Student Number: ____________________

St. Catherine's School
Waverley

2013

ASSESSMENT TASK 2

(30%)

Mathematics

Extension 2

- **Working time – 2 hours**
- **Reading time – 5 minutes**
- Attempt multiple choice questions (**Multiple choice nswer sheet attached**) and questions 8 to 11.on booklets provided.

General Instructions

- Start each question on a **new answer booklet.**
- Write using black or blue pen only.
- Board-approved calculators may be used.
- All necessary **working** must be shown.
- Marks may be deducted for careless or badly arranged work.

TEACHER'S USE ONLY

Multiple choice	/7
Question 8	/15
Question 9	/15
Question 10	/15
Question 11	/15
Total	/67
Total	/67

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Multiple Choice Questions (7 marks) 1 mark each

- Answer the following 7 multiple choice questions on the multiple choice answer sheet provided.

1. Let $z = 3 - i$. What is the value of $i\bar{z}$?

(A) $-1-3i$
(B) $-1+3i$
(C) $1-3i$
(D) $1+3i$

2. The points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the chord PQ subtends a right angle at $(0,0)$. Which of the following is the correct expression?

(A) $\tan\theta\tan\phi = -\frac{b^2}{a^2}$

(B) $\tan\theta\tan\phi = -\frac{a^2}{b^2}$

(C) $\tan\theta\tan\phi = \frac{b^2}{a^2}$

(D) $\tan\theta\tan\phi = \frac{a^2}{b^2}$

3. Consider the hyperbola with the equation $\frac{x^2}{144}-\frac{y^2}{25}=1$.

What are the equations of the directrices?

(A) $x=\pm\frac{13}{144}$

(B) $x=\pm\frac{13}{25}$

(C) $x=\pm\frac{25}{13}$

(D) $x=\pm\frac{144}{13}$

4. What is $-2+2\sqrt{3}i$ expressed in modulus-argument form?

(A) $2(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})$

(B) $4(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})$

(C) $2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$

(D) $4(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$

5. It is given that $3+i$ is a root of $P(z)=z^3+az^2+bz+10$ where a and b are real numbers. Which expression factorises $P(z)$ over the real numbers?

(A) $(z-1)(z^2+6z-10)$
(B) $(z-1)(z^2-6z-10)$
(C) $(z+1)(z^2+6z+10)$
(D) $(z+1)(z^2-6z+10)$

6 $|z-2i| = |z+2|$ is the locus of a point z. The cartesisn equation of this locus is.

A. $y = x$

B. $y = -x$

C. $y = 2x$

D. $y = -2x$

7 If $z = \cos\theta + i\sin\theta$, then $z + \frac{1}{z}$ is

A. $\frac{2}{\sin\theta}$

B. $2\cos\theta$

C. $2i\sin\theta$

D. $\frac{2i}{\sin\theta}$

Question 8

a) (i) Sketch clearly the locus of z: 1m

$$|z-(\sqrt{3}+i)| = 1$$

(ii) Show that $1 \le |z| \le 3$, for points on this locus. 2m

(ii)) Show that $0 \le \arg(z) \le \frac{\pi}{3}$, for points on this locus 2m

b) If $z_1 = 1+i$ and $z_2 = \sqrt{3}+i$, find

(i) $\frac{z_1}{z_2}$ in modulus-argument form 2m

(ii) $\frac{z_1}{z_2}$ in the form $a+ib$ 1m

(iii) Hence or otherwise state the exact value of $\cos\frac{\pi}{12}$ 1m

c The polynomial $P(x) = x^4 - 2x^3 + 8x^2 - 8x + 16$ has $1 + i\sqrt{3}$ as a zero. 3m

Express P(x) as the product of two quadratic factors.

d) (i) Express $z = 1 + \sqrt{3}\,i$ in modulus-argument form 1m

(ii) Hence or otherwise show that $z^7 - 64z = 0$ 2m

Question 9

a) A polynomial $P(x)$ is divided by $x^2 - a^2, a \neq 0$ and the remainder is $px + q$.
Write P(x) in terms of its divisor$x^2 - a^2$, and remainder $px + q$.

