Sydney Secondary College

SYDNEY SECONDARY COLLEGE

BLACKWATTLE BAY CAMPUS

2013

TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11-16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 2-6

10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II Pages 7-14

90 marks

- Attempt Questions 11-16
- Allow about 2 hours and 45 minutes for this section

STUDENT NUMBER/NAME: ....

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

---

1 Let $z = 1 + i$ and $w = 1 - 2i$

What is the value of zw ?

(A) $-1 - i$

(B) $-1 + i$

(C) $3 - i$

(D) $3 + i$

2 The equation $x^3 - y^3 + 3xy + 1 = 0$ defines y implicitly as a function of x.

What is the expression for the slope of the tangent to this curve ?

(A) $\frac{x^2+y}{y^2+x}$

(B) $\frac{x^2+y}{y^2-x}$

(C) $\frac{x^2-y}{y^2+x}$

(D) $\frac{x^2-y}{y^2-x}$

3 Given $z = 1 + i$, what is the modulus-argument form of $\bar{z}$ (z conjugate) ?

(A) $\sqrt{2}\left(cos\left(-\frac{\pi}{4}\right) + isin\left(-\frac{\pi}{4}\right)\right)$

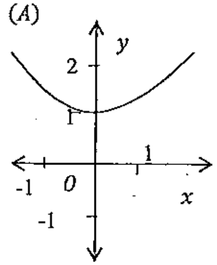
(B) $\sqrt{2}\left(cos\left(\frac{\pi}{4}\right) + isin\left(\frac{\pi}{4}\right)\right)$

(C) $\sqrt{2}\left(cos\left(\frac{\pi}{4}\right) - isin\left(\frac{\pi}{4}\right)\right)$

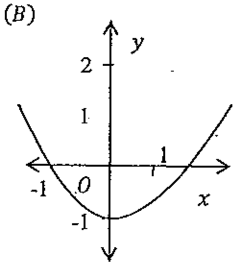
(D) $\sqrt{2}\left(cos\left(-\frac{\pi}{4}\right) - isin\left(-\frac{\pi}{4}\right)\right)$

4 If $f(x) = 1 - x^2$, which of the following graphs best represents $y = |f(x)|$?

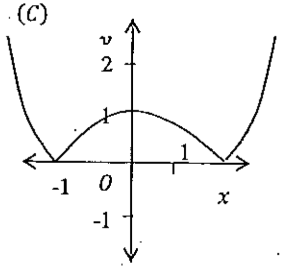
(A)



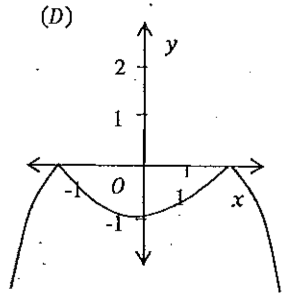
(B)



(C)



(D)



5 The equation $2x^3 - 7x + 1 = 0$ has roots α, β and γ.

What is the value of $\alpha^3 + \beta^3 + \gamma^3$?

(A) 0

(B) $-\frac{1}{2}$

(C) $\frac{43}{4}$

(D) $-\frac{3}{2}$

6 What is the focus of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$?

(A) $(\sqrt{5}, 0)$

(B) $(-\sqrt{13}, 0)$

(C) $(\frac{9\sqrt{13}}{13}, 0)$

(D) $(\frac{9\sqrt{5}}{5}, 0)$

7 Given that $u_n = \int tan^n x \; dx = \frac{tan^n x}{n-1} - u_{n-2}$, then

$\int tan^6 x \; dx = ?$

(A) $\frac{tan^4 x}{4} - \frac{tan^2 x}{2} + x + c$

(B) $\frac{tan^5 x}{5} - \frac{tan^3 x}{3} + \tan x - x + c$

(C) $\frac{tan^4 x}{4} - \frac{tan^2 x}{2} - x + c$

(D) $\frac{tan^5 x}{4} - \frac{tan^3 x}{3} + \tan x + x + c$

8 Find $\int sin^7\theta \cos\theta \; d\theta$

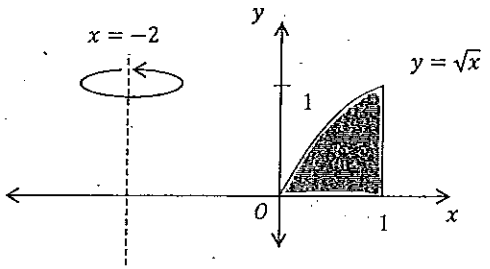
(A) $-\frac{1}{8} cos^8\theta + c$

(B) $-\frac{1}{8} sin^8\theta + c$

(C) $\frac{1}{8} cos^8\theta + c$

(D) $\frac{1}{8} sin^8\theta + c$

9 The region bounded by the curve $y = \sqrt{x}$, the x – axis and the line $x = 1$ is rotated about the line $x = -2$ to form a solid.



Which integral represents the volume of the solid ?

(A) $2\pi \int_0^1 (x+2)\sqrt{x}\ dx$

(B) $2\pi \int_0^1 (x-2)\sqrt{x}\ dx$

(C) $2\pi \int_0^1 (x+2)^2 x\ dx$

(D) $2\pi \int_0^1 (x+2)\, x\ dx$

10 A particle is moving in simple harmonic motion with velocity $v\ cm/s$.

If $v^2 = 48 + 16x - 4x^2$, where x is the displacement in centimetres, what is the amplitude of the motion ?

(A) $2\ cm$

(B) $4\ cm$

(C) $1\ cm$

(D) $8\ cm$

Section II

90 Marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) If $z = 1 + 2i$ and $w = 3 - i$, express in the form $a + bi$ where a and b are real

(i) $2z - w$ **1**

(ii) $z\overline{w}$ **1**

(b) Shade the region on the Argand diagram for which **2**

$$2 \le z + \bar{z} \le 6$$

(c) By completing the square, find $\int \frac{1}{x^2+2x+5}\,dx$ **2**

(d) (i) Write $z = \sqrt{3} + i$ in modulus-argument form. **2**

(ii) Hence, express z^5 in the form $x + yi$, where x and y are real numbers. **1**

(e) Evaluate $\int_0^1 \frac{x}{1+x^2}\,dx$, expressing your answer in exact form. **3**

(f) Sketch the following graphs, showing any intercepts and asymptotes.

(i) $y = \sqrt{1-x}$ **1**

(ii) $y = \frac{1}{\sqrt{1-x}}$ **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Find real numbers a and b such that $\frac{1}{x(x+1)} \equiv \frac{a}{x} + \frac{b}{x+1}$. **2**

(ii) Hence find $\int \frac{1}{x(x+1)}\ dx$ **1**

(b) Consider the hyperbola $H : 16x^2 - 9y^2 = 144$ with foci at S and S'.

(i) Find the coordinates of the foci **1**

(ii) What are the equations of the directrices ? **1**

(iii) Sketch H **2**

(iv) Find the gradient of the tangent to the curve at $P(3\sec\theta,\ 4\tan\theta)$ **2**

(v) Show that the equation of the tangent at P is $4x = (3\sin\theta)y + 12\cos\theta$ **2**

(c) If $I_n = \int_0^{\frac{\pi}{2}} cos^n x \quad dx$ (for $n = 0, 1, 2, 3, ...$) **2**

Show that for $n \geq 2 \quad I_n = \frac{n-1}{n}\ I_{n-2}$

(d) Find the locus of all points which satisfy $arg\left(\frac{z-1}{z-i}\right) = \frac{\pi}{4}$ **2**

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A particle of mass m falls vertically from rest, against a resistance which is proportional to its velocity v.

(i) Show that the time taken is given by $t = \frac{1}{k}\, log_e\left(\frac{g}{g-kv}\right)$ **3**

(ii) Show that $v = \frac{g}{k}\ (1 - e^{-kt})$ **3**

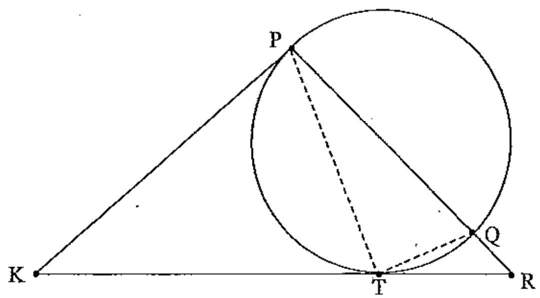
(iii) Show that $x = \frac{g}{k^2}\ (kt + e^{-kt} - 1)$ **3**

(iv) Find the terminal velocity **1**

(b) PQ is a diameter of a circle PQT.

