos\alpha)$

Similarly

$QS = (a - ae\cos\beta)$

∴ $PQ = (a - ae\cos\alpha) + (a - ae\cos\beta)$

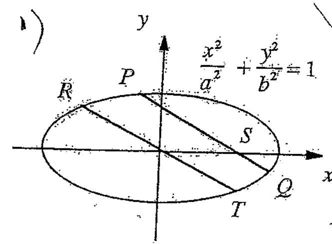
$= 2a - ae(\cos\alpha + \cos\beta)$

$= 2a - ae\left(2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\right)$

$= 2a - 2ae^2\cos^2\frac{\alpha+\beta}{2}$

$= 2a\left[1 - e^2\cos^2\frac{\alpha+\beta}{2}\right]$

c) i)



(ii) $(0,0)$ satisfies eqn of chord RT, Hence from a ii), if T has parameter ϕ, then

$\cos\frac{\theta-\phi}{2} = 0 \Rightarrow \frac{\theta-\phi}{2} = \pm\frac{\pi}{2}$

$\theta - \phi = \pm\pi$

∴ $\phi = \theta \pm \pi$

$\cos\phi = \cos(\theta+\pi)$
$= -\cos\theta$

$\sin\phi = \sin(\theta+\pi)$
$= -\sin\theta$

∴ Coords of T: $(a\cos\phi, b\sin\phi)$
$= (-a\cos\theta, -b\sin\theta)$

$RO = TO \Rightarrow RT = 2RO$

(iii) $RO^2 = a^2\cos^2\theta + b^2\sin^2\theta$

$= a^2\{\cos^2\theta + (1-e^2)\sin^2\theta\}$

$= a^2(1 - e^2\sin^2\theta)$

$RT^2 = 4RO^2$

$= 4a^2(1 - e^2\sin^2\theta)$

(iv) $PQ \parallel RT$.

Equating gradients:

$-\frac{b}{a}\cot\frac{\alpha+\beta}{2} = \frac{b}{a}\tan\theta$

$\tan\theta\tan\frac{\alpha+\beta}{2} = -1$

(v) $RT^2 = 2a\,PQ$

$RT^2 = 4a^2(1 - e^2\sin^2\theta)$

$PQ = 2a\left\{1 - e^2\cos^2\left(\frac{\alpha+\beta}{2}\right)\right\}$

but $\tan\theta\tan\frac{\alpha+\beta}{2} = -1$

$\tan^2\theta\tan^2\frac{\alpha+\beta}{2} = 1$

$\tan^2\theta\left(\sec^2\frac{\alpha+\beta}{2} - 1\right) = 1$

$\sec^2\frac{\alpha+\beta}{2} - 1 = \mathrm{cosec}^2\theta - 1$

$\sec^2\frac{\alpha+\beta}{2} = \mathrm{cosec}^2\theta$

$\cos^2\frac{\alpha+\beta}{2} = \sin^2\theta$

∴ $PQ = 2a\left\{1 - e^2\cos^2\frac{\alpha+\beta}{2}\right\}$

∴ $RT^2 = 2a\,PQ$