(i) Show that 2m

$p = \frac{1}{2a}(P(a) - P(-a))$ and

$q = \frac{1}{2}(P(a) + P(-a))$

(ii) Show that if P(x) is an even polynomial, then this reminder is a constant. 1m

(iii) Find the remainder when x^{100} is divided by $x^2 - 1$ 2m

(iv) If $P(x) = x^n - a^n$, and n is odd, find the remainder, when divided by $x^2 - a^2$ 3m

b) (i) Show that if α is a root of multiplicity m to the polynomial equation $P(x) = 0$, then α is a root of multiplicity $m - 1$ to $P'(x) = 0$ 2m

(ii) Explain why $P(x) = 0$, *where* $P(x) = px^3 + qx^2 + r$ cannot have a root of multiplicity 3 2m

(iii) If $px^3 + qx^2 + r = 0$ has a root of multiplicity 2, show that $4q^3 + 27p^2r = 0$ 3m

Question 10

a) (i) For the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

(i) Find the eccentricity 1m

(ii) Find the coordinates of the foci 1m

(iii) Find the equations of the directrices 1m

(iv) Sketch the ellipse showing the above features 1m

(v) Mark the point $P(4\cos\theta, 3\sin\theta)$ and clearly mark the position of the angle θ. 1m

(vi) Show that the equation of the tangent to the ellipse at P is given by $3\cos\theta\, x + 4\sin\theta\, y = 12$ 2m

b) If $2\sin^{-1}x = \cos^{-1}x$, 3m

using $\cos 2\alpha = 1 - 2\sin^2\alpha$ or otherwise, find the value of x.

c) Consider $P(x) = x^4 - 2Ax^3 + B$, where A and B are constants and $A \neq 0$. 3m

The roots are α, β, γ *and* $\alpha + \beta + \gamma$.

Show that $B = A^4$.

d) Let z_1 and z_2 be two complex numbers, where $z_1 = -1 + i$ and $z_2 = 1 + i\sqrt{3}$. 2m

Sketch the locus of z, where $\arg(z - z_1) = \arg(z - z_2)$

Question 11

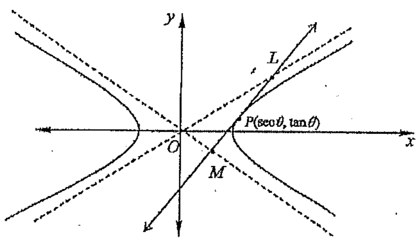
a) P $(2p, \frac{2}{p})$ and Q: $(2q, \frac{2}{q})$ are two distinct points on the rectangular hyperbola $xy = 4$, where $p \neq \pm q$

(i) Show that the equation of the tangent at P is $x + yp^2 = 4p$ 2m

(ii) Show that the tangents at P and Q intersect at M given by $(\frac{4pq}{p+q}, \frac{4}{p+q})$ 2m

(iii) Given that $pq = 1$, show that the locus of M is a straight line not including the origin. 3m

b) The diagram shows the hyperbola $x^2 - y^2 = 1$



(i) Show that the equation of the tangent to this hyperbola at the point P: $(\sec\theta, \tan\theta)$ is given by 2m

$$x \sec\theta - y \tan\theta = 1$$

(ii) The tangent cuts the asymptotes at L and M. 3m

Show that P is the midpoint of LM.

(iii) Show that the area of the triangle OLM is independent of P 3m

End of paper

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2013

HIGHER SCHOOL CERTIFICATE
Extension 2 MATHEMATICS
ASSESSMENT TASK 2 – 30%
Mid-Course Examination

Multiple Choice Answer Sheet

Colour in the correct oval completely

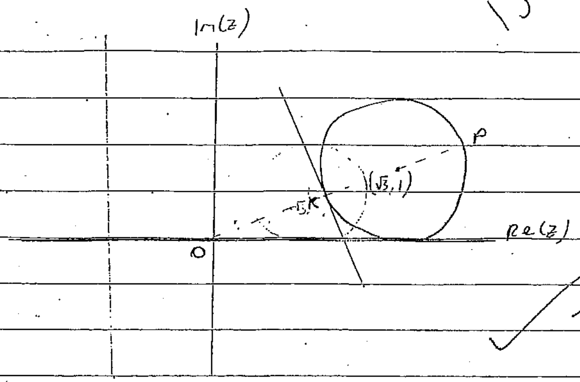
	A	B	C	D
1.			●	
2.		●		
3.				●
4.		●		
5.				●
6.		●		
7.		●		