The tangents at P and T meet at K.

The lines KT and PQ are produced to meet at R.



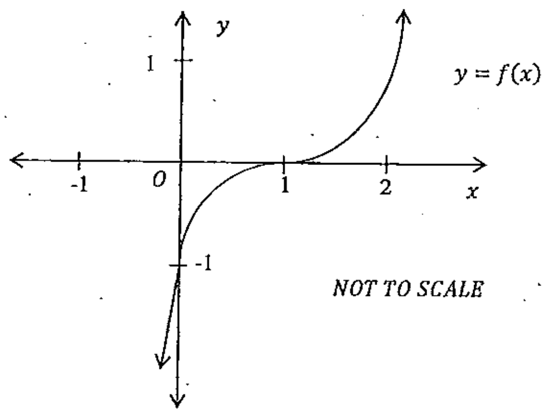
NOT TO SCALE

(i) Prove that $\angle TPK = 90° - \angle QTR$ **2**

(ii) Hence or otherwise prove that $\angle QTR = \frac{1}{2}\angle PKT$ **3**

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) The function $y = f(x)$ is shown below.



Draw separate one-third page sketches of :

(i) $y = |f(x)|$ **2**

(ii) $y = \sqrt{f(x)}$ **2**

(iii) $y = f(|x|)$ **2**

Question 14 continues on page 11

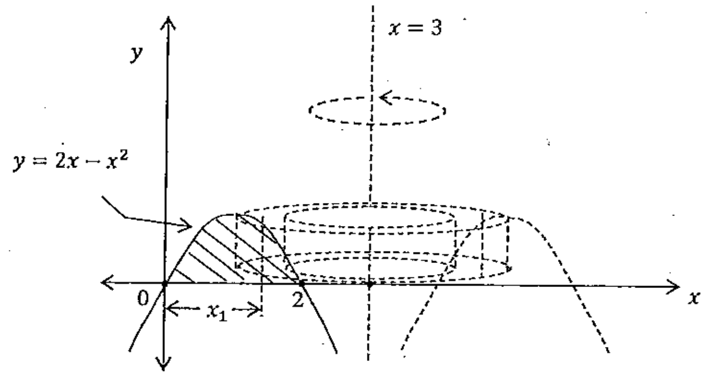
(b) The area bounded by the curve $y = 2x - x^2$ and the x axis (shaded) is rotated about the line $x = 3$.

A typical cylindrical shell of radius r and height y is shown at $x = x_1$

(i) What is the radius of the shell in terms of x ? **1**

(ii) What is the height of the shell in terms of x ? **1**

(iii) Find the volume of the solid using the method of cylindrical shells. **3**



(c) Suppose $x > 0$, $y > 0$. Prove that **4**

$$\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}$$

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) The roots of $x^3 + 5x^2 + 11 = 0$, are α, β, and γ.

(i) Find the polynomial equation whose roots are α^2, β^2, and γ^2. **3**

(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. **2**

(b) (i) If α is a multiple root of the polynomial equation $P(x) = 0$, show that $P'(\alpha) = 0$. **2**

(ii) Find all the roots of the equation $18x^3 + 3x^2 - 28x + 12 = 0$, given that two of the roots are equal. **3**

(c) During a lesson on sequences, the teacher asked the class :

" What is the next number in the sequence 1, 1, 1, ..." ?

Peter replied .. 'one'..

Helen replied .. 'it can be any number'..

The teacher then gave the class the following information -

" A sequence is defined to be a function, say $f(n)$, whose domain is the set of positive integers n (that is, $n = 1, 2, 3, ...$).
Now consider the sequence $1, 1, 1, ...$ where $f(1) = 1$, $f(2) = 1$, $f(3) = 1$ and so on.

Let $f(4) = k$ and $f(n) = an^3 + bn^2 + cn + d$ where k, a, b, c, d are constants".

Help the class by answering the following :

(i) By solving simultaneously, find expressions for a, b, c and d in terms of k. **3**

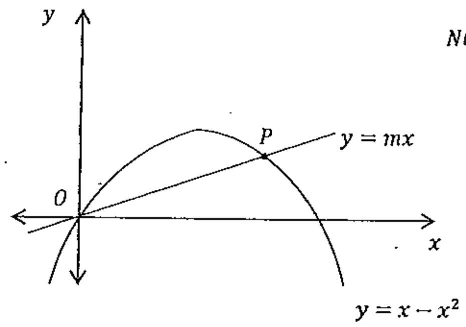
(ii) Hence express $f(n)$ in terms of k and n. **1**

(iii) Is Helen correct ? give reasons. **1**

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) The area bounded by the curve $y = x - x^2$ and the x axis, from $x = 0$ to $x = 1$ is to be divided equally by the line $y = mx$.

The line $y = mx$ and the curve $y = x - x^2$ intersect at $(0, 0)$ and $P(x, y)$.

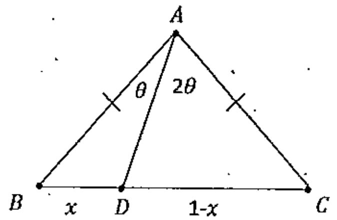


(i) Find the coordinates of P in terms of m. **3**

(ii) Calculate the value of m (leave answer in surd form). **4**

Question 16 continues on page 14

(b)



NOT TO SCALE

In the diagram ABC is a triangle in which $AB = AC$ and $BC = 1$.

The point D lies on BC such that $\angle BAD = \theta$, $\angle CAD = 2\theta$, $BD = x$ and $CD = 1 - x$.

(i) Use the sine rule in each of ΔADB and ΔADC to show that $\cos\theta = \dfrac{1-x}{2x}$ **4**

(ii) Hence show that $\dfrac{1}{3} < x < \dfrac{1}{2}$ **4**

End of paper

Section I

SOLUTIONS

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

---

1 Let $z = 1 + i$ and $w = 1 - 2i$

What is the value of zw ?

(A) $-1 - i$

(B) $-1 + i$

(C) $3 - i$

(D) $3 + i$

$$ZW = (1+i)(1-2i)$$
$$= 1 - 2i + i - 2i^2$$
$$= 1 - i + 2$$
$$ZW = 3 - i$$

(C)

2 The equation $x^3 - y^3 + 3xy + 1 = 0$ defines y implicitly as a function of x.

What is the expression for the slope of the tangent to this curve ?

(A) $\frac{x^2+y}{y^2+x}$

(B) $\frac{x^2+y}{y^2-x}$

(C) $\frac{x^2-y}{y^2+x}$

(D) $\frac{x^2-y}{y^2-x}$

→ Tangent slope means y'
→ Differentiate implicitly

$$3x^2 - 3y^2 \cdot y' + 3[1 \cdot y + x \cdot 1 \cdot y'] + 0 = 0$$
$$3x^2 - 3y^2 y' + 3y + 3xy' = 0$$
$$3y'(x - y^2) = -3(y + x^2)$$
$$y' = \frac{x^2 + y}{y^2 - x}$$

(B)

3 Given $z = 1 + i$, what is the modulus-argument form of $\bar{z}$ (z conjugate) ?

$r[\cos\theta + i\sin\theta]$

(A) $\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

(B) $\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

(C) $\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)\right)$

(D) $\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) - i\sin\left(-\frac{\pi}{4}\right)\right)$

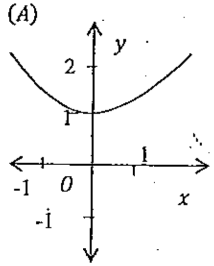
$\boxed{Z = 1 + i}$ $\therefore \bar{Z} = 1 - i$

$$|\bar{z}| = \sqrt{1^2 + 1^2}$$
$$= \sqrt{2}$$
$$\arg(\bar{z}) = \theta$$
$$\therefore \theta = -\frac{\pi}{4}$$
$$\bar{z} = \sqrt{2}\left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right]$$

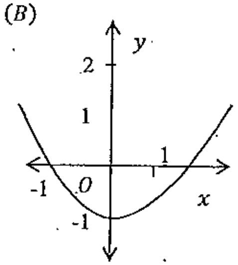
(A)

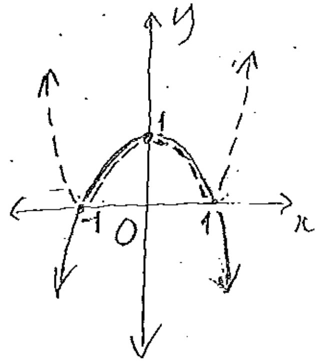
4 If $f(x) = 1 - x^2$, which of the following graphs best represents $y = |f(x)|$?

(A)

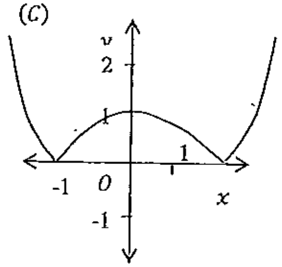


(B)

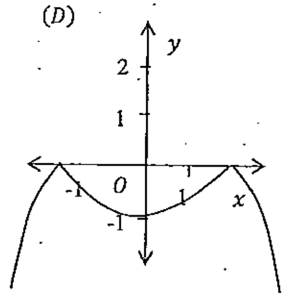




(C)



(D)



(C)

$- - - -$ $y = |f(x)|$
—— $y = f(x)$

5 The equation $2x^3 - 7x + 1 = 0$ has roots α, β and γ.

What is the value of $\alpha^3 + \beta^3 + \gamma^3$?

$\boxed{\Sigma\alpha = \frac{-b}{a} = \frac{0}{2} = 0}$

(A) 0

(B) $-\frac{1}{2}$

(C) $\frac{43}{4}$

(D) $-\frac{3}{2}$

Since α, β, γ are roots then each satisfies the equation. That is...

$2\alpha^3 - 7\alpha + 1 = 0$

$2\beta^3 - 7\beta + 1 = 0$

$2\gamma^3 - 7\gamma + 1 = 0$

Adding... $2(\alpha^3 + \beta^3 + \gamma^3) - 7(\alpha + \beta + \gamma) + 3 = 0$

$2(\alpha^3 + \beta^3 + \gamma^3) - 7(0) = -3$

$\alpha^3 + \beta^3 + \gamma^3 = -\frac{3}{2}$

D

6 What is the focus of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$?

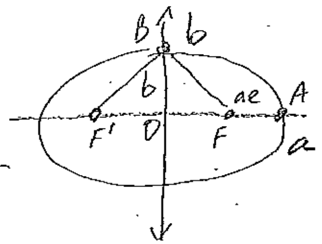
(A) $(\sqrt{5}, 0)$

(B) $(-\sqrt{13}, 0)$

(C) $(\frac{9\sqrt{13}}{13}, 0)$

(D) $(\frac{9\sqrt{5}}{5}, 0)$

$\boxed{a = 3 \quad b = 2}$



$ae = 3\sqrt{1 - \frac{4}{9}}$

$= 3\sqrt{\frac{5}{9}}$

$= \frac{3\sqrt{5}}{3}$

$= \sqrt{5}$

$\therefore F(\sqrt{5}, 0)$

A

$BF + BF' = 2a$ (sum of distances)

$\therefore BF = a$ (symmetry)

$\therefore OF^2 = a^2 - b^2$

$a^2e^2 = a^2 - b^2$

$e^2 = 1 - (\frac{b}{a})^2$

$e = \sqrt{1 - (\frac{b}{a})^2}$

7 Given that $u_n = \int \tan^n x\, dx = \frac{\tan^n x}{n-1} - u_{n-2}$, then

$\int \tan^6 x\, dx = ?$

$u_n = \int \tan^n x\, dx$

$= \frac{\tan^{n-1} x}{n-1} - u_{n-2}$

(A) $\frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + x + c$

(B) $\frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + c$

(C) $\frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} - x + c$

(D) $\frac{\tan^5 x}{4} - \frac{\tan^3 x}{3} + \tan x + x + c$

$\int \tan^6 x\, dx$

$= \frac{\tan^5 x}{5} - \int \tan^4 x\, dx$

$= \frac{1}{5}\tan^5 x - \left[\frac{\tan^3 x}{3} - \int \tan^2 x\, dx\right]$

$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \left[\frac{\tan^1 x}{1} - \int 1\, dx\right]$

$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + c$

B

8 Find $\int \sin^7\theta \cos\theta\, d\theta$

(A) $-\frac{1}{8}\cos^8\theta + c$

(B) $-\frac{1}{8}\sin^8\theta + c$

(C) $\frac{1}{8}\cos^8\theta + c$

(D) $\frac{1}{8}\sin^8\theta + c$

Let $u = \sin\theta$

$du = \cos\theta\, d\theta$

$\therefore \int \sin^7\theta \cos\theta\, d\theta$

$= \int u^7 \cdot du$

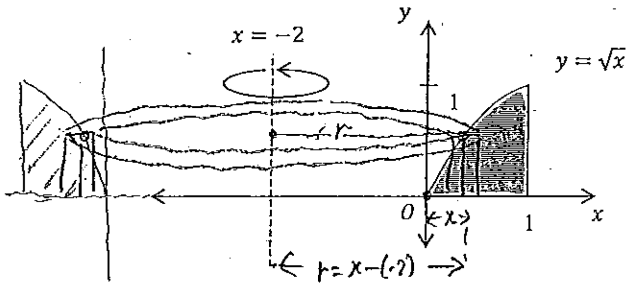
$= \frac{1}{8}u^8 + c$

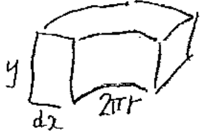
$= \frac{1}{8}\sin^8\theta + c$

D

9 The region bounded by the curve $y = \sqrt{x}$, the $x-$axis and the line $x = 1$ is rotated about the line $x = -2$ to form a solid.

typical shell.





$V = 2\pi r y \, \delta x$
$= 2\pi(x+2)\sqrt{x}\, \delta x$

$\therefore V = 2\pi \int_0^1 (x+2)\sqrt{x}\, dx$

Which integral represents the volume of the solid ?

(A) $2\pi \int_0^1 (x+2)\sqrt{x} \; dx$

(B) $2\pi \int_0^1 (x-2)\sqrt{x} \; dx$

(C) $2\pi \int_0^1 (x+2)^2 x \; dx$

(D) $2\pi \int_0^1 (x+2)\, x \; dx$

A

10 A particle is moving in simple harmonic motion with velocity v cm/s.

If $v^2 = 48 + 16x - 4x^2$, where x is the displacement in centimetres, what is the amplitude of the motion ?

(A) 2 cm

(B) 4 cm

(C) 1 cm

(D) 8 cm

Particle is stationary when $v=0$

or $v^2 = 0$

i.e. $-4(x^2-4x-12)=0$

$(x+2)(x-6)=0$

$x=-2, \; x=6$

Particle oscillates between $x=-2$ cm and $x=6$ cm

$\therefore$ Amplitude $= \frac{6-(-2)}{2}$

$= 4$ cm

B

QU. 11 ✓ = 1 mark **EXT 2 TRIAL 2013 - SOLUTIONS**

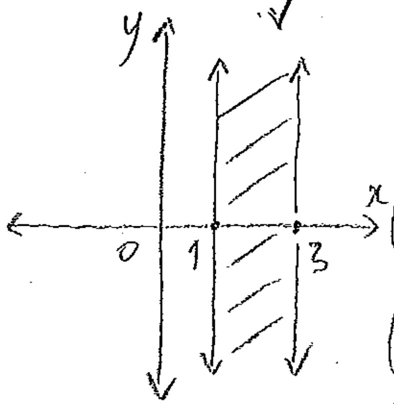
(a) $z = 1+2i \quad w = 3-i$

(i) $2z - w = 2(1+2i) - (3-i)$
$= 2 + 4i - 3 + i$
$= -1 + 5i$ ✓

(ii) $z\bar{w} = (1+2i)(3+i)$
$= 3 + i + 6i + 2i^2$
$= 1 + 7i$ ✓

(b) $z + \bar{z} = (x+yi) + (x-yi)$
$= 2x$

✓ $2 \le z + \bar{z} \le 6$
$2 \le 2x \le 6$
$1 \le x \le 3$



(c) $\displaystyle\int \frac{1}{x^2+2x+5}\,dx$

$= \displaystyle\int \frac{1}{(x+1)^2+2^2}\,dx$ ✓

$= \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$ ✓

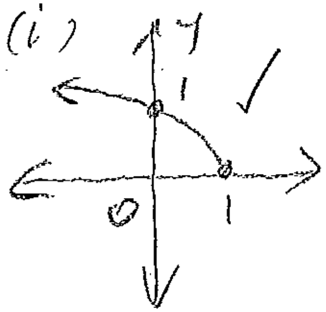
(d) (i) $z = \sqrt{3} + i \qquad r = 2$ ✓ $\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$\therefore \; z = 2\,\text{cis}\left(\frac{\pi}{6}\right)$ ✓