7/7

8 page writing booklet

15/15

8 a) i



2/2

ii) $(x-\sqrt{3})^2+(y-1)^2=1$

$y=\frac{1}{\sqrt{3}}x$

$x=\sqrt{3}\,y$

$(\sqrt{3}y-\sqrt{3})^2+(y-1)^2=1$

$3(y^2-2y+1)+y^2-2y+1=1$

$4y^2-8y+3=0$

$y=\frac{8\pm\sqrt{64-4.4.3}}{8}$

$=\frac{8\pm4}{8}$

$=\frac{3}{2},\ \frac{1}{2}$

$\therefore P\left(\frac{3\sqrt{3}}{2},\frac{3}{2}\right)\quad R\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

pl. see solution.

$OP=\sqrt{\left(\frac{3\sqrt{3}}{2}\right)^2+\left(\frac{3}{2}\right)^2}$

$=\sqrt{9}$

$=3$

2/2

$OR=\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2}$

$=1$

$\therefore\ 1\le|z|\le3$

iii $\quad y = kx \qquad (x-\sqrt{3})^2 + (y-1)^2 = 1$

$$(x-\sqrt{3})^2 + (kx-1)^2 = 1$$

$$x^2 - 2\sqrt{3}x + 3 + k^2x^2 - 2kx = 0$$

$$(k^2+1)x^2 - 2(\sqrt{3}+k)x + 3 = 0$$

$$\Delta = 4(\sqrt{3}+k)^2 - 4(k^2+1)\times 3 = 0$$

$$3 + 2\sqrt{3}k + k^2 - 3k^2 - 3 = 0$$

$$-2k^2 + 2\sqrt{3}k = 0$$

$$\cancel{\sqrt{3}k = k^2}$$

$$k^2 - \sqrt{3}k = 0$$

$$k(k-\sqrt{3}) = 0$$

$$k = 0, \sqrt{3}$$

$\therefore$ tangents to the circle at $y = 0$ —①

$y = \sqrt{3}x$ —②

$\arg(1) = 0$

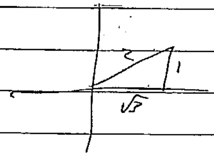
$\arg(2) = \tan^{-1}\sqrt{3}$

$= \frac{\pi}{3}$

$\therefore \quad 0 \le \arg(z) \le \frac{\pi}{3}$

(b) (i) $z_1 = 1+i$

$= \sqrt{2}\operatorname{cis}\frac{\pi}{4}$



$z_2 = \sqrt{3}+i$

$= 2\operatorname{cis}\frac{\pi}{6}$

$\frac{z_1}{z_2} = \frac{\sqrt{2}}{2}\operatorname{cis}\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$

$= \frac{\sqrt{2}}{2}\operatorname{cis}\frac{\pi}{12}$

(ii) $\frac{z_1}{z_2} = \frac{1+i}{\sqrt{3}+i} \times \frac{\sqrt{3}-i}{\sqrt{3}-i}$

$= \frac{\sqrt{3}+1}{4} + \frac{\sqrt{3}-1}{4}i$

8b) iii $\frac{\sqrt{2}}{2}\operatorname{cis}\frac{\pi}{12} = \frac{\sqrt{3}+1}{4} + \frac{\sqrt{3}-1}{4}i$

$\frac{\sqrt{2}}{2}\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{4}$

$\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

c) $P(x) = x^4 - 2x^3 + 8x^2 - 8x + 16$

$= (x^2-2x+4)(x^2+4)$

$(x-(1+i\sqrt{3}))(x-(1-i\sqrt{3}))$

$= x^2 - 2x + 1 + 3$

$= x^2 - 2x + 4$

d) i $z = 1+\sqrt{3}i$

$= 2\operatorname{cis}\frac{\pi}{3}$

ii $LHS = z^7 - 64z$

$= 2^7\operatorname{cis}\frac{7\pi}{3} - (64\times 2\operatorname{cis}\frac{\pi}{3})$