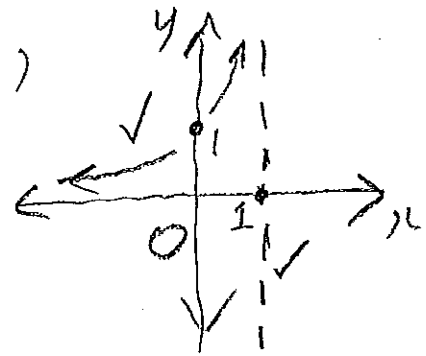
(ii) $z^5 = 2^5\,\text{cis}\left(\frac{5\pi}{6}\right)$ (De Moivre)
$= 32\left[-\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right]$
$= -16\sqrt{3} + 16i$ ✓

(e) $\displaystyle\int_0^1 \frac{x}{1+x^2}\,dx = \left[\frac{1}{2}\log_e(1+x^2)\right]_0^1 = \frac{1}{2}\log_e(2)$ ✓ ✓ ✓

(f) (i)



(ii)



Qu. 12 $\checkmark = 1$ mark EXT 2 TRIAL 2013 - SOLUTIONS

a) (i) $\dfrac{1}{x(x+1)} \equiv \dfrac{a}{x} + \dfrac{b}{x+1}$

$\therefore 1 \equiv a(x+1) + bx$

let $x = -1$

$1 = 0 + -b$

$b = -1$ $\checkmark$

Let $x = 0$

$1 = a(1) + 0$

$a = 1$ $\checkmark$

$\displaystyle\int \frac{1}{x(x+1)}\,dx = \int \frac{1}{x} - \frac{1}{x+1}\,dx$

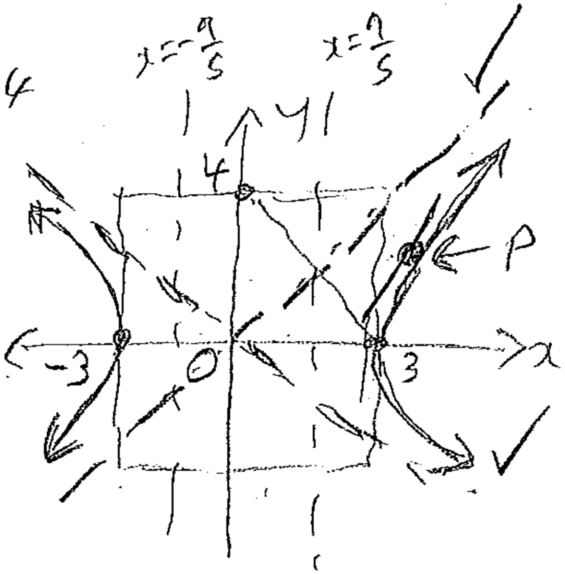
$= \log_e|x| - \log_e|x+1| + c$

$= \log_e\left|\dfrac{x}{x+1}\right| + c$ $\checkmark$

(b) $16x^2 - 9y^2 = 144$

$\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

$\therefore a = 3,\ b = 4$



(i) Foci: $(\pm 5, 0)$ $\checkmark$

$c^2 = a^2 + b^2$

$a^2e^2 = 9 + 16$

$ae = 5$

$e = \frac{5}{3}$

(ii) $x = \pm\frac{9}{5}$ $\checkmark$

(iii) $\rightarrow$

(iv) $16x^2 - 9y^2 = 144$

$32x - 18y \cdot y' = 0$ $\checkmark$

$y' = \dfrac{32x}{18y}$

at P: $\dfrac{dy}{dx} = \dfrac{32}{18} \cdot \dfrac{3}{4} \dfrac{\sec\theta}{\tan\theta}$

$= \dfrac{4}{3\sin\theta}$ $\checkmark$

(v) $y - y_1 = m(x - x_1)$ $\checkmark$

$y - 4\tan\theta = \dfrac{4}{3\sin\theta}(x - 3\sec\theta)$

$(3\sin\theta)y - 12\dfrac{\sin\theta}{\cos\theta}\sin\theta = 4x - \dfrac{12}{\cos\theta}$

$4x = (3\sin\theta)y - \dfrac{12\sin^2\theta}{\cos\theta} + \dfrac{12}{\cos\theta}$

$= (3\sin\theta)y - \dfrac{12}{\cos\theta}(\sin^2\theta - 1)$

$4x = (3\sin\theta)y + 12\cos\theta$ $\checkmark$

QU 12 $\quad \checkmark = 1$ mark

EXT 2 TRIAL 2013 - SOLUTIONS

(c) $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$

$$I_n = \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cdot \cos x \, dx$$

$y = uv$
$\frac{dy}{dx} = u'v + uv'$
$y = \int u'v + \int uv'$
$\int uv' = uv - \int u'v$

$u = \cos^{n-1} x \quad v' = \cos x$
$v = \sin x$
$u' = (n-1)\cos^{n-2} x \cdot (-\sin x)$

$\checkmark$ working (by parts)

$$= \left[\sin x \cdot \cos^{n-1} x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^2 x \cdot \cos^{n-2} x \, dx$$

$$= [0] + (n-1)\int_0^{\frac{\pi}{2}} (1-\cos^2 x)\cdot\cos^{n-2} x \, dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x - \cos^n x \, dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \, dx - (n-1)\int_0^{\frac{\pi}{2}} \cos^n x \, dx$$

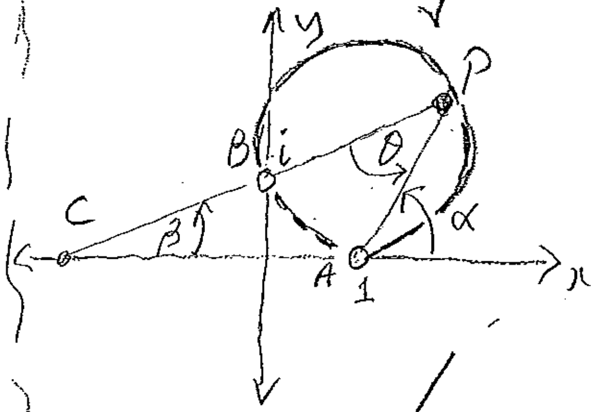
$$I_n = (n-1) I_{n-2} - (n-1) I_n$$
$$= nI_{n-2} - I_{n-2} - nI_n + I_n \quad \checkmark$$
$$nI_n = nI_{n-2} - I_{n-2}$$
$$I_n = \frac{(n-1)}{n} \cdot I_{n-2}$$

(d) $\arg\left(\frac{z-1}{z-i}\right) = \frac{\pi}{4}$

Geometrically

$\arg(z-1) - \arg(z-i) = \frac{\pi}{4}$

or $\alpha - \beta = \frac{\pi}{4}$ $\checkmark$



$\checkmark$

The locus is the arc AB passing through P. Minor arc AB is not part of the locus.

Algebraically

$$\frac{z-1}{z-i} = \frac{x+yi-1}{x+yi-i}$$

$$= \frac{(x-1)+yi}{x+(y-1)i} \times \frac{x-(y-1)i}{x-(y-1)i}$$

$$= \frac{x^2+y^2-x-y+(x+y-1)i}{x^2+(y-1)^2} \quad \checkmark$$

$Z = X + Yi$, say

$\arg z = \theta$, $\tan\theta = \frac{y}{x}$ } recall

$\arg Z = \frac{\pi}{4}$

$\therefore 1 = \frac{Y}{X}$

or $Y = X$

$$\therefore \frac{x^2+y^2-x-y}{x^2+(y-1)^2} = \frac{(x+y-1)}{x^2+(y-1)^2}$$

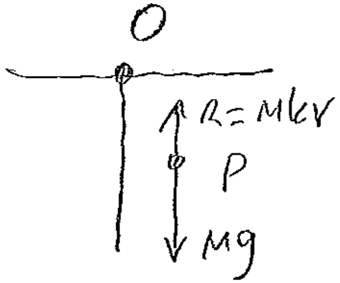
$$(x-1)^2 + (y-1)^2 = 1 \quad \checkmark$$

$\therefore$ Circle, centre $(1,1)$ radius 1 unit

exclude $(1,0)$ $(0,1)$ and minor arc

QU. 13 $\quad$ ✓ = 1 mark

EXT 2 TRIAL 2013 – SOLUTIONS



(a)