$= 128\operatorname{cis}\frac{\pi}{3} - 128\operatorname{cis}\frac{\pi}{3}$

$= 0$

$= RHS$

8 page writing booklet

$\frac{14}{15}$

9a) $P(x) = (x^2 - a^2)Q(x) + px + q$

i ~~$P(a) = (a^2-a^2)Q(a) + ap+q$~~
~~$= -ap+q$~~
~~$P(-a) = (a^2-a^2)Q(-a) - ap + q$~~
~~$= -ap+q$~~

~~$P(a) = (x^2-a^2)$~~
~~$P(a) = (a^2-a^2)Q(a) + pa+q$~~
~~$P(-a) = (a^2-a^2)Q(-a) = pa+q$~~
~~$pa+q = -pa+q$~~

$P(a) = (a^2-a^2)Q(a) + R(a)$
$= ap+q$
$P(-a) = -ap+q$
$P(a) - P(-a) = ap+q+ap-q$
$= 2ap$
$\therefore RHS = 2ap \times \frac{1}{2a}$
$= p$
$= LHS$

2/2

ii $RHS = \frac{1}{2}(P(a) + P(-a))$
$= \frac{1}{2}(ap+q - ap+q)$
$= \frac{2q}{2}$
$= q$
$= LHS$

DO NOT TEAR ANY PAGES FROM THIS BOOKLET

9a) ii if $P(x)$ is even
$P(x) = P(-x)$
$\therefore P(a) = P(-a)$
$p = \frac{1}{2a}(P(a) - P(-a))$
$= 0$
$\therefore$ remainder $= q$ = constant

1/1

iii $x^{100} = (x^2-1)Q(x) + px+q$
$P(x) = x^{100}$
$P(a) = a^{100}$ $\quad P(1) = 1^{100}$
$= 1$

~~$p = \frac{1}{2a}(a^{100}$~~
$q = \frac{1}{2}(a^{100} + a^{100})$
$q = a^{100}$
$a = 1$
$q = 1$
$\therefore R(x) = 1$

2/2

iv $P(x) = x^n - a^n$
$P(x) = (x^2-a^2)Q(x) + px+q$
~~$P(a)$~~ $P(a) = ap+q = 0$
$P(-a) = -ap+q = (-a)^n - a^n$
$= -a^n - a^n$
$-ap+q = -2a^n$
$\therefore R(x) = -2a^n$

2/3

9b) i) $P(x) = (x-\alpha)^m Q(x)$

$P'(x) = Q(x)m(x-\alpha)^{m-1} + (x-\alpha)^m Q'(x)$

~~$= (x-\alpha)[Q(x)m(x-\alpha)^{m-2} + Q'(x)(x-\alpha)^{m-1}]$~~

$\therefore$ ~~$P'(\alpha) = 0$~~

$= (x-\alpha)^{m-1}(Q(x)m + (x-\alpha)Q'(x))$

$\therefore (x-\alpha)^{m-1}$ is a root of $P'(x)$

2/2

ii) $P(x) = px^3 + qx^2 + r$

$P'(x) = 3px^2 + 2qx$

$P''(x) = 6px + 2q \neq 0$

~~$x = -\frac{2q}{6p} = -\frac{q}{3p}$~~

$6px + 2q \neq 0$ why?

$x \neq -\frac{q}{3p}$

~~$x = -\frac{2q}{6p} = -\frac{q}{3p}$~~

1/2

~~$px^3 + qx^2 + r$~~

~~$px^3 + qx^2 = -r$~~

~~$x^2(p+q) = -r$~~

$\therefore P(x) = 0$

~~$p\left(-\frac{q}{3p}\right)^3 + q\left(-\frac{q}{3p}\right)^2 + r = -\frac{q^3}{27p^2} + \frac{q^3}{9p}$~~

iii) $P(x) = px^3 + qx^2 + r = 0$

$P'(x) = 3px^2 + 2qx = 0$

$x(3px + 2q) = 0$

$x = -\frac{2q}{3p}$

$P\left(-\frac{2q}{3p}\right) = p\left(-\frac{2q}{3p}\right)^3 + q\left(-\frac{2q}{3p}\right)^2 + r = 0$