(i) $Ma = Mg - R$

$= Mg - Mkv$

$a = g - kv$

$\dfrac{dv}{dt} = g - kv$

$\dfrac{dt}{dv} = \dfrac{1}{g-kv}$ ✓

$t = \int \dfrac{1}{g-kv}\,dv$

$= -\dfrac{1}{k}\log_e(g-kv) + c$ ✓

$t=0,\ v=0$

$0 = -\dfrac{1}{k}\log_e(g) + c$

$c = \dfrac{1}{k}\log_e(g)$

$\therefore t = \dfrac{1}{k}\log_e\left(\dfrac{g}{g-kv}\right)$ ✓

(ii) $kt = \log_e\left(\dfrac{g}{g-kv}\right)$

$e^{kt} = \dfrac{g}{g-kv}$ ✓

$\dfrac{g-kv}{g} = e^{-kt}$ ✓

$kv = g(1-e^{-kt})$

$v = \dfrac{g}{k}(1-e^{-kt})$ ✓

(iii) $\dfrac{dx}{dt} = \dfrac{g}{k}(1-e^{-kt})$

$x = \dfrac{g}{k}\int 1-e^{-kt}\,dt$ ✓

$= \dfrac{g}{k}\left[t + \dfrac{1}{k}e^{-kt}\right] + c$

$x=0,\ t=0,\quad \therefore c = -\dfrac{g}{k^2}$ ✓

$\therefore x = \dfrac{g}{k^2}\left[kt + e^{-kt} - 1\right]$ ✓

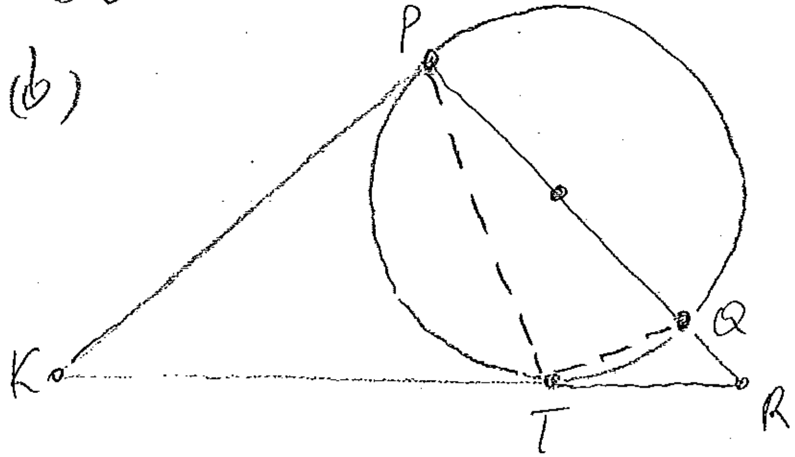
(iv) $\lim\limits_{t\to\infty}(v) = \dfrac{g}{k}(1-0)$

$= \dfrac{g}{k}$ ✓

QU.13 ✓ = 1 mark

EXT 2 TRIAL 2013 – SOLUTIONS

(b)



(i) $\angle PTQ = 90^\circ$ ($\angle$ in semi-circle) (1)

✓ $\angle KTP = 180^\circ - \angle PTQ - \angle QTR$ (2) (straight $\angle$)

$= 180^\circ - 90^\circ - \angle QTR$ (using (1))

$= 90^\circ - \angle QTR$ (3)

✓ $\angle TPK = \angle KTP$ ($\because KP = KT$ (4) (tangents from external pt; base $\angle$'s of isosceles ΔKTP)

$= 90^\circ - \angle QTR$

(ii) $\angle QTR \overset{✓}{=} 90^\circ - \angle KTP$ (using (3))

$= 90^\circ - [180^\circ - (\angle PKT + \angle TPK)]$ ($\angle$ sum ΔKPT)

✓ $= 90^\circ - [180^\circ - [\angle PKT + (90^\circ - \angle QTR)]]$ (using (4))

$= 90^\circ - [180^\circ - \angle PKT - 90^\circ + \angle QTR]$

$\angle QTR = \angle PKT - \angle QTR$

$\therefore 2\angle QTR = \angle PKT$ ✓

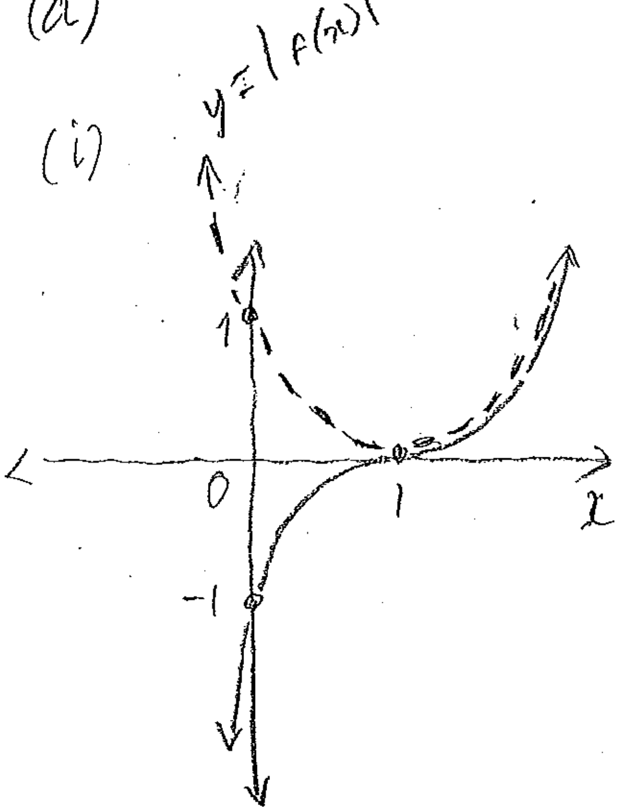
$\angle QTR = \frac{1}{2}\angle PKT$

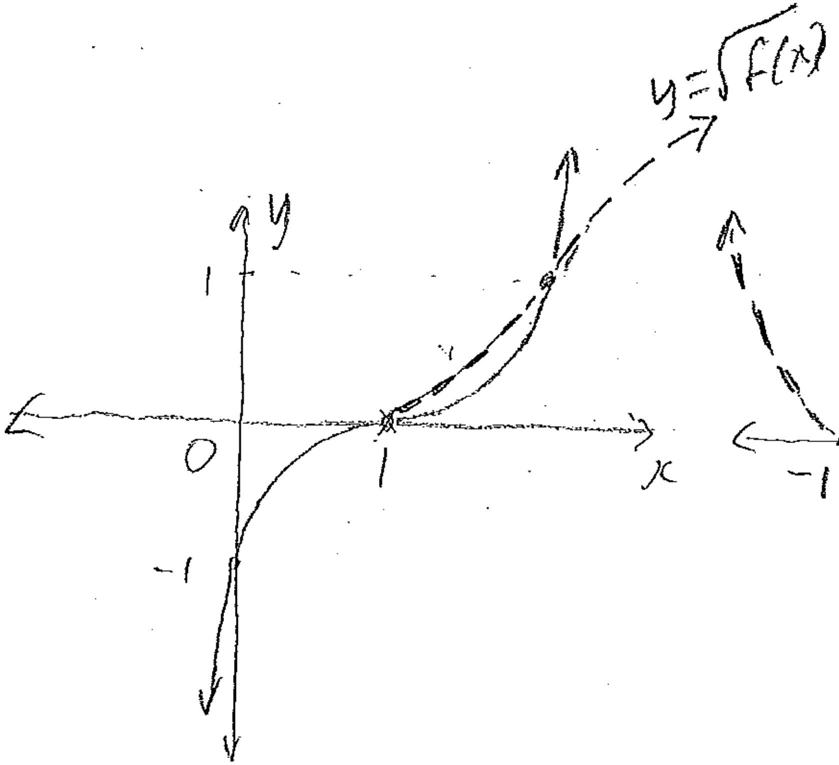
Qu. 14 $\quad \checkmark = 1$ mark $\quad$ EXT 2 TRIAL 2013 - SOLUTIONS

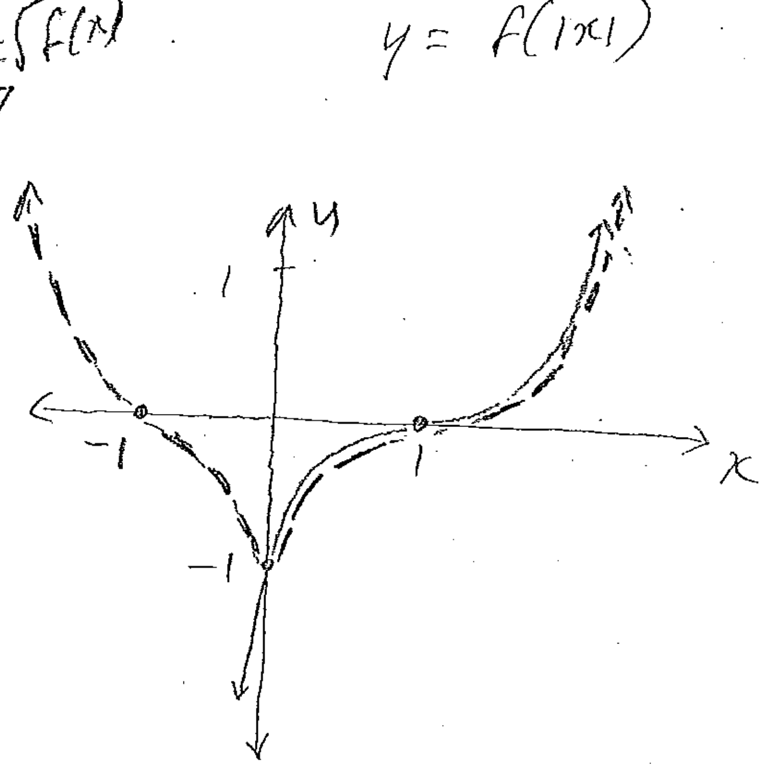
$\boxed{y = f(x)}$

(a)

(i)







✓ for intercepts at $x = 0$, $x = 1$

✓ for shape

✓ for $\sqrt{f(x)} > f(x)$ when $0 < f(x) < 1$

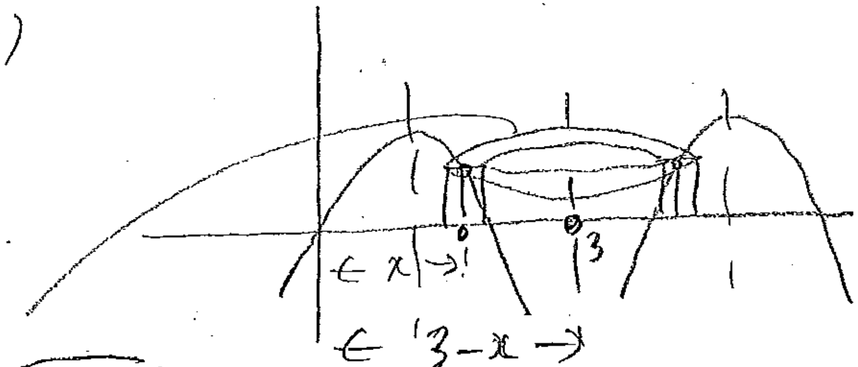
✓ for intercept at $x = 1$ and intersection at approx. $(1, 1)$

✓ for intercepts

✓ for 'reflection'

QV. 14 $\checkmark = 1$ mark EXT 2 TRIAL 2013 - SOLUTIONS

(b)



$V = 2\pi r h \, dx$ ✓

$= 2\pi(3-x)(2x-x^2)\,dx$

$V = 2\pi\int_0^2 (3-x)(2x-x^2)\,dx$

$= 2\pi\int_0^2 x^3 - 5x^2 + 6x \; dx$ ✓

$= 2\pi\left[\frac{1}{4}x^4 - \frac{5}{3}x^3 + 3x^2\right]_0^2$ ✓

$= 2\pi\left[\left(\frac{8}{3}\right)\right]$

$= \frac{16\pi}{3}$ units^3 ✓

(c) $\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}$ $(x>0,\ y>0)$

Need to show either as above or

rearranging... $\frac{1}{x} + \frac{1}{y} - \frac{4}{x+y} \geq 0$

$LHS = \frac{1}{x} + \frac{1}{y} - \frac{4}{x+y}$

$= \frac{y+x}{xy} - \frac{4}{(x+y)}$ ✓

$= \frac{(x+y)^2 - 4xy}{xy(x+y)}$ ✓

$= \frac{x^2+y^2-2xy}{xy(x+y)}$ ✓

$= \frac{(x-y)^2}{xy(x+y)}$ ✓

≥ 0

Note $LHS = 0$ if $x = y$

$x>0,\ y>0$ (data)

$\therefore xy > 0$ (product of '+ves'

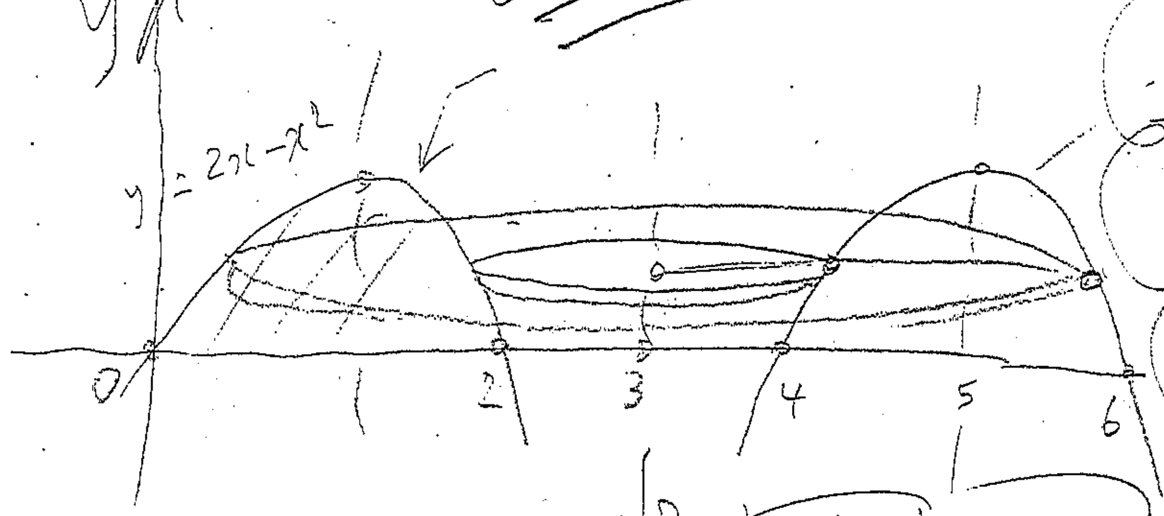
also $(x+y) > 0$ (sum of '+ves'

also $(x-y)^2 \geq 0$ ($\because X^2 \geq 0$

$\therefore \frac{(x-y)^2}{xy(x+y)} \geq 0$ (Product, quotient of '+ves')

Qu. 5 (b)

OR



$y = -(x-4)(x-6)$

$-y = x^2 - 10x + 24$

$x^2 - 10x = -y - 24$

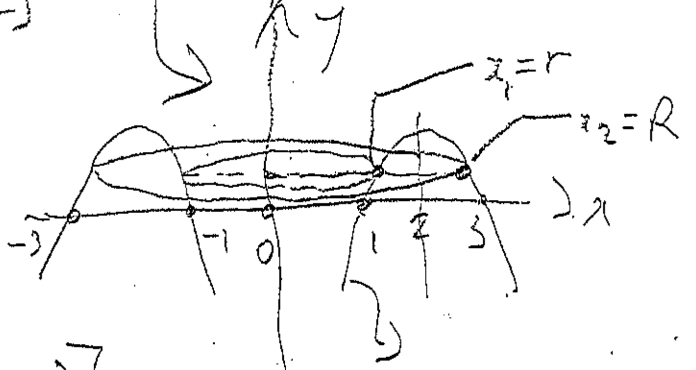
$(x-5)^2 = -y + 1$

$(x-5) = \pm\sqrt{1-y}$

$x = 5 \pm \sqrt{1-y}$

OR

translate to Origin



$y = -(x-1)(x-3)$

$y = -x^2 + 4x - 3$

$x^2 - 4x = -y - 3$

$(x-2)^2 = -y - 3 + 4$

$(x-2)^2 = -y + 1$

$x - 2 = \pm\sqrt{1-y}$

$x = 2 \pm \sqrt{1-y}$

$\therefore\ x_1 = r = 2 - \sqrt{1-y}$

$x_2 = R = 2 + \sqrt{1-y}$

Using washers:

$V = \pi R^2 - \pi r^2$

$= \pi\left(2+\sqrt{1-y}\right)^2 - \pi\left(2-\sqrt{1-y}\right)^2$

$= \pi\left[\left(\cancel{4} + 4\sqrt{1-y} + \cancel{1-y}\right) - \left(\cancel{4} - 4\sqrt{1-y} + \cancel{1-y}\right)\right]$