$-\frac{8q^3}{27p^2} + \frac{4q^3}{9p^2} + r = 0$

~~$-8q^3 + 12q^3 + r = 0$~~

$-8q^3 + 12q^3 + 27p^2 r = 0$

$4q^3 + 27p^2 r = 0$

3/3

12/15

10a) i) $9 = 16(1-e^2)$

$e^2 = \frac{7}{16}$

$e = \frac{\sqrt{7}}{4}$

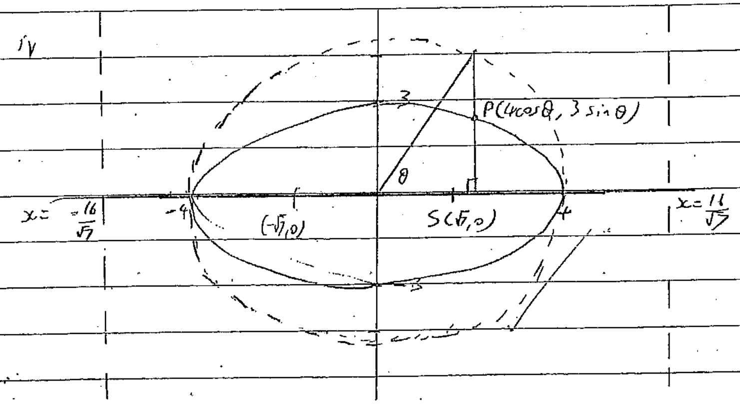
ii) $S(\pm ae, 0)$

$S(\pm\sqrt{7}, 0)$

iii) $x = \pm\frac{a}{e}$

$x = \pm\frac{16}{\sqrt{7}}$

iv



10(a) vi $\frac{x^2}{16} + \frac{y^2}{9} = 1$

$\frac{2x}{16} + \frac{2y}{9} \times \frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{9x}{16y}$

$= \frac{-3\cos\theta}{4\sin\theta}$

$y - 3\sin\theta = \frac{-3\cos\theta}{4\sin\theta}(x - 4\cos\theta)$

$4y\sin\theta - 12\sin^2\theta = -3x\cos\theta + 12\cos^2\theta$

$3x\cos\theta + 4y\sin\theta = 12(\cos^2\theta + \sin^2\theta)$

$\cancel{3x\cos\theta + 4y}$

$3x\cos\theta + 4y\sin\theta = 12$

b) $2\sin^{-1}x = \cos^{-1}x$

$\sin^{-1}(x) = \frac{1}{2}\cos^{-1}(x)$

let $\sin^{-1}x = \alpha$

$x = \sin\alpha$

$\frac{1}{2}\cos^{-1}(x) = \beta$

$\cos^{-1}(x) = 2\beta$

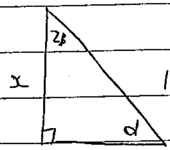
$x = \cos 2\beta$

$\cancel{\alpha = 2\beta}$

$\cos 2\beta = 1 - 2\sin^2\beta$

$1 - 2\sin^2\beta = \sin\alpha$

$\cancel{2\sin^2\beta = \sin(\frac{\pi}{2} - 2\beta)}$



why?

$\alpha + 2\beta = \frac{\pi}{2}$

$\alpha = \frac{\pi}{2} - 2\beta$

$2\alpha = \beta$

$3\beta = \frac{\pi}{2}$

$\beta = \frac{\pi}{6}$

$\alpha = \frac{\pi}{3}$

$x = \sin\frac{\pi}{3}$

$\cancel{x = \frac{1}{2}}$

$x = \frac{\sqrt{3}}{2}$

10(c) $P(x) = x^4 - 2Ax^3 + B$ Show $B = A^4$

sum of roots

$\alpha + \beta + \gamma + \alpha + \beta + \gamma = 2A$

$\alpha + \beta + \gamma = A$

$\cancel{\alpha(\alpha+\beta+\gamma) + \beta(\alpha+\beta+\gamma) + \gamma(\alpha+\beta+\gamma)}$

Sum of roots 4x

$\alpha\beta\gamma(\alpha+\beta+\gamma) = B$

$\cancel{A^4 = (\alpha+\beta+\gamma)^4}$

$\cancel{(\alpha+\beta+\gamma)^3 =}$

$P(\alpha) = \alpha^4 - 2A\alpha^3 + B = 0$

$P(\beta) = \beta^4 - 2A\beta^3 + B = 0$

$\alpha\beta\gamma(\alpha+\beta+\gamma) = B$

$\cancel{(\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2)} = \alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$

$\alpha\beta\gamma A = B$

$\cancel{P(\alpha+\beta+\gamma)}$

$\cancel{(\alpha+\beta+\gamma)^3 = (\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\alpha\gamma+\beta\gamma))}$

$\cancel{= \alpha^3+\beta^3+\gamma^3+2(\alpha^2\beta+\alpha^2\gamma+\alpha\beta\gamma)}$

sum of roots 2x

$\alpha(\alpha+\beta+\gamma) + \beta(\alpha+\beta+\gamma) + \gamma(\alpha+\beta+\gamma) + \alpha\beta + \alpha\gamma + \beta\gamma = 0$

$\cancel{(\alpha+\beta+\gamma)^2 + \alpha\beta + \alpha\gamma + \beta\gamma = 0}$

$\cancel{\alpha\beta + \alpha\gamma + \beta\gamma = -(\alpha+\beta+\gamma)^2}$

$\cancel{(\alpha\beta + \alpha\gamma + \beta\gamma)^2 = (\alpha+\beta+\gamma)^4}$

$\cancel{\alpha^2\beta^2 + \alpha^2\beta\gamma + \alpha\beta^2\gamma}$ $(\alpha+\beta)(\alpha+\beta+\gamma) + \gamma(\alpha+\beta+\gamma) + \alpha\beta + \alpha\gamma + \beta\gamma = 0$

$(\alpha+\beta+\gamma)(\alpha+\beta+\gamma) + \alpha\beta + \alpha\gamma + \beta\gamma = 0$

$(\alpha+\beta+\gamma)^2 = -(\alpha\beta + \alpha\gamma + \beta\gamma)$

Sum of roots

$3x$

$\alpha\beta\gamma + \alpha\beta(\alpha+\beta+\gamma) + \alpha\gamma(\alpha+\beta+\gamma) + \beta\gamma(\alpha+\beta+\gamma) = 0$

$\alpha\beta\gamma + (\alpha\beta+\alpha\gamma+\beta\gamma)(\alpha+\beta+\gamma) = 0$