$= \pi\left[\,8\sqrt{1-y}\,\right]$

$= 8\pi\sqrt{1-y}$

$V = 8\pi\int_0^1 \sqrt{1-y}$

$= 8\pi\left[-\frac{2}{3}(1-y)^{\frac{3}{2}}\right]_0^1$

$= 8\pi\left[-(0) + \left(\frac{2}{3}\right)\right]$

$= \frac{16\pi}{3}$ units^3

QU.15 ✓ = 1 mark

EXT 2 TRIAL 2013 – SOLUTIONS

(a) If α, β, γ are zeros of
$P(x) = x^3 + 5x^2 + 11$

(i) then $\alpha^3 + 5\alpha^2 + 11 = 0$ (1)

or $\beta^3 + 5\beta^2 + 11 = 0$ (2)

or $\gamma^3 + 5\gamma^2 + 11 = 0$ (3) ✓

Let $\alpha^2 = M$ $\therefore$ $\alpha = \pm\sqrt{M}$

hence $(\pm\sqrt{M})^3 + 5 \cdot M + 11 = 0$

$(\pm\sqrt{M})^3 = -(5M + 11)$ ✓

$M^3 = 25M^2 + 110M + 121$ ✓

$M^3 - 25M^2 - 110M - 121 = 0$

or $x^3 - 25x^2 - 110x - 121 = 0$ ✓

(ii) (1) + (2) + (3) :

$\alpha^3 + \beta^3 + \gamma^3 + 5(\alpha^2 + \beta^2 + \gamma^2) + 33 = 0$ ✓

$\alpha^3 + \beta^3 + \gamma^3 = -5(25) - 33$

$\alpha^3 + \beta^3 + \gamma^3 = -158$ ✓

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$
$= 25 - 2(0)$
$= 25$

$\alpha + \beta + \gamma = \frac{-b}{a}$
$= -5$

$\Sigma \alpha\beta = \frac{c}{a}$
$= 0$

QU.15   ✓ = 1 mark

EXT 2 TRIAL 2013 – SOLUTIONS

(b)

(i) Let $P(x)=(x-\alpha)^n Q(x)$

then $P'(x)=n(x-\alpha)^{n-1}.Q(x)+Q'(x).(x-\alpha)^n$ ✓

$\therefore P'(\alpha)=n(\alpha-\alpha)^{n-1}Q(\alpha)+Q'(\alpha)(\alpha-\alpha)^n$ ✓

$=0+0$

$=0$

(ii) Let $P(x)=18x^3+3x^2-28x+12$

since $x=\alpha$, say, is a double root

then $P'(\alpha)=0$ (proved above)

$P'(x)=54x^2+6x-28$

$54x^2+6x-28=0$ ✓

$27x^2+3x-14=0$

$9x \quad 7$

$3x \quad -2$

$(9x+7)(3x-2)=0$

By inspection.. $x=-\frac{7}{9}$, $x=\frac{2}{3}$ ...

Only one can be a root of original equation (why?)

[Recall if $x=\alpha$ is a root of $P(x)$ of multiplicity 'n' then $x=\alpha$ is a root of $P'(x)$ of multiplicity "$n-1$".

Testing... $P(\frac{2}{3})=0$ ✓

$\therefore P(x)=(x-\frac{2}{3})^2Q(x)$

$=(3x-2)^2Q(x)$

$=(3x-2)^2(ax+c)$

$(x-\frac{2}{3})^2Q(x)=$

$(3x-2)^2Q(x)=$

by inspection $a=2$ $c=3$

$\therefore$ roots are $\frac{2}{3}, \frac{2}{3}, -\frac{3}{2}$ ✓

OR use "roots" $2\alpha+\beta=-\frac{1}{6}$ $\quad 2\alpha\beta+\alpha^2=-\frac{14}{9}$

$\beta=-\frac{1}{6}-2\alpha \rightarrow 2\alpha(-\frac{1}{6}-2\alpha)+\alpha^2=-\frac{14}{9}$

$\rightarrow 27\alpha^2+3\alpha-14=0$

$\alpha=\frac{2}{3}$ $\quad \alpha=-\frac{7}{9}$ and so on

EXT 2 TRIAL 2013 - SOLUTIONS

QU. 15

(c)

(i) $f(x) = ax^3 + bx^2 + cx + d$

$f(1) = 1 = f(2) = f(3)$; $f(4) = k$

$\therefore$ we have these equations:

$$a + b + c + d = 1 \qquad (1)$$

$$8a + 4b + 2c + d = 1 \qquad (2)$$

$$27a + 9b + 3c + d = 1 \qquad (3)$$

$$64a + 16b + 4c + d = k \qquad (4)$$

(2) − (1): (eliminate 'd')

$$7a + 3b + c = 0 \qquad (5)$$

(3) − (2): (eliminate 'd')

$$19a + 5b + c = 0 \qquad (6)$$

(4) − (3): (eliminate 'd')

$$37a + 7b + c = k - 1 \qquad (7)$$

(7) − (6): (eliminate 'c')

$$18a + 2b = k - 1 \qquad (8)$$

(6) − (5): (eliminate 'c')

$$12a + 2b = 0 \qquad (9)$$

(8) − (9): (eliminate 'b')

$$6a + 0 = k - 1$$

$$\therefore a = \frac{1}{6}(k-1) \qquad (10)$$

Sub (10) into (9):

$$2b = -2(k-1)$$

$$b = 1 - k \qquad (11)$$

Sub (10), (11) into (5):

$$c = -7a - 3b$$

$$= -\frac{7}{6}(k-1) - 3(1-k)$$

$$= \frac{-7(k-1) - 18(1-k)}{6}$$

$$= \frac{11k - 11}{6}$$

$$c = \frac{11}{6}(k-1) \qquad (12)$$

QU.15 $\checkmark = 1$ mark EXT 2 TRIAL 2013 - SOLUTION

(c)

Sub (10), (11), (12) into (1):

$$d = -a - b - c + 1$$

$$= -\frac{1}{6}(k-1) - \frac{6(1-k)}{6} - \frac{11(k-1)}{6} + \frac{6}{6}$$

$$= \frac{1-k+6k-6-11k+11+6}{6}$$

$$= \frac{12-6k}{6}$$

$$d = 2-k \qquad (13)$$

$\checkmark\checkmark$ for working

$\checkmark$ for expressions a, b, c, d

(ii) $\therefore f(n) = \frac{1}{6}(k-1)n^3 + (1-k)n^2 + \frac{11}{6}(k-1)n + (2-k)$ $\checkmark$

(iii) We test in turn $f(1), f(2), f(3), f(4)$

$$f(1) = \frac{1}{6}(k-1) + (1-k) + \frac{11}{6}(k-1) + 2-k$$

$$= \frac{(k-1)+6(1-k)+11(k-1)+6(2-k)}{6}$$

$$= \frac{\cancel{k}-\cancel{1}+6-6k+11\cancel{k}-\cancel{11}+\cancel{12}-6k}{6}$$

$$= 1$$

$$f(2) = \frac{8(k-1)+24(1-k)+22(k-1)+6(2-k)}{6}$$

$$= 1$$

$$f(3) = \frac{27(k-1)+54(1-k)+33(k-1)+6(2-k)}{6}$$

$$= 1$$

$$f(4) = \frac{64(k-1)+96(1-k)+44(k-1)+6(2-k)}{6}$$

$$= k \quad \checkmark$$

Since $f(4) = k$, where 'k' is any constant, we accept that the 'next number' is any number, 'k', including '1'.

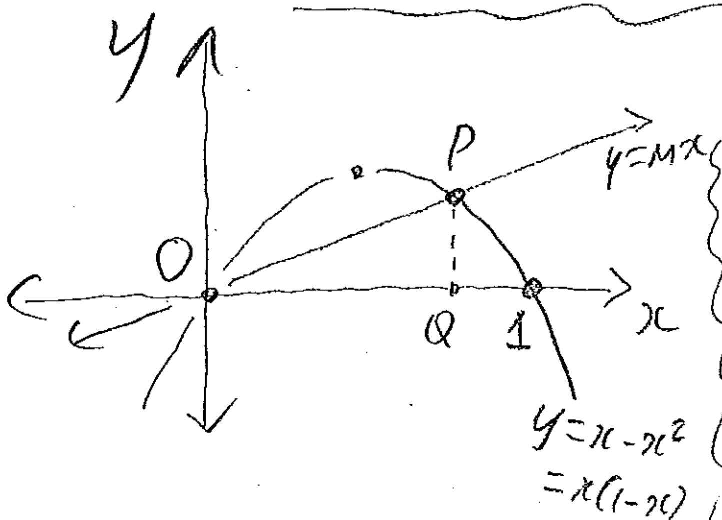
$\therefore$ Helen is correct (Peter is also correct!)