$\cancel{\alpha\beta\gamma +}$ $B + A(\alpha\beta+\alpha\gamma+\beta\gamma) = 0$

$A(\alpha\beta\gamma) = B$

$A(\alpha\beta\gamma) + A(\alpha\beta+\alpha\gamma+\beta\gamma) = 0$

$A(\alpha\beta\gamma+\alpha\beta+\alpha\gamma+\beta\gamma) = 0$

$A \neq 0 \quad \alpha\beta\gamma+\alpha\beta+\alpha\gamma+\beta\gamma = 0$

$\cancel{B = -(\alpha\beta+\alpha\gamma+\beta\gamma)}$

$\cancel{= (\alpha+\beta+\gamma)^2}$

$\alpha\beta\gamma = -(\alpha\beta+\alpha\gamma+\beta\gamma)$ × $1\frac{1}{2}$

$= (\alpha+\beta+\gamma)^2$ $\frac{}{3}$

$A^2 = \alpha\beta\gamma$

$A^4 = (\alpha\beta\gamma)^2$

$=$

$A^2 = \alpha\beta\gamma \qquad A^2 = \alpha\beta\gamma$

$A(\alpha\beta\gamma) = B \qquad A(\alpha\beta\gamma) = B$

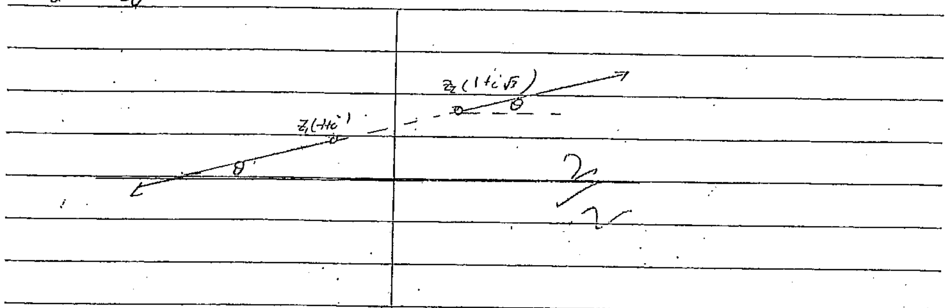
$\cancel{\alpha\beta\gamma = \frac{B}{A}} \qquad A^2(\alpha\beta\gamma)^2 = B^2$

$A^2 = \frac{B}{A} \qquad A \; A^2 = \frac{B}{A^2}$

$\cancel{B = A^3} \qquad A^4 = B$

$x^4 - 2Ax^3 + 0x^2 + 0x + B$

10d z_1



2

8 page writing booklet

$14\frac{1}{2}$ / 15

11a) i $xy = 4$

$y + x\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{y}{x}$

$= -\frac{1}{p^2}$

$y - \frac{2}{p} = -\frac{1}{p^2}(x - 2p)$

$\cancel{p^2y =}$

$p^2y - 2p = -x + 2p$

$x + yp^2 = 4p$ 2

ii $x + q^2y = 4q$ — ①

$x + p^2y = 4p$ — ②

①−② $y(q^2-p^2) = 4(q-p)$

$y = \frac{4(q-p)}{(q+p)(q-p)}$

$y = \frac{4}{q+p}$ 2

$x = 4p - p^2\left(\frac{4}{p+q}\right)$

$= \frac{4p(p+q) - 4p^2}{p+q}$

$= \frac{4pq}{p+q}$

$\therefore M\left(\frac{4pq}{p+q}, \frac{4}{p+q}\right)$

a) iii

$x = \frac{4pq}{p+q} \qquad y = p+q$

$pq = 1$

$\cancel{y = \frac{4}{p+q}}$

$x = \frac{4}{p+q} \Rightarrow y = \frac{4}{p+q}$ ✓

$\therefore y = x$ straight line.

since $p+q \neq 0$

$y \neq 0$

$x \neq 0$

$y \neq 0$ 3.

$p+q \neq 0$

$\therefore$ excludes origin

b) i $\quad x^2 - y^2 = 1$

$2x - 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{x}{y}$

$= \frac{\sec\theta}{\tan\theta}$

$= \frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}$

$= \frac{1}{\sin\theta}$

$y - \tan\theta = \frac{1}{\sin\theta}(x - \sec\theta)$ $1\frac{1}{2}/2$

$y\sin\theta - \tan\theta\sin\theta = x - \sec\theta$

$x - y\sin\theta = \frac{\sin^2\theta}{\cos\theta} - \frac{1}{\cos\theta}$

$\frac{x}{\cos\theta} - y\frac{\sin\theta}{\cos\theta} = \tan^2\theta - \sec^2\theta$

$x\sec\theta - y\tan\theta = 1$ ✓

11 b) ii $\quad x\sec\theta - y\tan\theta = 1$

$y = x$

$x\sec\theta - x\tan\theta = 1$

$x(\sec\theta - \tan\theta) = 1$

$x = \frac{1}{\sec\theta - \tan\theta}$

$L\left(\frac{1}{\sec\theta - \tan\theta}, \frac{1}{\sec\theta - \tan\theta}\right)$ ✓

$y = -x$

$x\sec\theta + x\tan\theta = 1$

$x = \frac{1}{\sec\theta + \tan\theta}$

$M\left(\frac{1}{\sec\theta + \tan\theta}, \frac{-1}{\sec\theta + \tan\theta}\right)$ ✓

let $P(x_1, y_1)$

$x_1 = \frac{\frac{1}{\sec\theta+\tan\theta} + \frac{1}{\sec\theta-\tan\theta}}{2}$

$= \frac{\sec\theta - \tan\theta + \sec\theta + \tan\theta}{2(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)}$

$= \frac{2\sec\theta}{2(\sec^2\theta - \tan^2\theta)}$

$= \sec\theta$ ✓   3.