$\checkmark = 1$ mark

EXT 2 TRIAL 2013 - SOLUTION

QU.16

(a)



(i) Intersection point

$P(x, mx)$ and $y = x - x^2$

$\therefore \; mx = x - x^2$ $\checkmark$

$x^2 + (m-1)x = 0$ $\checkmark$

$x[x + (m-1)] = 0$ $\checkmark$

$\therefore \; x = 0$ or $x = 1-m$

$\therefore \; y = 0$ $\quad$ $y = m(1-m)$

$\therefore \; P((1-m), m(1-m))$ $\checkmark$

(ii) We require, $A_1 = A_2$, where..

$$A_1 = \int_0^{1-m} [(x - x^2) - mx] \, dx$$

$$A_2 = \frac{1}{2} OQ \cdot QP + \int_{1-m}^{1} x - x^2 \, dx$$

$$A_1 = \int_0^{1-m} (1-m)x - x^2 \, dx \quad \checkmark$$

$$= \left[\frac{(1-m)}{2} x^2 - \frac{1}{3} x^3 \right]_0^{1-m}$$

$$= \left[\frac{1}{6} (1-m)^3 \right] \text{ units}^2$$

$\checkmark$ working

$$A_2 = \frac{1}{2} \times (1-m) \times m(1-m) + \int_{1-m}^{1} x - x^2 \, dx \quad \checkmark$$

$$= \frac{1}{2} (1-m)^2 + \left(\frac{1}{2} - \frac{1}{3} \right) - \left(\frac{1}{2} (1-m)^2 - \frac{1}{3} (1-m)^3 \right)$$

$$= -\frac{1}{6} \left[(1-m)^3 - 1 \right]$$

$A_1 = A_2 \quad \therefore \; \frac{1}{6} (1-m)^3 = -\frac{1}{6} (1-m)^3 + \frac{1}{6}$

$$2(1-m)^3 = 1$$

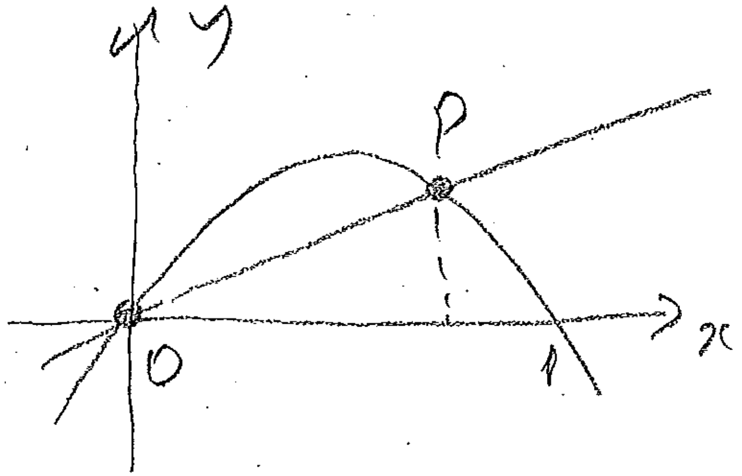
$$1 - m = \frac{1}{\sqrt[3]{2}}$$

$$m = 1 - \frac{1}{\sqrt[3]{2}} \quad \checkmark$$

QU.16 ✓ = 1 mark EXT 2 TRIAL 2013 - SOLUTION

(a)

(ii) ALTERNATIVE METHOD



'Half the area'

$= \frac{1}{2}\int_0^1 x - x^2 \, dx$

$= \frac{1}{2}\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1$

$= \frac{1}{2}\left[\left(\frac{1}{2}-\frac{1}{3}\right)-(0)\right]$

$= \frac{1}{12}$ units^2

$\therefore \int_0^{1-M} x - x^2 - Mx \, dx = \frac{1}{12}$

$\left[\frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{M}{2}x^2\right]_0^{1-M} = \frac{1}{12}$

$\left[\frac{3x^2 - 2x^3 - 3Mx^2}{6}\right]_0^{1-M} = \frac{1}{12}$

$\left[\frac{3x^2(1-M) - 2x^3}{6}\right]_0^{1-M} = \frac{1}{12}$

$\frac{1}{6}\left(\left[3(1-M)^2(1-M) - 2(1-M)^3\right] - 0\right) = \frac{1}{12}$

$3(1-M)^3 - 2(1-M)^3 = \frac{1}{2}$

$(1-M)^3 = \frac{1}{2}$

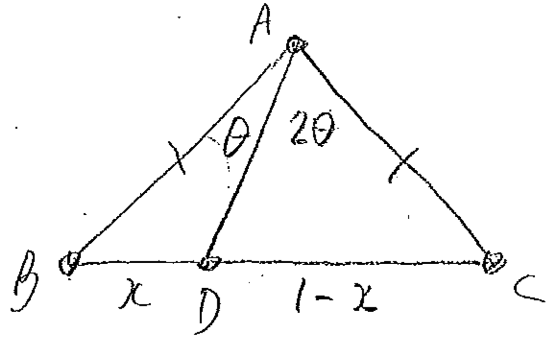
$1 - M = \frac{1}{\sqrt[3]{2}}$

$M = 1 - \frac{1}{\sqrt[3]{2}}$

QU.16 $\checkmark = 1$ mark

EXT 2 TRIAL 2013 - SOLUTIONS

(b)



$AB = AC$ (data)

$= p$, say

$\angle ABD = \angle ACD$ (base $\angle$'s of isosceles ΔABC)

$= \alpha$, say

$2\alpha + \theta + 2\theta = 180^\circ$ ($\angle$ sum ΔABC)

$2\alpha = 180^\circ - 3\theta$

$\alpha = \frac{1}{2}(180^\circ - 3\theta)$

$\therefore \angle ADB = 180^\circ - (\alpha + \theta)$ ($\angle$ sum ΔADB)

$= 180^\circ - \alpha - \theta$

$= 180^\circ - \frac{1}{2}(180^\circ - 3\theta) - \theta$

$= 180^\circ - 90^\circ + \frac{3}{2}\theta - \theta$

$= 90^\circ + \frac{1}{2}\theta$

$= [90^\circ - (\frac{1}{2}\theta)]$

Also.. $\angle ADC = \alpha + \theta$ (exterior $\angle$ of ΔADB)

$\checkmark$ working

From ΔADB:

$$\frac{x}{\sin\theta} = \frac{p}{\sin(90^\circ - \frac{1}{2}\theta)} \checkmark$$

i.e. $$\boxed{\frac{x}{\sin\theta} = \frac{p}{\cos(\frac{1}{2}\theta)} \quad (1)}$$

$$\therefore \boxed{p = \frac{x\cos(\frac{1}{2}\theta)}{\sin\theta} \quad (2)}$$

From ΔADC:

$$\frac{1-x}{\sin 2\theta} = \frac{p}{\sin(\alpha+\theta)} \checkmark$$

$\alpha + \theta$

$= 90^\circ - \frac{3}{2}\theta + \theta$

$= 90^\circ - \frac{1}{2}\theta$

i.e. $$\frac{1-x}{\sin 2\theta} = \frac{p}{\sin(90^\circ - \frac{1}{2}\theta)}$$

$$\boxed{\frac{1-x}{\sin 2\theta} = \frac{p}{\cos(\frac{1}{2}\theta)} \quad (3)}$$

$$\therefore p = \frac{(1-x)\cos(\frac{1}{2}\theta)}{\sin 2\theta} = \frac{(1-x)\cos(\frac{1}{2}\theta)}{2\sin\theta\cos\theta} \quad (4)$$

Equate (2), (4):

$$\therefore \frac{(1-x)\cancel{\cos(\frac{1}{2}\theta)}}{2\cancel{\sin\theta}\cos\theta} = \frac{x\cancel{\cos(\frac{1}{2}\theta)}}{\cancel{\sin\theta}} \checkmark$$

$$x = \frac{1}{2}\frac{(1-x)}{\cos\theta}$$

$2x\cdot\cos\theta = 1 - x$

$$\therefore \cos\theta = \frac{1-x}{2x}$$

QU.16 ✓ = 1 mark EXT2 TRIAL - SOLUTIONS

(b)

(ii)

$0^\circ < 3\theta < 180^\circ$ ✓

$0^\circ < \theta < 60^\circ$

$\therefore \quad \frac{1}{2} < \cos\theta < 1$ ✓

$\frac{1}{2} < \frac{1-x}{2x} < 1$

$\frac{1-x}{2x} > \frac{1}{2}$

$2(1-x) > 2x \quad (x>0)$

$2-2x > 2x$

$4x < 2$

$x < \frac{1}{2}$ ✓

$\frac{1-x}{2x} < 1$

$1-x < 2x \quad (x>0)$

$3x > 1$

$x > \frac{1}{3}$ ✓

$\therefore \quad \frac{1}{3} < x < \frac{1}{2}$