$y_1 = \frac{\frac{1}{\sec\theta-\tan\theta} - \frac{1}{\sec\theta+\tan\theta}}{2}$

$= \frac{\sec\theta + \tan\theta - \sec\theta + \tan\theta}{2(\sec^2\theta - \tan^2\theta)}$

$= \tan\theta$ ✓

$\therefore P(\sec\theta, \tan\theta)$

P is the midpoint of LM.

11b) $y = x$ ~~$m = \frac{\pi}{4}$~~

$y = -x$

$m_1 \times m_2 = -1$

$\therefore OL \perp OM$

$$OL^2 = \left(\frac{1}{\sec\theta - \tan\theta}\right)^2 + \left(\frac{1}{\sec\theta - \tan\theta}\right)^2$$

$$= \frac{2}{\sec^2\theta + \tan^2\theta - 2\sec\theta\tan\theta}$$

$$= \frac{2\cos^2\theta}{\cos^2\theta + \sin^2\theta - 2\sin\theta}$$

$$= \frac{2\cos^2\theta}{\sin^2\theta - 2\sin\theta}$$

$$OM^2 = \left(\frac{1}{\sec\theta + \tan\theta}\right)^2 + \left(\frac{-1}{\sec\theta + \tan\theta}\right)^2$$

$$= \frac{2}{\sec^2\theta + \tan^2\theta + 2\sec\theta\tan\theta} = 2\cos^2\theta$$

~~Are~~

$$\text{Area} = \frac{1}{2} \times OL \times OM$$

$$= \frac{1}{2} \times \sqrt{\frac{4}{(\sec^2\theta + \tan^2\theta - 2\sec\theta\tan\theta)(\sec^2\theta + \tan^2\theta + 2\sec\theta\tan\theta)}}$$

~~$\sec^4\theta +$~~

$$OM^2 = \frac{2}{\frac{1}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} + \frac{2\sin\theta}{\cos^2\theta}}$$

$$= \frac{2\cos^2\theta}{1 + \sin^2\theta + 2\sin\theta}$$

$$OL^2 = \frac{2}{\frac{1}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} - \frac{2\sin\theta}{\cos^2\theta}}$$

$$= \frac{2\cos^2\theta}{1 + \sin^2\theta - 2\sin\theta}$$

$$OM^2 \times OL^2 = \frac{4\cos^4\theta}{(\sin^2\theta + 2\sin\theta + 1)(\sin^2\theta - 2\sin\theta + 1)}$$

$$= \frac{4\cos^4\theta}{\sin^4\theta + 2\sin^2\theta - 4\sin^2\theta + 1}$$

$$= \frac{4\cos^4\theta}{\sin^4\theta + 2\sin^2\theta + 1}$$

$$= \frac{4\cos^4\theta}{(\sin^2\theta + 1)(\sin^2\theta + 1)}$$

$$= \frac{4\cos^4\theta}{(1 - \cos^2\theta + 1)(1 - \cos^2\theta + 1)}$$

$$= \frac{4\cos^4\theta}{(2 - \cos^2\theta)^2}$$

$$= \frac{4\cos^4\theta}{\sin^4\theta - 2\sin^2\theta + 1}$$

$$= \frac{4\cos^4\theta}{(\sin^2\theta - 1)^2}$$

$$= \frac{4\cos^4\theta}{(1 - \cos^2\theta - 1)^2}$$

$$= \frac{4\cos^4\theta}{\cos^4\theta}$$

$= 4$ $\therefore$ ~~independent of θ & P~~

$OM \times OL = 2$ since OM & OL are lengths

$\therefore$ Area $= \frac{1}{2} \times OM \times OL$

$= \frac{1}{2} \times 2$

$= 1$

$\therefore$ independent of θ & P

well done!

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